DEPARTMENT OF ENERGETICS

POLITECNICO DI TORINO

Turbulence: Basic Physics and Engineering Modeling

Numerical Heat Transfer

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Outline of this Section

- Fundamental Considerations
  - Introduction
  - Length and Time Scales in Turbulence
- Modeling of Turbulence
  - Reynolds Averaged Navier – Stokes (RANS)
  - Direct Numerical Simulation (DNS)
  - Large Eddy Simulation (LES)
- RANS models and Boundary Layers
  - k–ε closure models
  - law of the wall

da Vinci Sketch of Turbulent Flow

"… the smallest eddies are almost numberless, and large things are rotated only by large eddies and not by small ones, and small things are turned by small eddies and large."

The Reynolds Experiment

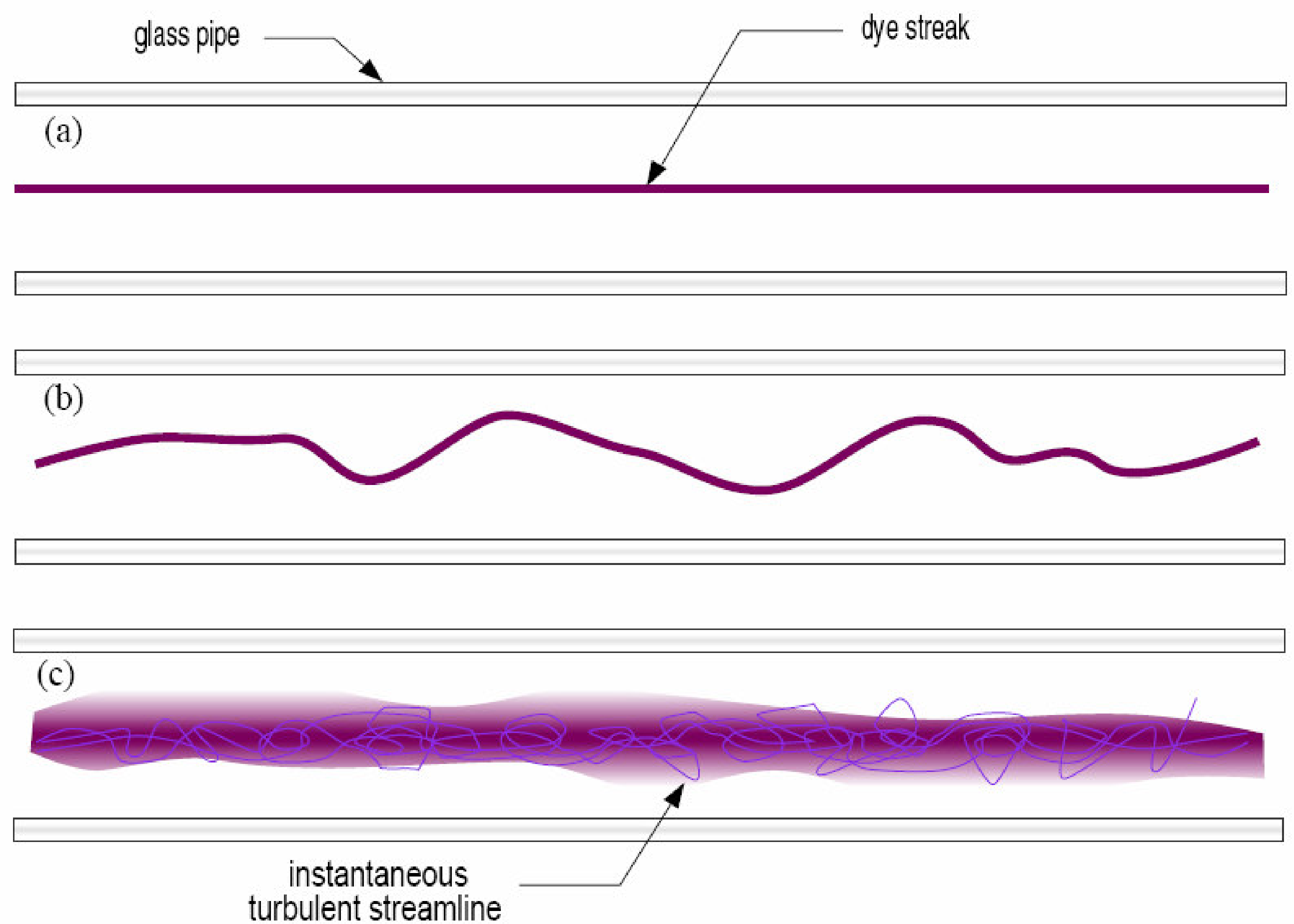
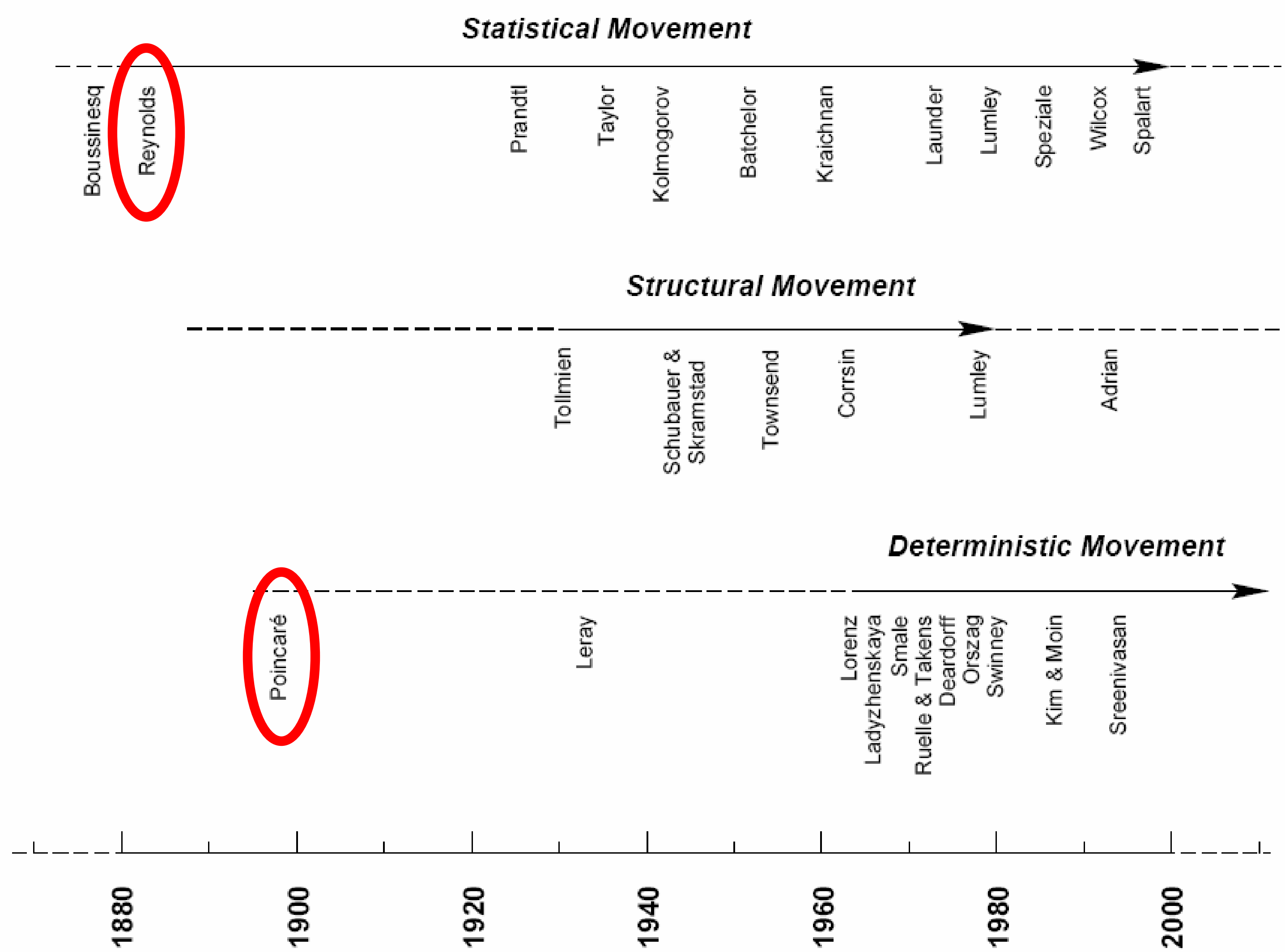


Figure 1.2: The Reynolds experiment; (a) laminar flow, (b) early transitional (but still laminar) flow, and (c) turbulence.

Three Approaches to Turbulence



Modern View of Turbulence

"Turbulence is any chaotic solution to the 3-D Navier–Stokes equations that is sensitive to initial data and which occurs as a result of successive instabilities of laminar flows as a bifurcation parameter is increased through a succession of values."

- Random is not (deterministic) chaos
- Navier – Stokes system of equations may exhibit turbulent solutions
- Turbulent solutions show sensitivity to initial conditions
- The sequence of bifurcations is usually quite short (steady → periodic → quasiperiodic → turbulent)

Main Features of Turbulence

1. disorganized, chaotic, seemingly random behavior;
2. nonrepeatability (*i.e.*, sensitivity to initial conditions);
3. extremely large range of length and time scales (but such that the smallest scales are still sufficiently large to satisfy the continuum hypothesis);
4. enhanced diffusion (mixing) and dissipation (both of which are mediated by viscosity at molecular scales);
5. three dimensionality, time dependence and rotationality (hence, potential flow cannot be turbulent because it is by definition irrotational);
6. intermittency in both space and time.

Navier – Stokes System of Equations

- Navier – Stokes (NS) equations for incompressible flow:

$$\nabla \cdot \boldsymbol{U} = 0\,,$$
$$\boldsymbol{U}_t + \boldsymbol{U} \cdot \nabla \boldsymbol{U} = -\nabla P + \nu \Delta \boldsymbol{U} + \boldsymbol{F}_B\,.$$

- The critical parameter driving the succession of bifurcations and ruling the transition from laminar to turbulent flow is the Reynolds number, namely

$$\mathrm{Re} = UL / \nu$$

where U is a velocity scale, L is a typical length scale and ν is the kinematic viscosity.

Existence (E), Uniqueness (U), Regularity

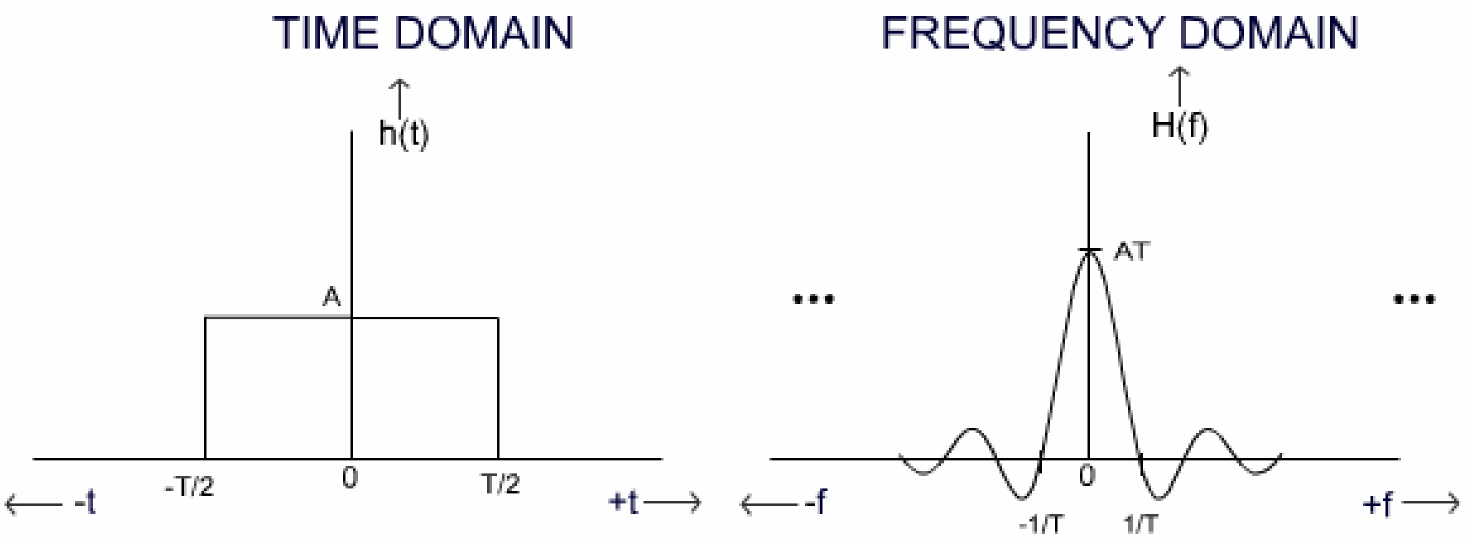
	2D	3D
Strong Solutions	E & U	E?, U
Weak Solutions	E & U	E, U?

- Even though it should happen that one day it is proven that long-time strong solutions do not exist, we will still rely on the weak solutions, which exist all the time, but may be not unique.
- Attempting to employ high-order numerical methods for discretization of NS equations in strong form is likely doomed to fail.

Fourier Transform (in Time)

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-i\omega t}\, dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{i\omega t}\, d\omega$$



$h(t) = A \quad |t| < T/2$

$\quad = A/2 \quad |t| = T/2$

$\quad = 0 \quad |t| > T/2$

$\longleftrightarrow$

$H(f) = AT \dfrac{\sin(\pi T f)}{\pi T f}$

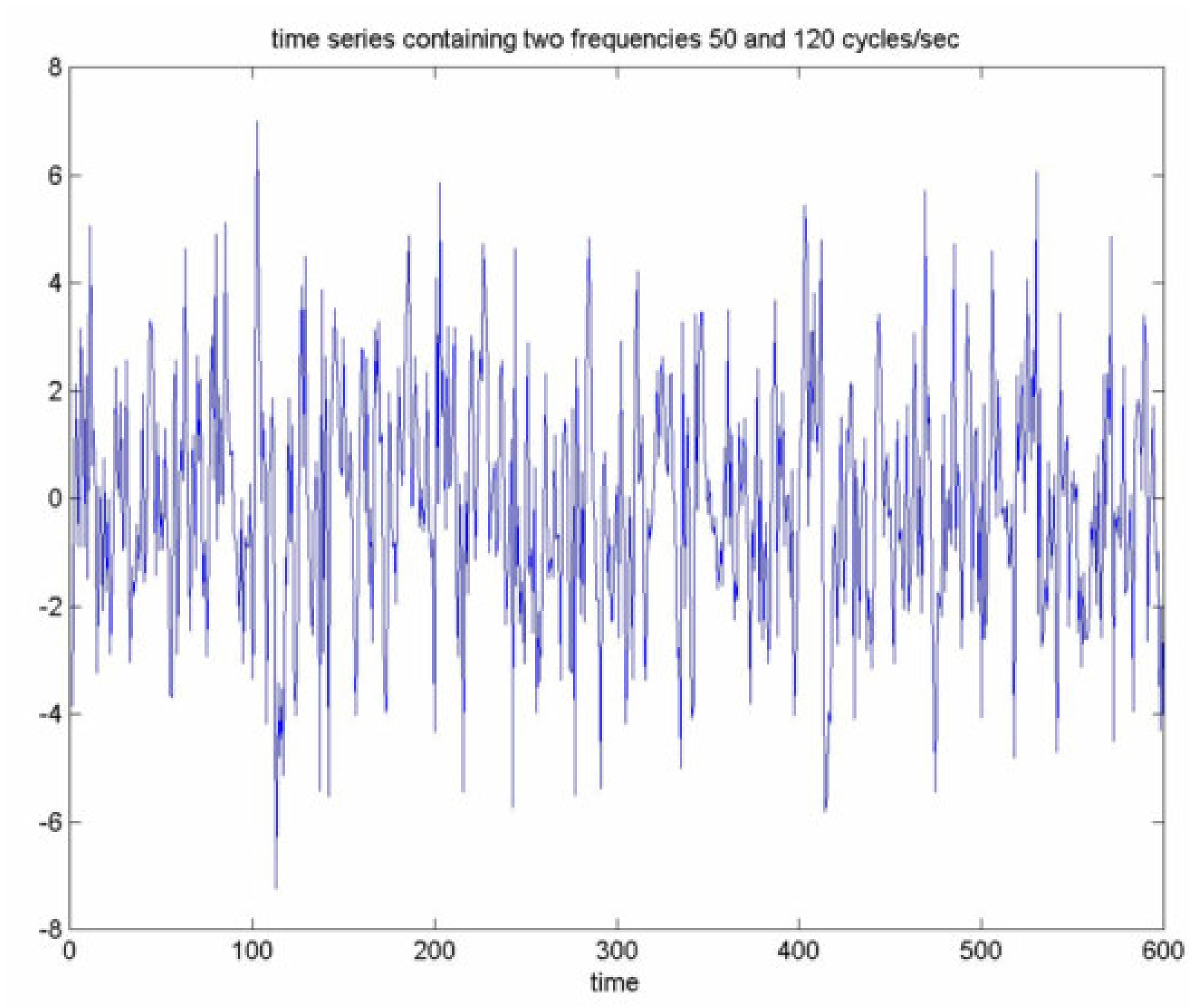
Autocovariance, Frequency Spectrum

$$R_{ff}(\tau) = \int_{-\infty}^{\infty} f(t)\overline{f}(t-\tau)\, dt$$

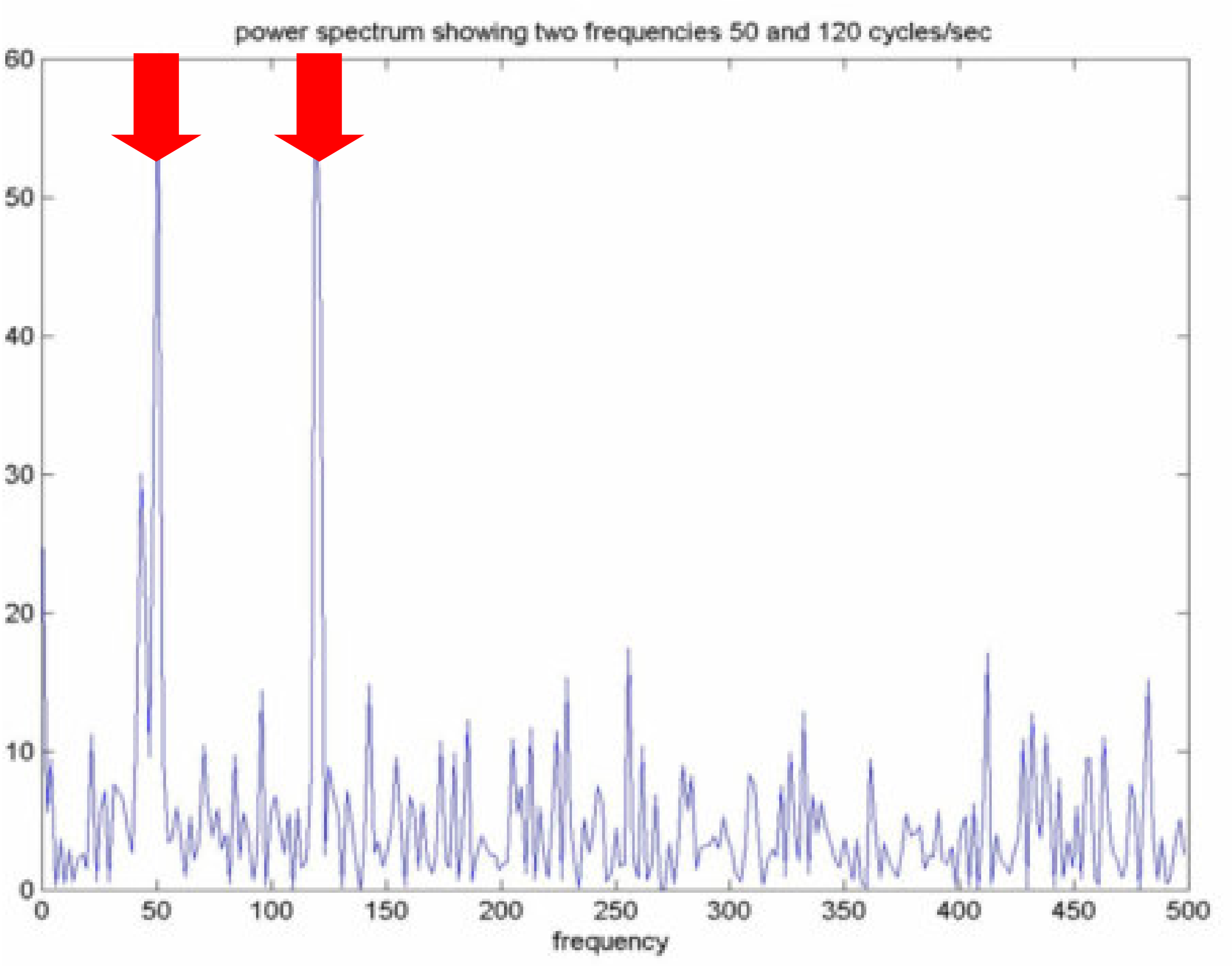
$$S(f) = \int_{-\infty}^{\infty} R(\tau)\, e^{-2\,\pi\, i\, f\, \tau}\, d\tau$$

- Autocovariance (which is proportional to autocorrelation) is a function that provides a measure of how well a signal remembers what it has happened.
- The frequency spectrum allows to identify the leading oscillating components of a signal, in order to identify periodicity in pseudo-random behavior.

Example: Pseudo-Random Signal



Example: Pseudo-Random Signal



Time → Space: Taylor Hypothesis

Taylor's hypothesis. As noted in Def. 1.99, Taylor's hypothesis can be employed to deduce spatial information about turbulent fluctuations using time series of measurements at a single point, or at a sequence of points at which measurements have not been taken simultaneously. The preceding discussions of length and time scales suggest that these can typically be related through some velocity scale, and this is what is involved when invoking Taylor's hypothesis. In particular, as described in [7], measurements of fluctuating velocities are sometimes collected by traversing a probe through the flow field so rapidly that the nature of the turbulence does not change significantly during the measurement process. This permits construction of spatial derivatives of the fluctuating quantities at an ostensibly fixed time. If the speed of traversal U of the probe is sufficiently high, then a fluctuating velocity signal $u'(t)$ at a fixed location can be identified with fluctuations at a different location a distance x away by substituting $t = x/U$. This is often termed a "frozen turbulence" approximation, and it is shown in Hinze [6], among other places, that $|u'|/U \ll 1$ must hold for results obtained from Taylor's hypothesis to be valid.

Two-point, Wavenumber Spectrum

$$R_{i,j}(\mathbf{r}) = \int u_j(\mathbf{x}) u_i(\mathbf{x}+\mathbf{r}) d\mathbf{x}$$

$$\phi_{i,j}(\mathbf{k}) = \frac{1}{(2\pi)^3} \int exp(-i\mathbf{k}.\mathbf{r}) R_{i,j}(\mathbf{r}) d\mathbf{r}$$

- The concept of stationary is somehow ambiguous in turbulence → it is better to introduce the concept of homogenous (statistics invariant of translations) and isotropic (statistics invariant of translations, rotations and reflections) turbulence.
- The wavenumber spectrum tensor is the Fourier transform of the two–point one-time autocovariance.

Energy Spectrum

$$E = 1/2 \int u_i(\mathbf{x}) u_i(\mathbf{x}) d\mathbf{x} \tag{4.5}$$

can be written in terms of the spectrum $\phi_{i,j}(\mathbf{k})$

$$E = \frac{1}{2} \int \phi_{i,i}(\mathbf{k}) d\mathbf{k} = \int E(\mathbf{k}) d\mathbf{k} \tag{4.6}$$

where $\phi_{i,j}(\mathbf{k})$ is the Fourier transform of the velocity correlation tensor $R_{i,j}(\mathbf{r})$:

$$\phi_{i,j}(\mathbf{k}) = \frac{1}{(2\pi)^3} \int exp(-i\mathbf{k}.\mathbf{r}) R_{i,j}(\mathbf{r}) d\mathbf{r} \; ; \; R_{i,j}(\mathbf{r}) = \int u_j(\mathbf{x}) u_i(\mathbf{x} + \mathbf{r}) d\mathbf{x} \tag{4.7}$$

$R_{i,j}(\mathbf{r})$ tells us how velocities at points separated by a vector $\mathbf{r}$ are related. If we know these two point velocity correlations, we can deduce $E(\mathbf{k})$. Hence the energy spectrum has the information content of the two-point correlation.

$E(\mathbf{k})$ contains directional information. More usually, we want to know the energy at a particular scale $k = \sqrt{\mathbf{k}.\mathbf{k}}$ without any interest in separating it by direction. To find $E(k)$, we integrate over the spherical shell of radius k (in 3-dimensions):

$$E = \int E(\mathbf{k}) d\mathbf{k} = \int_0^\infty \oint E(\mathbf{k}) d\sigma dk = \int_0^\infty E(k) dk \tag{4.8}$$

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Length and Time Scales in Turbulence

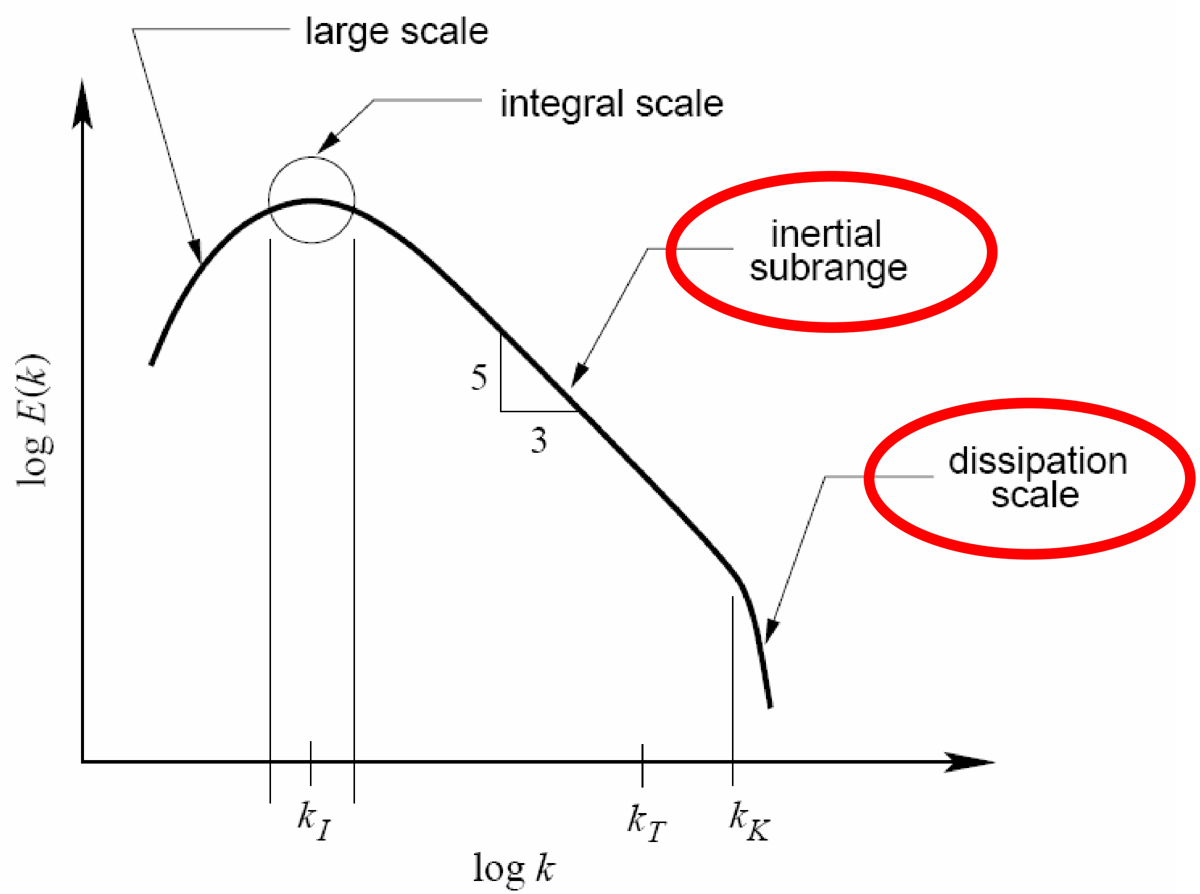


Figure 1.5: Turbulence energy wavenumber spectrum.

Integral Scale (ℓ)

$$\ell = \frac{1}{\|u'\|_{L^2}^2} \int_{-\infty}^{\infty} u'(\boldsymbol{x},t)u'(\boldsymbol{x}+\boldsymbol{r},t)\,d\boldsymbol{r}$$

$$\ell = \frac{|u'|^3}{\varepsilon} \qquad \varepsilon = 2\nu\|\boldsymbol{S}\|^2$$

- The integral scale is defined by a single wavenumber, which corresponds to the maximum in turbulence energy.
- It can be expressed either by means of autocorrelation function or by means of turbulence energy dissipation rate ε, which depends on the symmetric part of the stress tensor.

Taylor (Micro) Scale (λ)

$$\lambda^2 = \frac{\langle |u'|^2 \rangle}{\langle \|\boldsymbol{S}\|^2 \rangle}, \qquad \lambda = \left[\frac{\nu \langle |u'|^2 \rangle}{\varepsilon} \right]^{1/2}$$

- It does not involve a clear physical meaning → it is considered for historical reasons.
- This scale is roughly consistent with the Kolmogorov inertial subrange scales, but it usually overestimates the actual dissipative eddy size

Kolmogorov (Micro) Scale (η)

$$\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$$

We now find expressions for the smallest scales of turbulence. These were derived by Kolmogorov under the assumption that at these scales mainly dissipation would be important, so the only two physical parameters needed to describe behavior from a dimensional standpoint are viscosity ν and dissipation rate ε of turbulence kinetic energy.

$$\frac{\eta}{\ell} \sim \left(\frac{\nu^3}{|u'|^3\ell^3}\right)^{1/4} \sim Re_\ell^{-3/4}$$

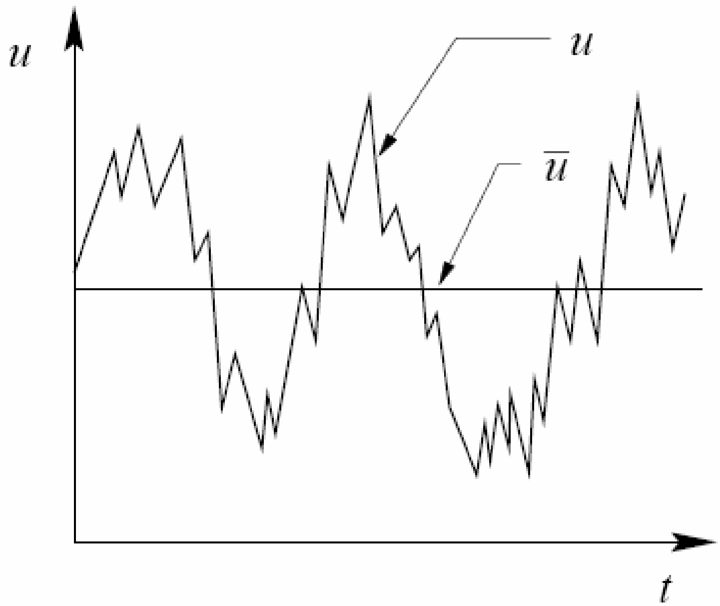
Very huge computational demand !!

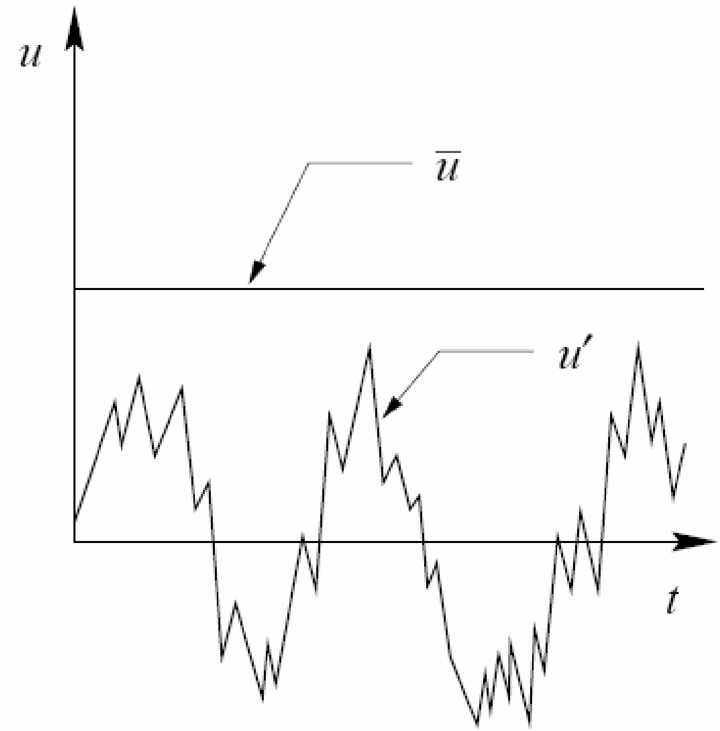
Reynolds Decomposition

Then the *Reynolds decomposition* of $u(\boldsymbol{x}, t)$ is

$$u(\boldsymbol{x}, t) = \overline{u}(\boldsymbol{x}) + u'(\boldsymbol{x}, t) \, ,$$

where $u'(\boldsymbol{x}, t)$ is termed the "fluctuating part."





Derivation of RANS equations 1/2

$$\nabla \cdot \boldsymbol{U} = \nabla \cdot (\overline{\boldsymbol{u}} + \boldsymbol{u}') = \nabla \cdot \overline{\boldsymbol{u}} + \nabla \cdot \boldsymbol{u}' = 0\,. \qquad (2.2)$$

Then averaging this equation results in

$$\nabla \cdot \overline{\overline{\boldsymbol{u}}} + \nabla \cdot \overline{\boldsymbol{u}'} = 0\,, \qquad (2.3)$$

and from Eq. (1.26) we deduce that

$$\nabla \cdot \overline{\boldsymbol{u}} = 0\,. \qquad (2.4)$$

Then it follows from the far right-hand side of Eq. (2.2) that

$$\nabla \cdot \boldsymbol{u}' = 0 \qquad (2.5)$$

also holds.

We remark that if the averaging performed in Eq. (2.3) had been omitted, we might conclude that only

$$\nabla \cdot \overline{\boldsymbol{u}} = -\nabla \cdot \overline{\boldsymbol{u}'}$$

Derivation of RANS equations 2/2

$$(\overline{u}+u')_t + (\overline{u}+u')\cdot\nabla(\overline{u}+u') = -\nabla(\overline{p}+p') + \nu\Delta(\overline{u}+u') .$$

Now recall that $\overline{u}$ is independent of t, by definition, so the first term in this expression is identically zero. Then upon expansion of the dot product on the left-hand side we have

$$u'_t + \overline{u}\cdot\nabla\overline{u} + \overline{u}\cdot\nabla u' + u'\cdot\nabla\overline{u} + u'\cdot\nabla u' = -\nabla(\overline{p}+p') + \nu\Delta(\overline{u}+u') .$$

$$\overline{u'_t} + \overline{\overline{u}\cdot\nabla\overline{u}} + \overline{\overline{u}\cdot\nabla u'} + \overline{u'\cdot\nabla\overline{u}} + \overline{u'\cdot\nabla u'} = -\nabla\overline{(\overline{p}+p')} + \nu\Delta\overline{(\overline{u}+u')} . \qquad (2.6)$$

$$\nabla\cdot\overline{u}^2 + \nabla\cdot\overline{u'^2} = -\nabla\overline{p} + \nu\Delta\overline{u} , \qquad (2.9)$$

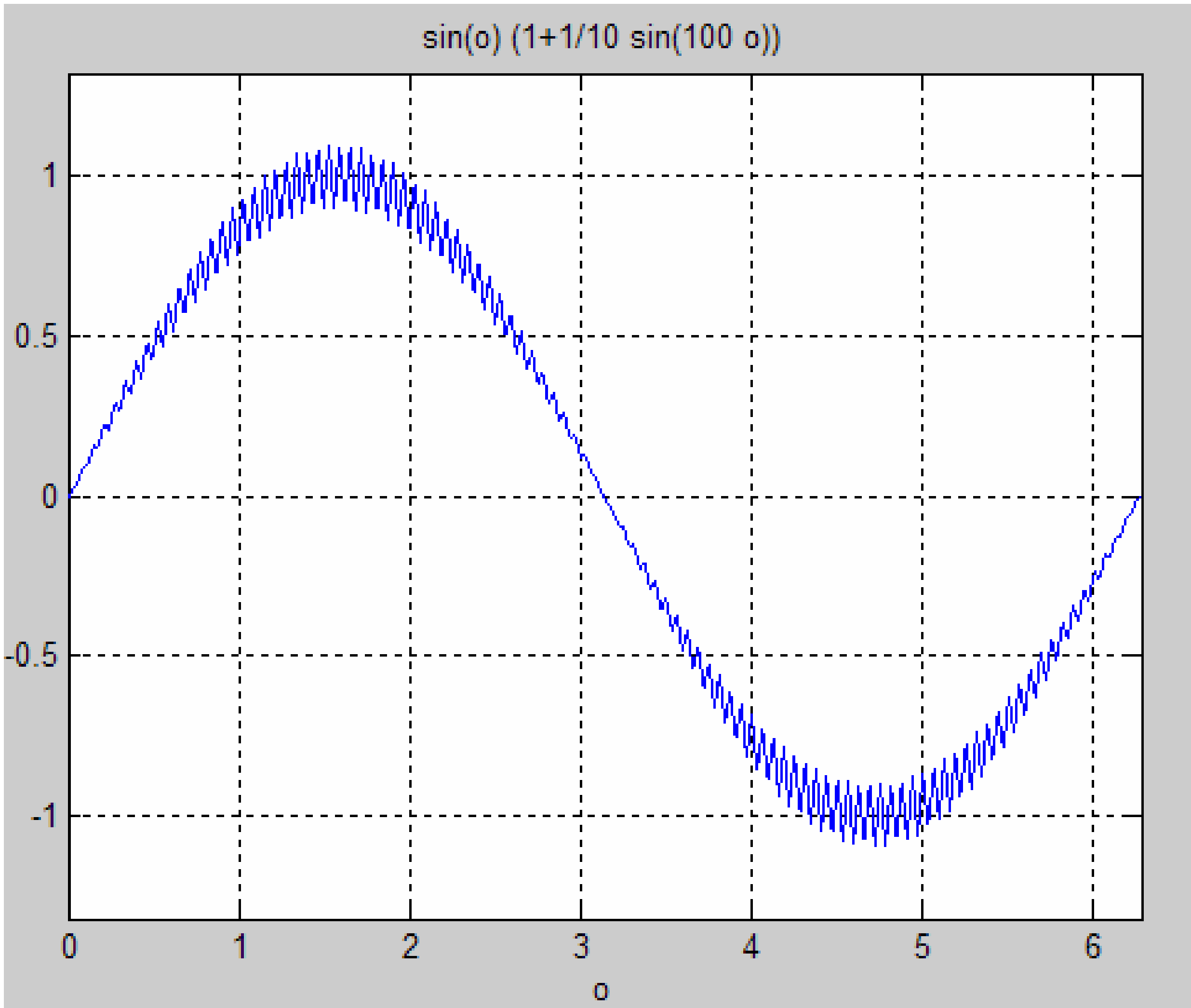
- Closure problem: this vector equations contain more unknowns than equations → the situation is the same in case one considers the higher order statistical equations for averaged quantities

Reynolds (Stress) Tensor

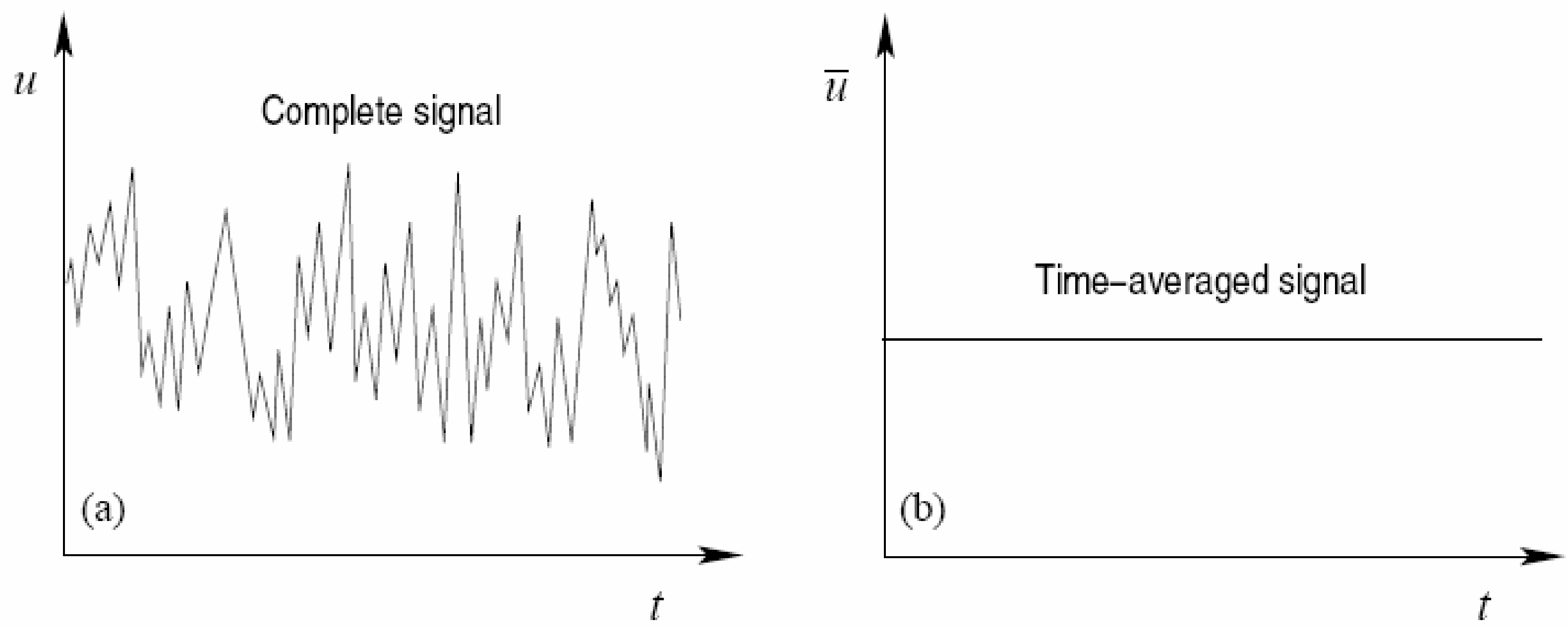
$$\overline{\mathbf{u}'^2} = \begin{pmatrix} \overline{u'^2} & \overline{u'v'} & \overline{u'w'} \\ \overline{u'v'} & \overline{v'^2} & \overline{v'w'} \\ \overline{u'w'} & \overline{v'w'} & \overline{w'^2} \end{pmatrix}$$

- Closure problem: The so-called RANS closure model provide a proper dependence of the unknown tensor on the main flow characteristics (average quantities and/or their gradients) → these models involve some heuristic content, which must be verified before applying it to the considered application.

Multiple Time Scales



General Problems of RANS 1/4



$$physics \longrightarrow statistics$$

Many-to-one mapping !!

Unknown quantities are recursively affecting themselves

General Problems of RANS 2/4

$$u(\boldsymbol{x},t) = \overline{u}(\boldsymbol{x}) + u'(\boldsymbol{x},t)\,. \tag{2.22}$$

$$u(\boldsymbol{x},t) = \sum_{|\boldsymbol{k}|\geq 0}^{\infty} a_{\boldsymbol{k}}(t)\varphi_{\boldsymbol{k}}(\boldsymbol{x})\,.$$

$$u'(\boldsymbol{x},t) = \sum_{\boldsymbol{k}} \left(a_{\boldsymbol{k}}(t) - \overline{a_{\boldsymbol{k}}}\right)\varphi_{\boldsymbol{k}}(\boldsymbol{x})\,. \tag{2.23}$$

The Reynolds fluctuation contains all other modes of the Fourier representation but the first one → There is a lot of physics here, we are neglecting !!

General Problems of RANS 3/4

So consider a fluctuating temperature $\theta'(x,t)$ at a fixed location x and its interaction with a component of fluctuating velocity, say $u'(x,t)$. Now recall that, by definition, both θ' and u' have zero mean; but in the case of θ', the fluctuations will remain close to this mean value for long periods with an ocassional brief departure that may be of quite large magnitude. The consequence of this will be a temperature-velocity correlation such that $u'\theta' \ll \mathcal{O}(1)$, implying that the turbulent heat fluxes have essentially no effect on the mean temperature. But in reality, the turbulent fluctuations could produce very significant local in space and instantaneous in time effects. RANS models are unable, in general, to reproduce such phenomena, and in fact do not attempt to do so. Rather, the goal of RANS modeling in such situations is simply to produce averaged scalar fluxes whose overall effect is close to a "smearing" over time of the actual physics.

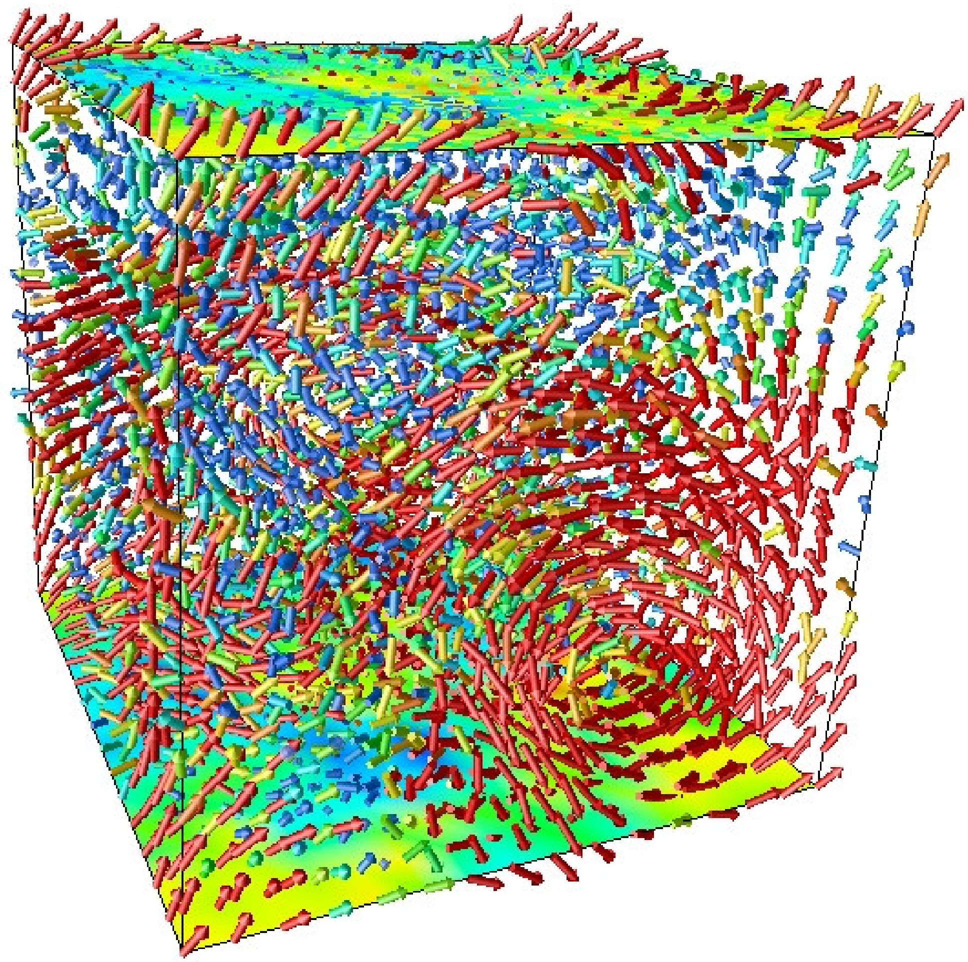
It is not possible to model interactions at the small scales between flow and other quantities (transported scalars) !!

General Problems of RANS 4/4

In this final subsection we consider what is possibly the most important question related to RANS solutions. To put this into the proper setting we first think in terms of time-averaged experimental results. Since it is now almost universally accepted that the N.–S. equations embody all the physics of turbulence, it is reasonable to view experimental time series as solutions (albeit, analog) to the N.–S. equations, subject, of course, to measurement errors. Hence, averaging of any such time series yields a time-averaged solution to the N.–S. equations. Then, with respect to the RANS equations, the natural question to consider is whether solutions to these equations equal the time-averaged solutions to the N.–S. equations. Failure to demonstrate such equality would obviously raise serious questions regarding use of RANS formulations in general, and it is well known that comparisons of RANS solutions with experimental data have, from the earliest calculations to the present, always shown discrepancies.

Conceptually impossible to recover the exact results of NS system, without using the exact expression for the Reynolds stresses !!

Direct Numerical Simulation (DNS)

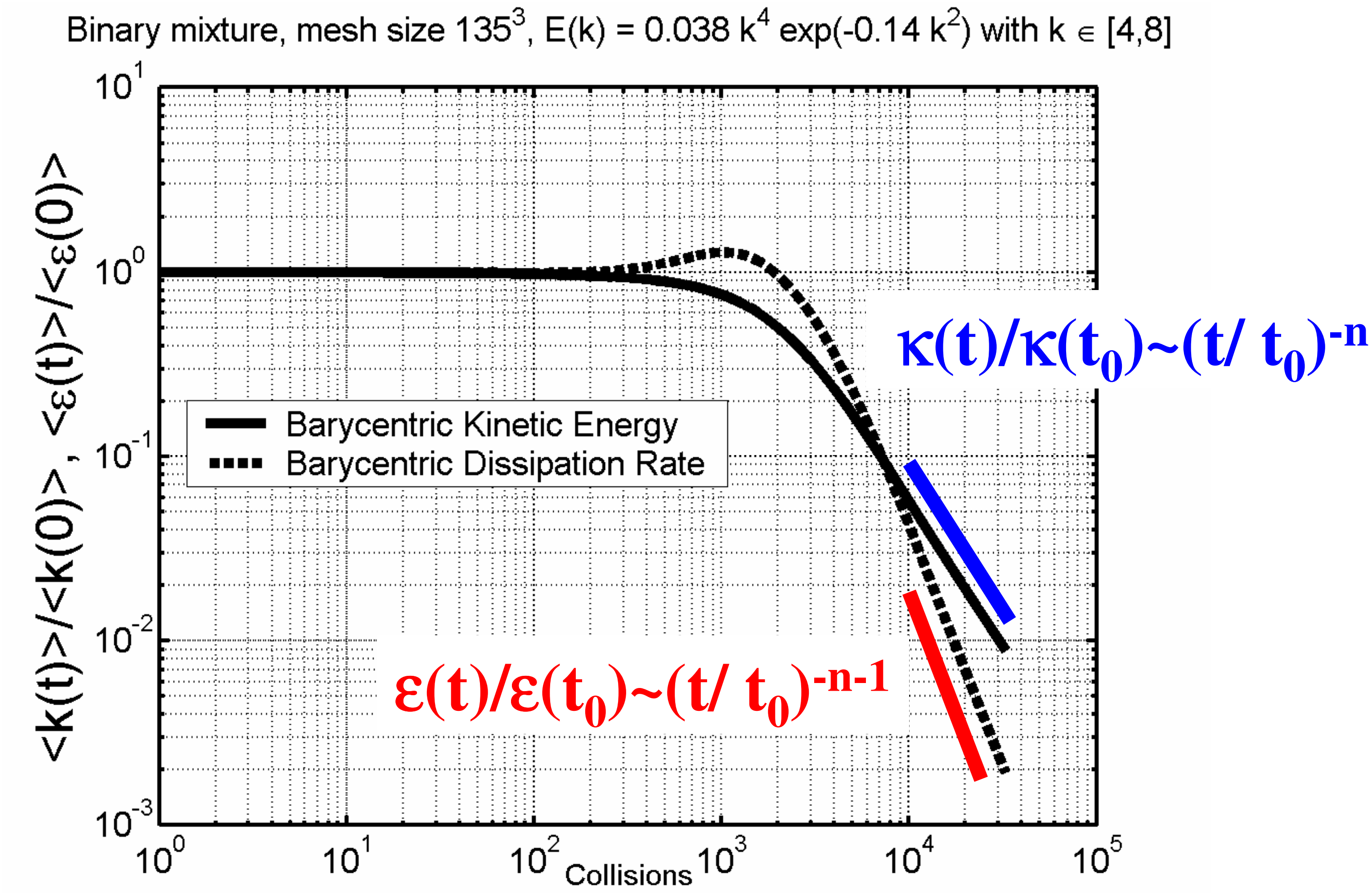


Federico Toschi PhD, IAC-CNR

Direct Numerical Simulation (DNS)

- Direct numerical simulation consists in solving the Navier – Stokes equations, resolving all the scales of motion, with initial and boundary conditions appropriate to the flow considered.
- There is no closure problem !!
- The computational demand is very huge (for the current computational resources), as clearly pointed out by the Kolmogorov estimation of turbulence scales → parallel computing is essential.
- Problems in defining accurate initial conditions, consistent with reality.

Decay of Homogenous Isotropic Turb.



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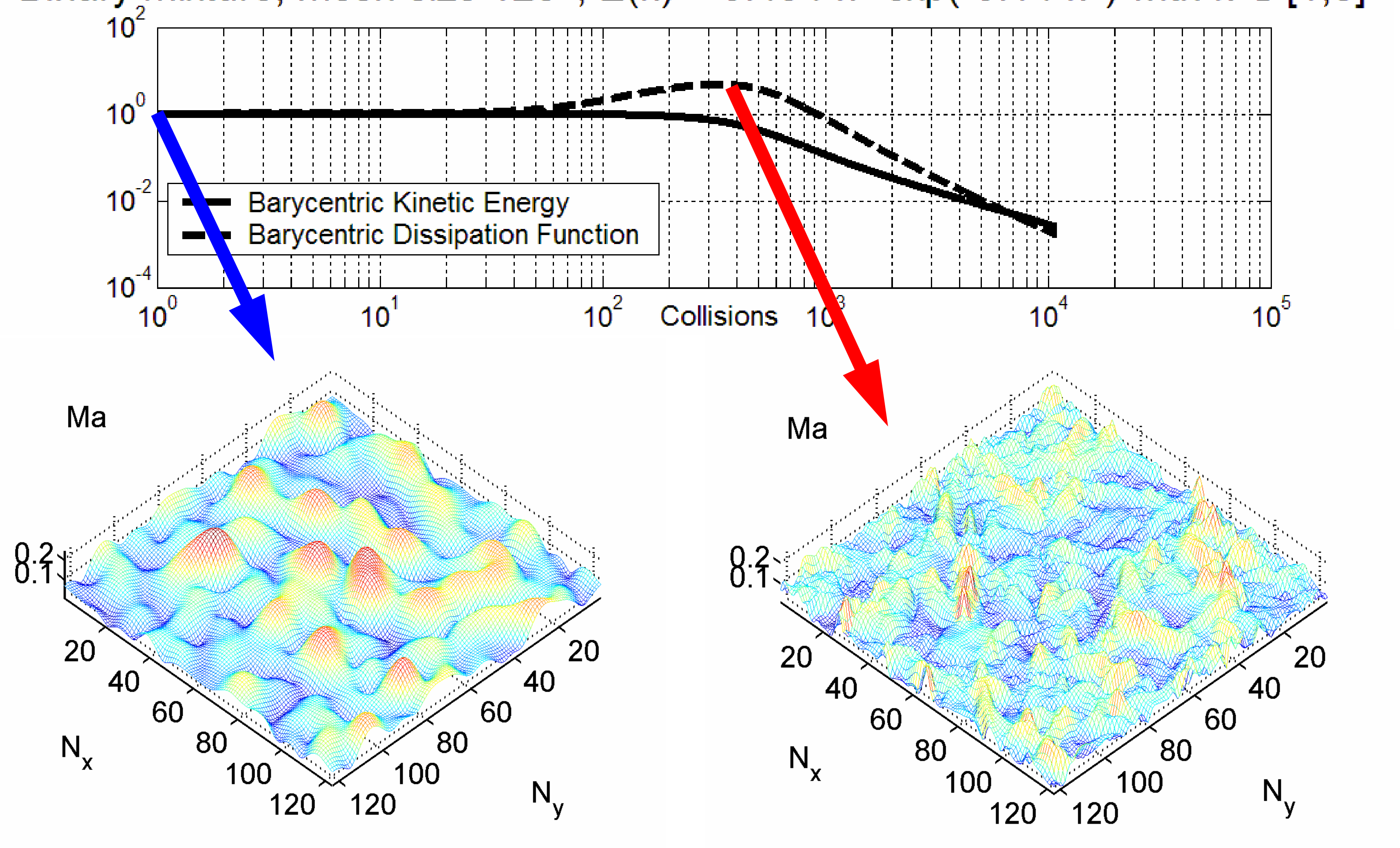
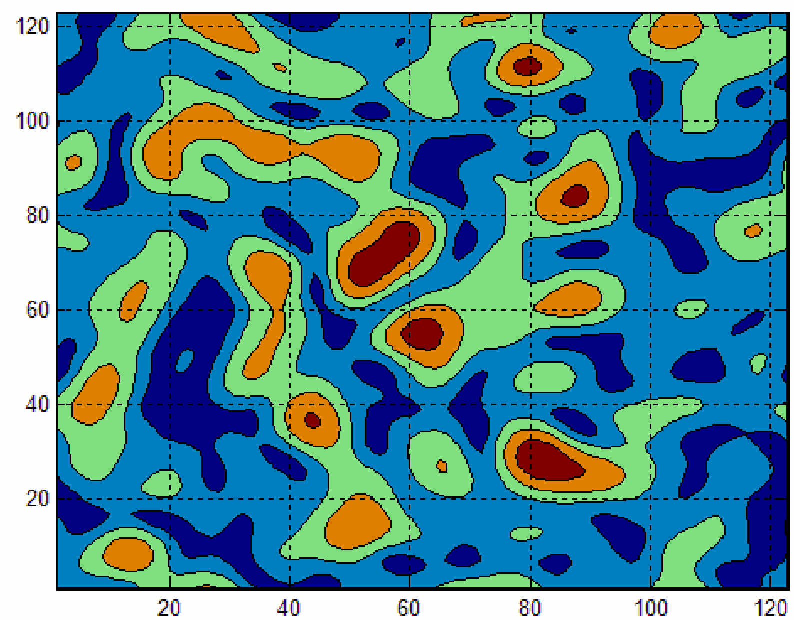
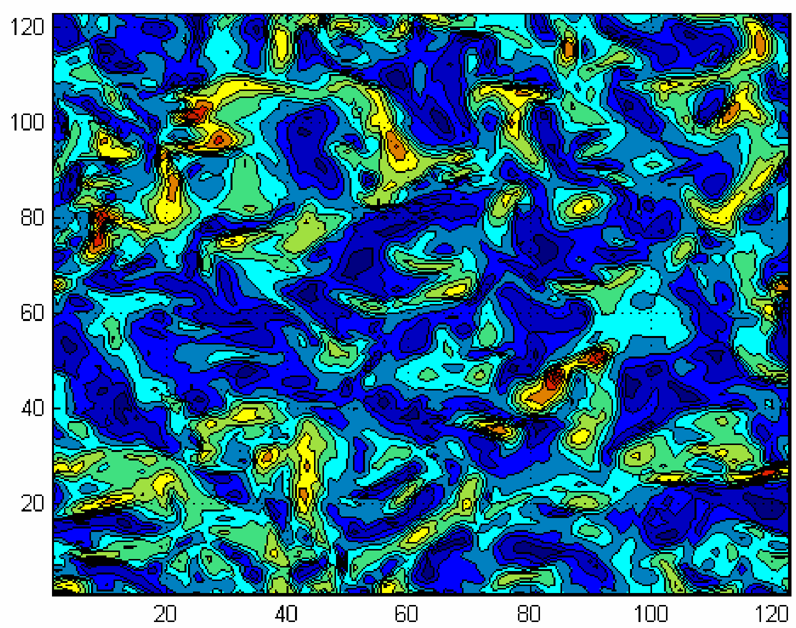
Decay of Homogenous Isotropic Turb.

Binary mixture, mesh size 123^3, $E(k) = 0.494\ k^4 \exp(-0.14\ k^2)$ with $k \in [1,8]$

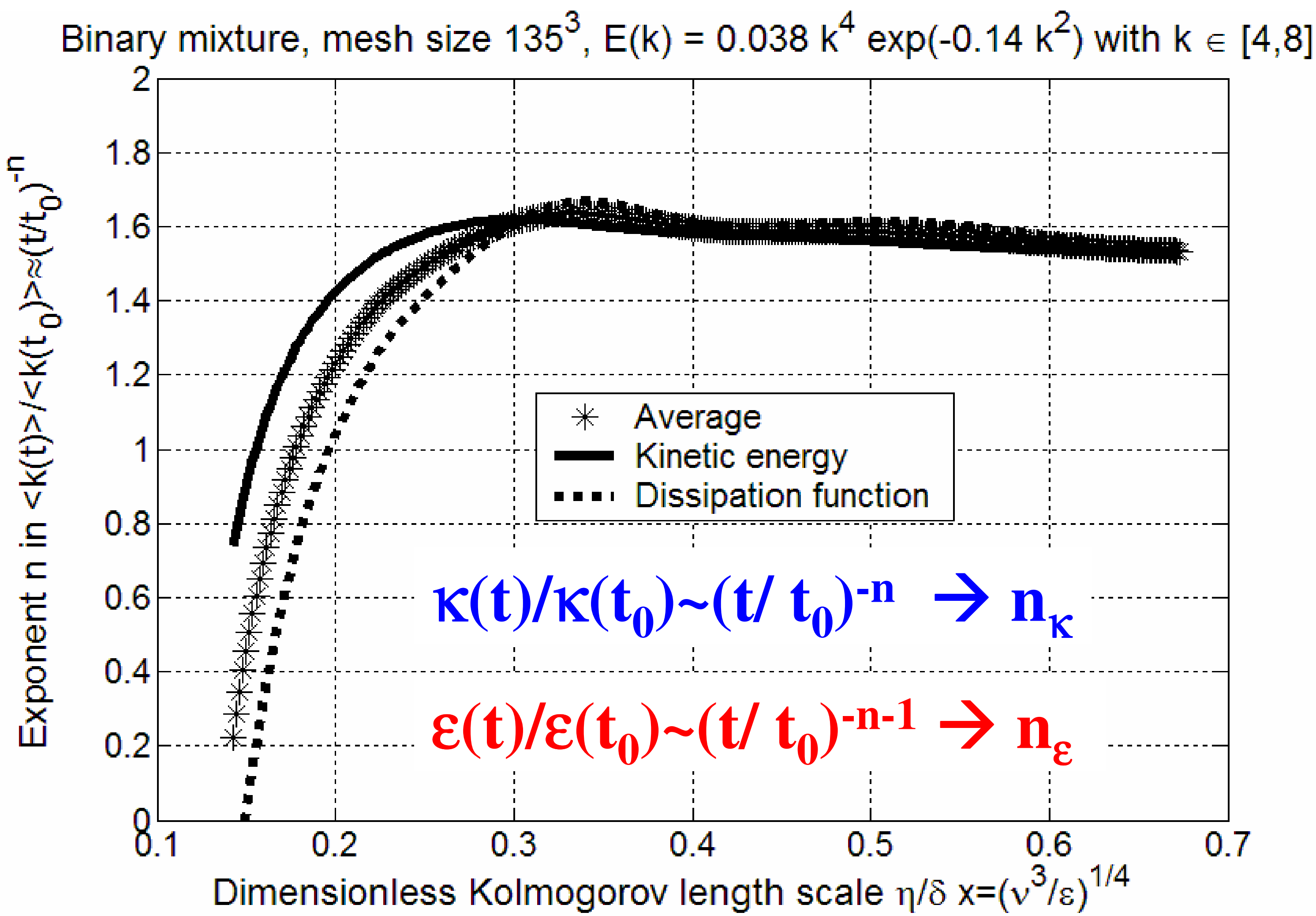


Decay of Homogenous Isotropic Turb.





Optimal Resolution



Large Eddy Simulation (LES)

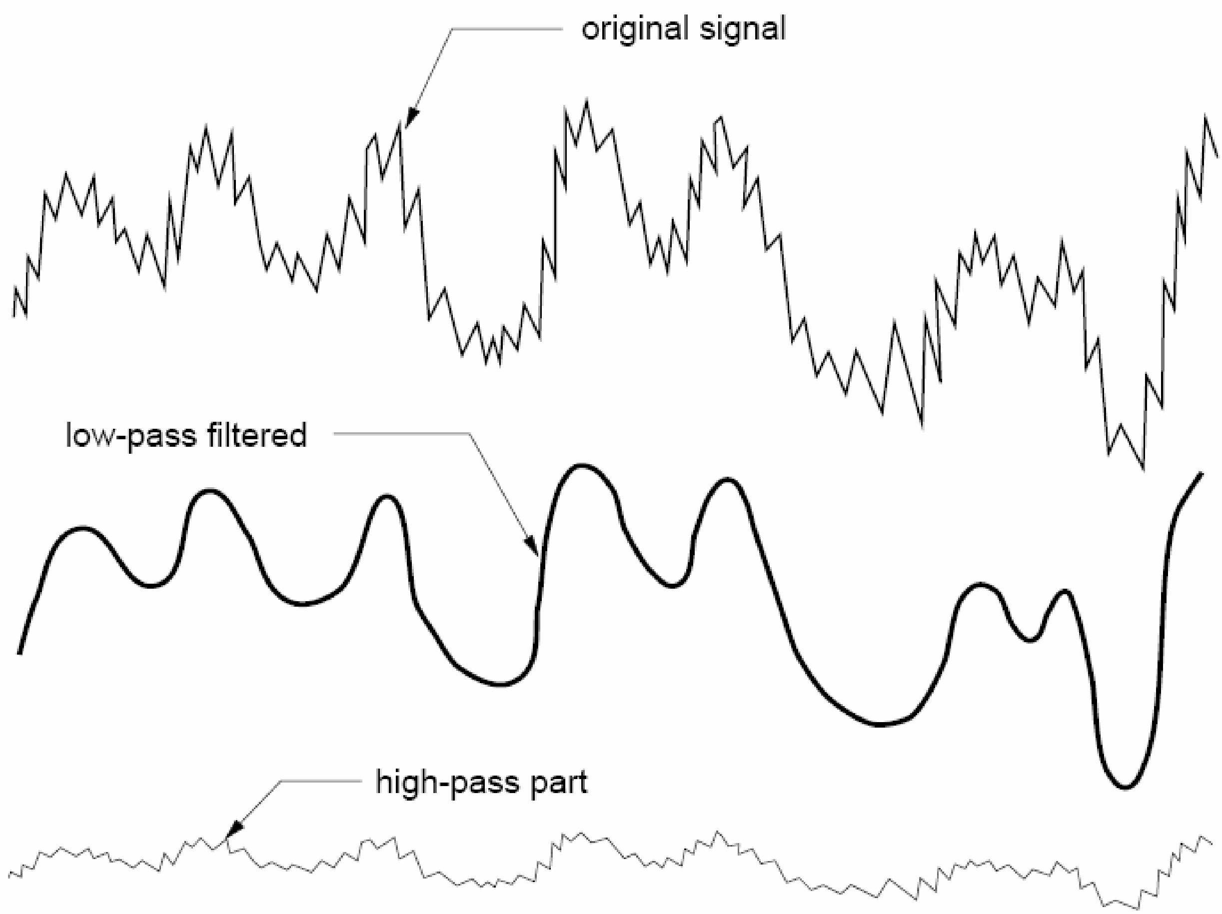


Figure 1.6: Low-pass and high-pass filtered parts of a signal.

Large Eddy Simulation (LES)

i) DNS requires no modeling, but it demands resolution from the large scales all the way through the beginning of the dissipation scales. This results in total arithmetic scaling at least as Re^3.

ii) LES requires modeling of part of the inertial subrange and into the beginning of the dissipation scales. The amount of required modeling is set by the amount of resolution that can be afforded, but it is unlikely that total arithmetic will scale worse than Re^2.

iii) RANS requires modeling of everything from the integral scales into the dissipation range. As a consequence, total arithmetic is at most a weak function of Re.

- Large Eddy Simulation (LES) is a turbulence computational method lying somewhere in between RANS and DNS (also in terms of computational demand)

Large Eddy Simulation (LES)

$$U(x,t) = \widetilde{u}(x,t) + u'(x,t)\,. \tag{3.1}$$

We remind the reader of two key points associated with this form of decomposition. The first is that '$\sim$' represents a spatial filter that should be considered a low-pass filter which, in principle, removes all wavenumbers in the Fourier representation of U above those supported by the chosen discretization of the governing equations. Formally, we have

$$\widetilde{U}(x,t) = \int_{\Omega_i} G(x|\xi)\, U(\xi,t)\, d\xi \equiv \widetilde{u}(x,t)\,, \tag{3.2}$$

where the *filter kernel* G is often taken to be a Gaussian, and Ω_i is a subdomain of the solution domain Ω such that the volume of Ω_i is approximately h^3 with h being the discrete step size of the numerical approximation.

- It is based on a spatial filtering, which excludes the smallest scales in dissipation range because they are assumed independent of the particular considered flow

LES Decomposition

$$u'(x,t) = U(x,t) - \widetilde{u}(x,t)\,.$$

$$\widetilde{\widetilde{u}}(x,t) \neq \widetilde{u}(x,t) \qquad \text{and} \qquad \widetilde{u'}\,(x,t) \neq 0\,,$$

$$U(x,t) = \sum_{|\boldsymbol{k}|\geq 0}^{\boldsymbol{k}_c} a_{\boldsymbol{k}}(t)\varphi_{\boldsymbol{k}}(x) \; + \sum_{|\boldsymbol{k}|=\boldsymbol{k}_c+1}^{\infty} a_{\boldsymbol{k}}(t)\varphi_{\boldsymbol{k}}(x)\,.$$

- The fundamental information of the flow are not lost and the approximation converges to the exact solution by increasing the resolution

Filtered Momentum Equation

$$\frac{\partial \widetilde{\boldsymbol{u}}}{\partial t} + \nabla \cdot \left(\widetilde{\boldsymbol{U}\boldsymbol{U}} \right) = -\nabla \widetilde{p} + \nu \Delta \widetilde{\boldsymbol{u}} \,. \tag{3.9}$$

$$\widetilde{(\widetilde{u} + u')\,(\widetilde{v} + v')} = \widetilde{\widetilde{u}\,\widetilde{v}} + \widetilde{\widetilde{u} v'} + \widetilde{\widetilde{v} u'} + \widetilde{u' v'} \,.$$

$$L_{ij} \equiv \widetilde{\widetilde{u}\,\widetilde{v}} - \widetilde{u}\,\widetilde{v} \,, \qquad \text{(Leonard stress)}$$

$$C_{ij} \equiv \widetilde{\widetilde{u} v'} + \widetilde{\widetilde{v} u'} \qquad \text{(cross stress)}$$

$$R_{ij} \equiv \widetilde{u' v'} \,, \qquad \text{(Reynolds stress)}$$

- The subgrid part of a LES representation consists of high-pass filtering of the solution, thus carrying information only from the modes above cut-off wavenumber

Subgrid – Scale (SGS) Model

$$\widetilde{\boldsymbol{u}}_t + \nabla\cdot(\widetilde{\boldsymbol{u}}\widetilde{\boldsymbol{u}}) = -\nabla\widetilde{p} + \nu\Delta\widetilde{\boldsymbol{u}} - \nabla\cdot\boldsymbol{\tau}_{SGS}\,, \tag{3.12}$$

$$\tau_{SGS,ij} \equiv L_{ij} + C_{ij} + R_{ij}\,. \tag{3.13}$$

- At least from the standpoint of maintaining NS invariances, it is probably best to model SGS stress as a single entity.
- Time derivative is rigorously correct (the filtering is spatial).
- Spatial filtering is defined in such a way that LES → DNS.
- From the mathematical point of view, filtering eliminates aliasing arising from under resolution imposed by coarse grids of practical discretization.

Smagorinsky Model

$$\tau_{SGS} = -2\nu_{SGS}\widetilde{S}\,, \tag{3.14}$$

$$\nu_{SGS} = (C_S\Delta)^2|\widetilde{S}|\,. \tag{3.15}$$

- Early LESs were often performed with resolutions nearly as fine as employed for DNS → the SGS viscosity becomes very small, and contributions from the model are rather minimal
- The Smagorinsky model is completely dissipative → no backscattering of the turbulence kinetic energy (which can be up to 1/3)
- Modern models allow to automatically tune the constant C_S → Dynamic models by Germano et al. (POLITO)

RANS Models

There are several different ways by means of which RANS models are classified. One of the more common is in terms of the number of additional PDEs one must solve beyond those of the Navier–Stokes equations. Thus, as described earlier in Def. 1.108, one considers zero-equation, one-equation and two-equation models, and as noted by Wilcox [92] there are also $\frac{1}{2}$-equation models which include a single ordinary differential equation in their formulation. But this does not exhaust the possibilties. Indeed, the "second-moment closures," so named because they include equations for each of the second-moment statistical quantities comprising the Reynolds stress tensor, employ at least five additional PDEs, and in some forms might include seven. In principle, one could construct third- and higher-moment closures, although this is almost never done, even theoretically. In part because of all these various possible formulations and associated terminologies, turbulence models are often simply called "algebraic" if they include no additional ODEs or PDEs, and otherwise they are termed "differential."

Boussinesq Hypothesis

Newton's law of viscosity which might be paraphrased as "shear stress is proportional to strain rate, with viscosity being the constant of proportionality." That is,

$$\tau = \mu \frac{\partial u}{\partial y}\,, \tag{2.44}$$

The turbulent flow case, which might be expressed as

$$-\overline{u'v'} = \nu_T \left(\frac{\partial \overline{u}}{\partial y} + \frac{\partial \overline{v}}{\partial x} \right)\,, \tag{2.45}$$

where ν_T is the *turbulent eddy viscosity*, is not an empirically-supported physical result.

$$\nu_T \equiv -\frac{\overline{u'v'}}{\frac{1}{2}\left(\frac{\partial \overline{u}}{\partial y} + \frac{\partial \overline{v}}{\partial x} \right)}\,, \tag{2.46}$$

we see that ν_T cannot be a constant, except for extremely simple flows. Moreover, it is also easy to see that ν_T should actually be a tensor if we are to employ this in 3D. This is all very different from the situation for the physical viscosity ν, as should be obvious.

Instantaneous Response (?!)

It should first be observed that turbulence tends to "remember" its past history, at least for short times. This is, of course, one of the strongest arguments against randomness (but not necessarily against stochasticity) of turbulence, and indeed, the Taylor hypothesis would never work if turbulence did not however briefly remember its past. Equation (2.45) does not reflect this. It is clear that in a flow field exhibiting mean strain rate this equation predicts nonzero turbulent stresses (which is qualitatively correct); but as soon as the strain is removed, this same relationship instantaneously predicts zero turbulent shear stress. This does not at all coincide with experimental observations,

- Instantaneous decay to zero of turbulent stress, when the deformation goes to zero, violates the basic notion of causality: an effect can neither precede, nor be simultaneous with, its cause.
- Reynolds stresses, which come from averaging the nonlinear advective terms, are replaced with linear diffusive terms → we alter the bifurcation sequence.

Total Kinetic Energy

$$\boldsymbol{U}\cdot(\boldsymbol{U}_t+\boldsymbol{U}\cdot\nabla\boldsymbol{U})=\boldsymbol{U}\cdot(-\nabla p+\nu\Delta\boldsymbol{U})\,. \tag{2.58}$$

A straightforward calculation shows that

$$\frac{\partial}{\partial t}\left(\frac{1}{2}|\boldsymbol{U}|^2\right)+\boldsymbol{U}\cdot\nabla\left(\frac{1}{2}|\boldsymbol{U}|^2\right)=-\nabla\cdot(p\boldsymbol{U})+\nu\Delta\left(\frac{1}{2}|\boldsymbol{U}|^2\right)\,,$$

where

$$\frac{1}{2}|\boldsymbol{U}|^2=\frac{1}{2}(u^2+v^2+w^2)\equiv K\,, \tag{2.59}$$

the total kinetic energy per unit mass. Thus, the above can be expressed as

$$K_t+\boldsymbol{U}\cdot\nabla K=-\nabla\cdot(p\boldsymbol{U})+\nu\Delta K\,. \tag{2.60}$$

Mean Flow Kinetic Energy

$$\begin{aligned} K &= \frac{1}{2}\left[(\overline{u}+u')^2+(\overline{v}+v')^2+(\overline{w}+w')^2\right] \\ &= \frac{1}{2}\left(\overline{u}^2+\overline{v}^2+\overline{w}^2\right)+\frac{1}{2}\left(u'^2+v'^2+w'^2\right)+\overline{u}u'+\overline{v}v'+\overline{w}w' \\ &\equiv \overline{k}+k'+\overline{\boldsymbol{U}}\cdot\boldsymbol{U}'\,. \end{aligned} \tag{2.61}$$

$$\nabla\cdot\overline{\boldsymbol{u}}^2 = -\nabla\overline{p}+\nu\Delta\overline{\boldsymbol{u}}-\boldsymbol{R}(\boldsymbol{u}',\boldsymbol{u}')\,, \tag{2.28}$$

We can also construct an energy equation analogous to Eq. (2.60) for the mean flow in exactly the same way, *viz.*, form the dot product of the mean velocity vector with the RANS equations. This results in

$$\overline{\boldsymbol{u}}\cdot\nabla\overline{k} = -\nabla\cdot(\overline{p}\,\overline{\boldsymbol{u}})+\nu\Delta\overline{k}-\overline{\boldsymbol{u}}\cdot\boldsymbol{R}(\boldsymbol{u}',\boldsymbol{u}')\,. \tag{2.63}$$

Turbulent Kinetic Energy

Formal time averaging of (2.60) after substituting the decomposition obtained in Eq. (2.61) yields

$$\overline{\boldsymbol{U}\cdot\nabla(\overline{k}+k')}+\overline{\boldsymbol{U}\cdot\nabla(\overline{\boldsymbol{U}}\cdot\boldsymbol{U}')}=-\overline{\nabla\cdot(p\boldsymbol{U})}+\nu\overline{\Delta(\overline{k}+k')}\,,$$

and after considerable manipulation this can be expressed as

$$\overline{\boldsymbol{u}}\cdot\nabla(\overline{k}+k)+\nabla\cdot\left(\overline{\boldsymbol{u}'k'}\right)=-\nabla(\overline{p}\,\overline{\boldsymbol{u}})-\nabla\cdot\left(\overline{p'\boldsymbol{u}'}\right)+\nu\Delta\overline{k}+\nu\Delta k\,. \tag{2.64}$$

We observe that Eq. (2.64) is now steady state, and that it is expressed in terms of both mean and fluctuating quantities. Subtracting Eq. (2.63) from this leads to an equation for the turbulence kinetic energy:

$$\boxed{\overline{\boldsymbol{u}}\cdot\nabla k+\nabla\cdot\left(\overline{\boldsymbol{u}'k'}\right)=-\nabla\cdot\left(\overline{p'\boldsymbol{u}'}\right)+\nu\Delta k+\overline{\boldsymbol{u}}\cdot\boldsymbol{R}(\boldsymbol{u}',\boldsymbol{u}')\,.} \tag{2.65}$$

Turbulent Kinetic Energy Budget

$$\overline{\boldsymbol{u}}\cdot\nabla k + \nabla\cdot\left(\overline{\boldsymbol{u}'k'}\right) = -\nabla\cdot\left(\overline{p'\boldsymbol{u}'}\right) + \nu\Delta k + \overline{\boldsymbol{u}}\cdot\boldsymbol{R}(\boldsymbol{u}',\boldsymbol{u}')\,. \tag{2.65}$$

$$\overline{u_j}\frac{\partial k}{\partial x_j} = -\frac{\partial}{\partial x_j}\left(\overline{p'u'_j} + \frac{1}{2}\overline{u'_iu'_iu'_j} - 2\nu\overline{u'_is'_{ij}}\right) - \overline{u'_iu'_j}\,\overline{s_{ij}} - 2\nu\overline{s'_{ij}s'_{ij}}, \tag{2.66}$$

where p' is fluctuating pressure divided by density, and the $\overline{s_{ij}}$ and s'_{ij} represent components of mean and fluctuating strain rate tensors, respectively:

$$\overline{s_{ij}} = \frac{1}{2}\left(\frac{\partial\overline{u_i}}{\partial x_j} + \frac{\partial\overline{u_j}}{\partial x_i}\right), \qquad \text{and} \qquad s'_{ij} = \frac{1}{2}\left(\frac{\partial u'_i}{\partial x_j} + \frac{\partial u'_j}{\partial x_i}\right). \tag{2.67}$$

- Turbulence Diffusion Transfer (T)
- Turbulence Production (P) → P= ε Equilibrium Turbulence
- Turbulence Dissipation (ε)

Simplified Turbulent Kinetic Energy Eq.

$$\frac{\partial k}{\partial t} + \overline{u_j}\frac{\partial k}{\partial x_j} = -\overline{u_i' u_j'}\frac{\partial \overline{u_i}}{\partial x_j} - \varepsilon + \frac{\partial}{\partial x_j}\left[\left(\nu + \nu_T/\sigma_k\right)\frac{\partial k}{\partial x_j}\right]. \qquad (2.75)$$

- Since we have little in the way of sound theory for modeling the velocity triple correlation, we (arbitrarily) combine this with the pressure diffusion term and model these together in the diffusion term of kinetic energy.
- There is no physical justification for this → there is no reason why the turbulent quantities must satisfy an advection – diffusion equation

Standard k – ε RANS Model

$$\nabla \cdot \overline{\boldsymbol{u}} = 0\,, \tag{2.77a}$$

$$\overline{\boldsymbol{u}}_t + \overline{\boldsymbol{u}} \cdot \nabla \overline{\boldsymbol{u}} = -\nabla \overline{p} + \nabla \cdot [(\nu + \nu_T) \nabla \overline{\boldsymbol{u}}]\ , \tag{2.77b}$$

$$k_t + \overline{\boldsymbol{u}} \cdot \nabla k = \mathcal{P} - \varepsilon + \nabla \cdot [(\nu + \nu_T/\sigma_k) \nabla k]\ , \tag{2.77c}$$

$$\varepsilon_t + \overline{\boldsymbol{u}} \cdot \nabla \varepsilon = C_{\varepsilon 1} \frac{\varepsilon}{k} \mathcal{P} - C_{\varepsilon 2} \frac{\varepsilon^2}{k} + \nabla \cdot [(\nu + \nu_T/\sigma_\varepsilon) \nabla \varepsilon]\ , \tag{2.77d}$$

with production $\mathcal{P}$ given by

$$\mathcal{P} = -\overline{u_i' u_j'} \frac{\partial u_i}{\partial x_j}\,, \tag{2.78a}$$

$$-\overline{u_i' u_j'} = 2 \nu_T \overline{s_{ij}} - \frac{2}{3} k \delta_{ij}\,, \tag{2.78b}$$

and

$$\nu_T = C_\nu \frac{k^2}{\varepsilon}\,. \tag{2.79}$$

$$C_\nu = 0.09\,, \quad C_{\varepsilon 1} = 1.44\,, \quad C_{\varepsilon 2} = 1.92\,, \quad \sigma_k = 1.0\,, \quad \sigma_\varepsilon = 1.3\,.$$

Standard k – ε RANS Model: Remarks

- The model equations cannot be integrated all the way to a solid boundary → The law of the wall must be employed to provide velocity boundary conditions away from solid boundaries (equations for k and ε are not valid in viscous sublayer; ε^2/k is singular close to the wall).
- The model equations show a strong coupling with each other (existence and uniqueness of solution?).
- Assignment of boundary conditions is quite difficult for k and ε (in particular for inlet conditions), because they are statistical quantities more than physical.
- Some improvements have been developed for this kind of models (for example RNG models, based on Renormalization Group Theory).

Standard k – ε RANS Model: Performance

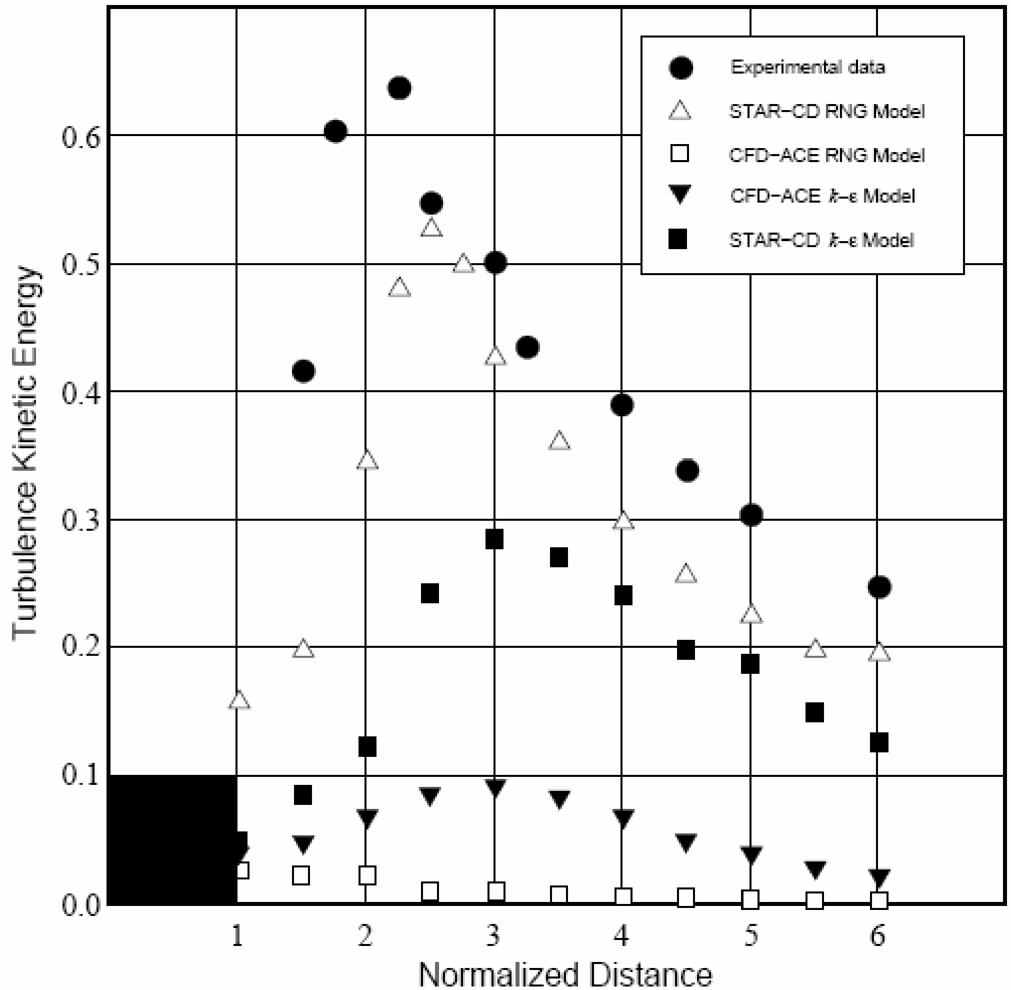


Figure 2.3: Comparison of various k–ε models for flow over a rib of square cross section.

Law of the Wall for Wall-bounded Flows

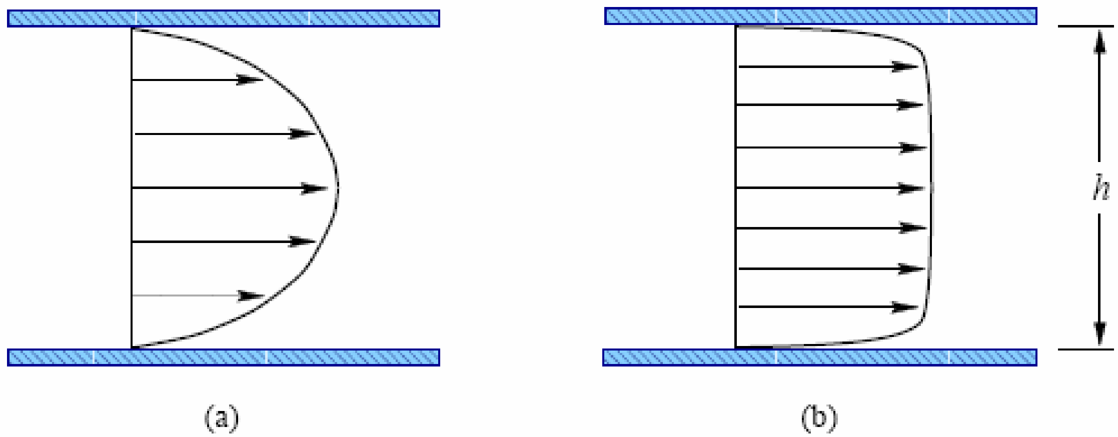


Figure 1.7: Comparison of laminar and turbulent velocity profiles in a duct; (a) laminar, and (b) turbulent.

- (At least) Two length scales are required to describe the turbulent profile, because of the sharp deformation at the wall
- The velocity profile is linear in the viscous sublayer, while it is nearly constant in the outer region → Hence a third profile is needed to math the previous ones.

Length Scales

Let $\overline{u}(y)$ denote the time mean velocity, and let u_τ denote a velocity scale for the inner region. (This should generally correspond to turbulent velocity fluctuations and might, for example, be the square root of the turbulence kinetic energy. Here, as the notation suggests, and will be evident later, we use the friction velocity.) Now observe that the two length scales are a large advective scale associated with $\overline{u}$ (say, $h/2$, the half-height of the duct) and a viscous scale corresponding to u_τ, *viz.*, ν/u_τ. In order for an intermediate scale to make sense, it must be the case that the ratio of these two length scales be large; *i.e.*,

$$\frac{h/2}{\nu/u_\tau} = \frac{hu_\tau}{2\nu} \gg 1\,.$$

Then we are able to identify a range of distances y from the wall(s) such that

$$\frac{yu_\tau}{\nu} \gg 1\,, \quad \text{and simultaneously} \quad \frac{y}{h} \ll 1\,.$$

Log Law

$$\frac{\partial \overline{u}}{\partial y} = C_1 u_\tau / y \,,$$

where C_1 is a constant that ultimately will be determined from experimental data. This can be directly integrated to yield

$$\frac{\overline{u}}{u_\tau} = C_1 \ln y + C_2 \,, \tag{1.70}$$

where C_2 is an integration constant which also will need to be found from experimental data.

Modulo a few details which will be supplied later, Eq. (1.70) is the well-known "log law" that matches the inner to the outer layer. As noted in [7], the range of length scales over which the log law is valid corresponds to the inertial subrange of the Kolmogorov theory or, equivalently, to approximately the Taylor microscales.

$$u_\tau = \sqrt{\frac{\tau_w}{\rho}} \,. \tag{1.71}$$

Furthermore, we define

$$y_+ \equiv \frac{y u_\tau}{\nu} \,, \qquad \text{and} \qquad u_+ \equiv \overline{u}/u_\tau \tag{1.72}$$

Law of the Wall

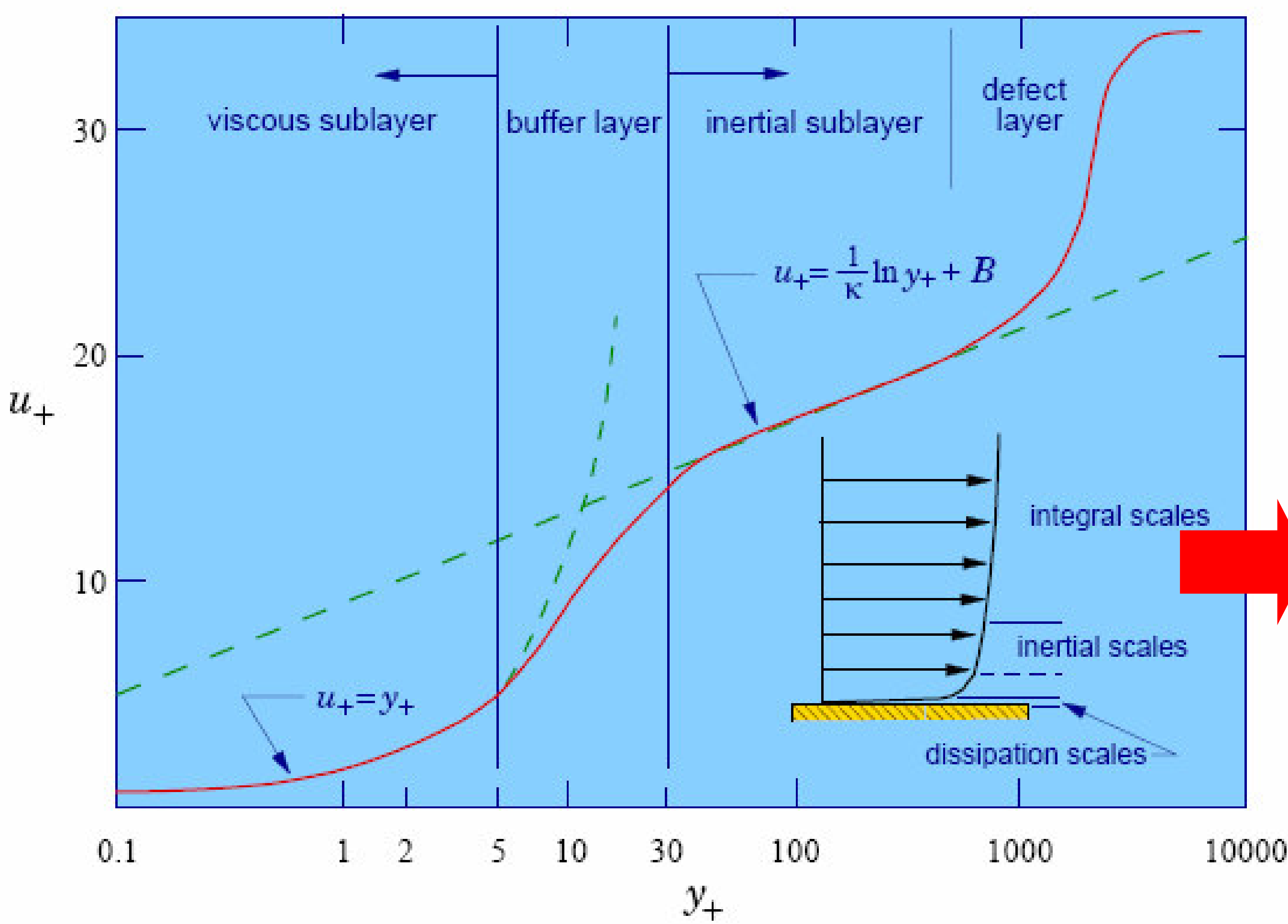


Figure 1.8: Law of the wall.

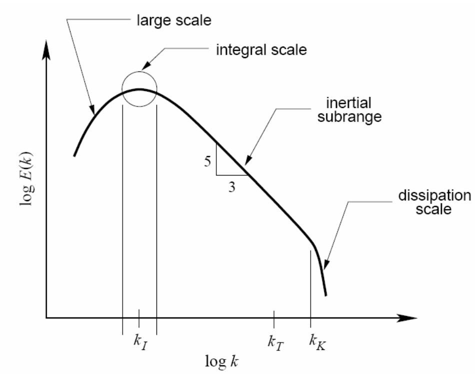


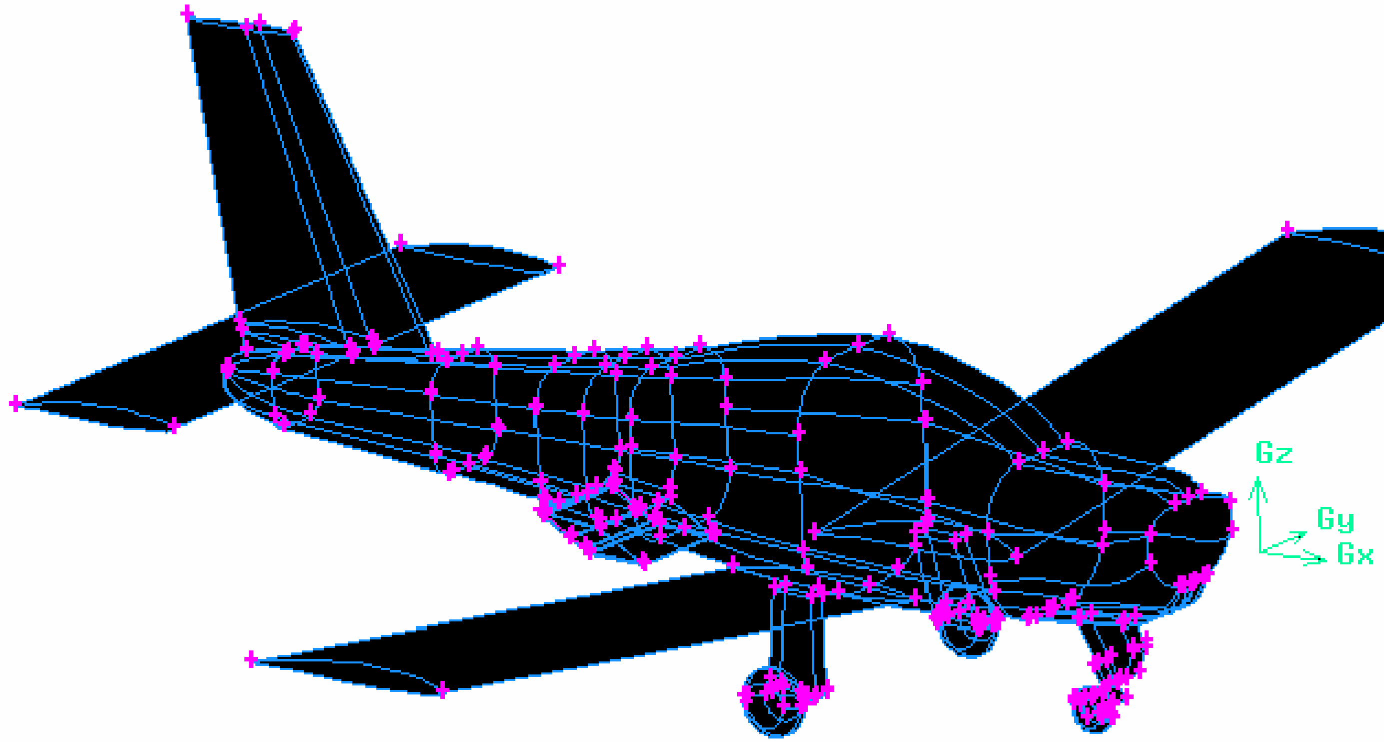
Figure 1.5: Turbulence energy wavenumber spectrum.

Law of the Wall: Application

- The log law formula can be used to compute the velocity boundary condition at the outer edge of the buffer layer, or even farther from the wall if extremely coarse gridding in employed, instead of solving the equations in a very thin boundary layer.
- At the same time, we reduce the computational demand and avoid the problems due to singularities of k – ε models close to the wall.
- The simple law of the wall described here is not valid for non similar boundary layers, and furthermore cannot be used accurately in the presence of flow separation and/or not fully–developed flows.

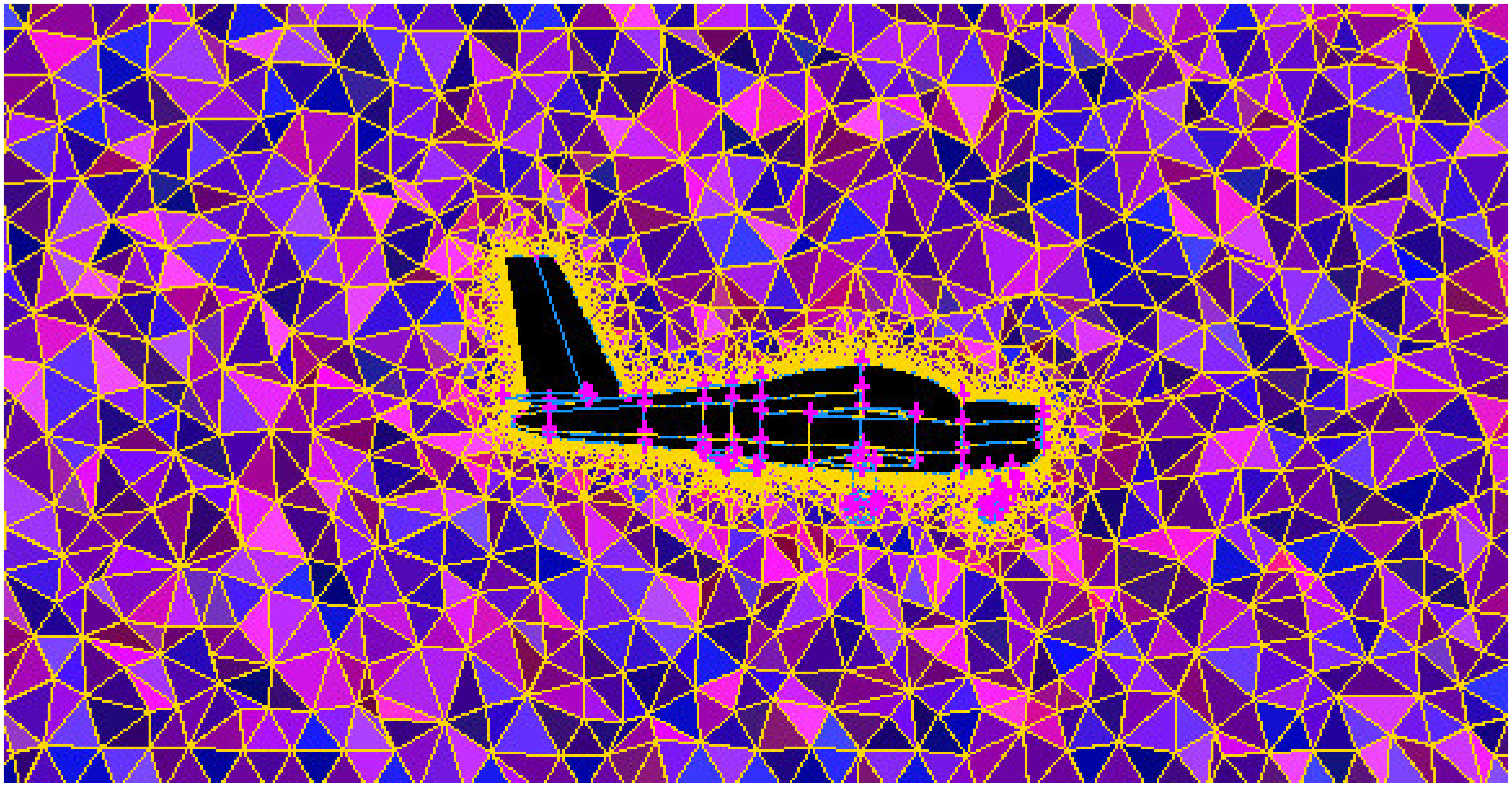
POLITECNICO DI TORINO

Validation of Prototype Installation



Case study provided by Ilaria Giolo

Validation of Prototype Installation



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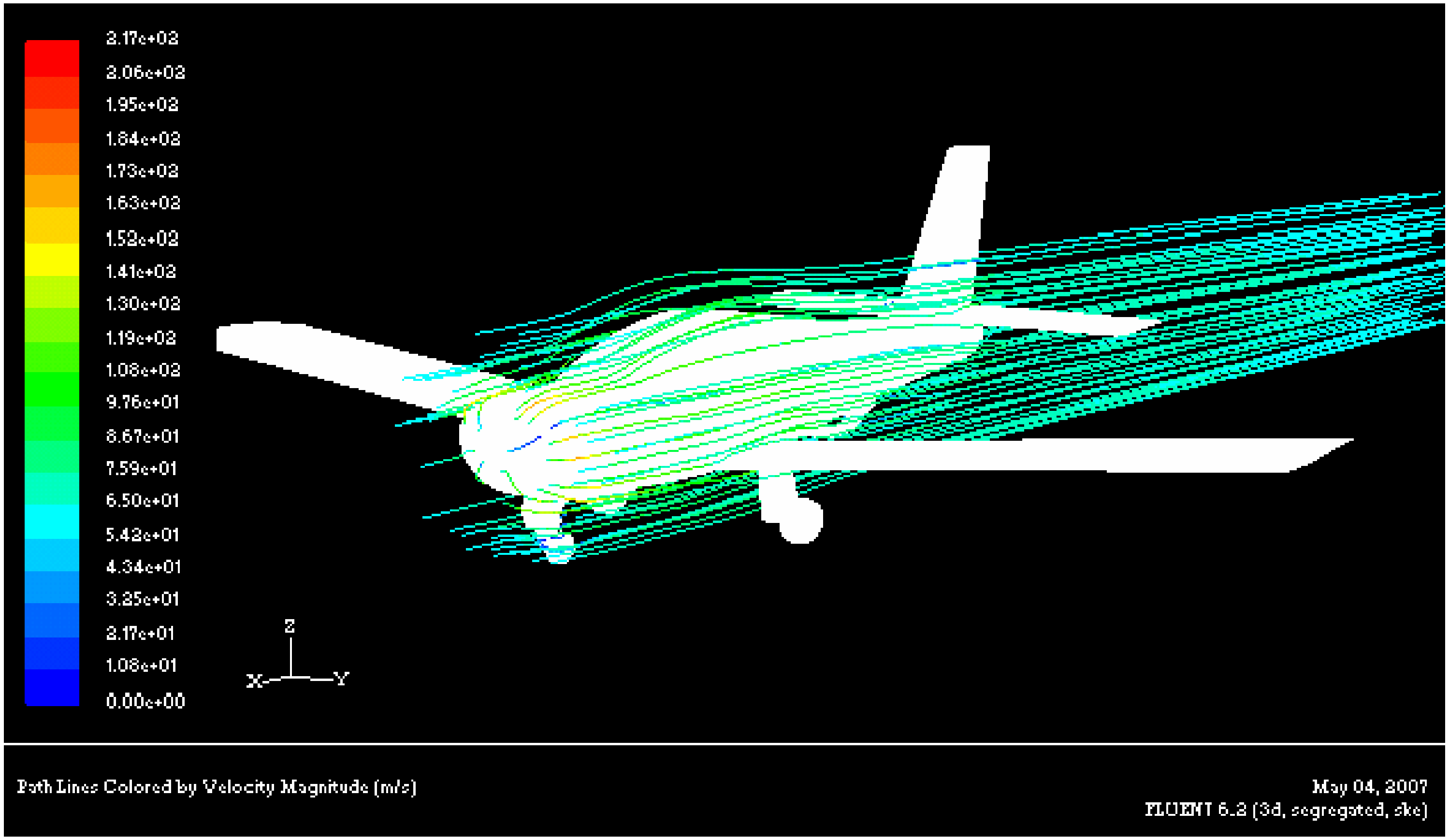
Validation of Prototype Installation



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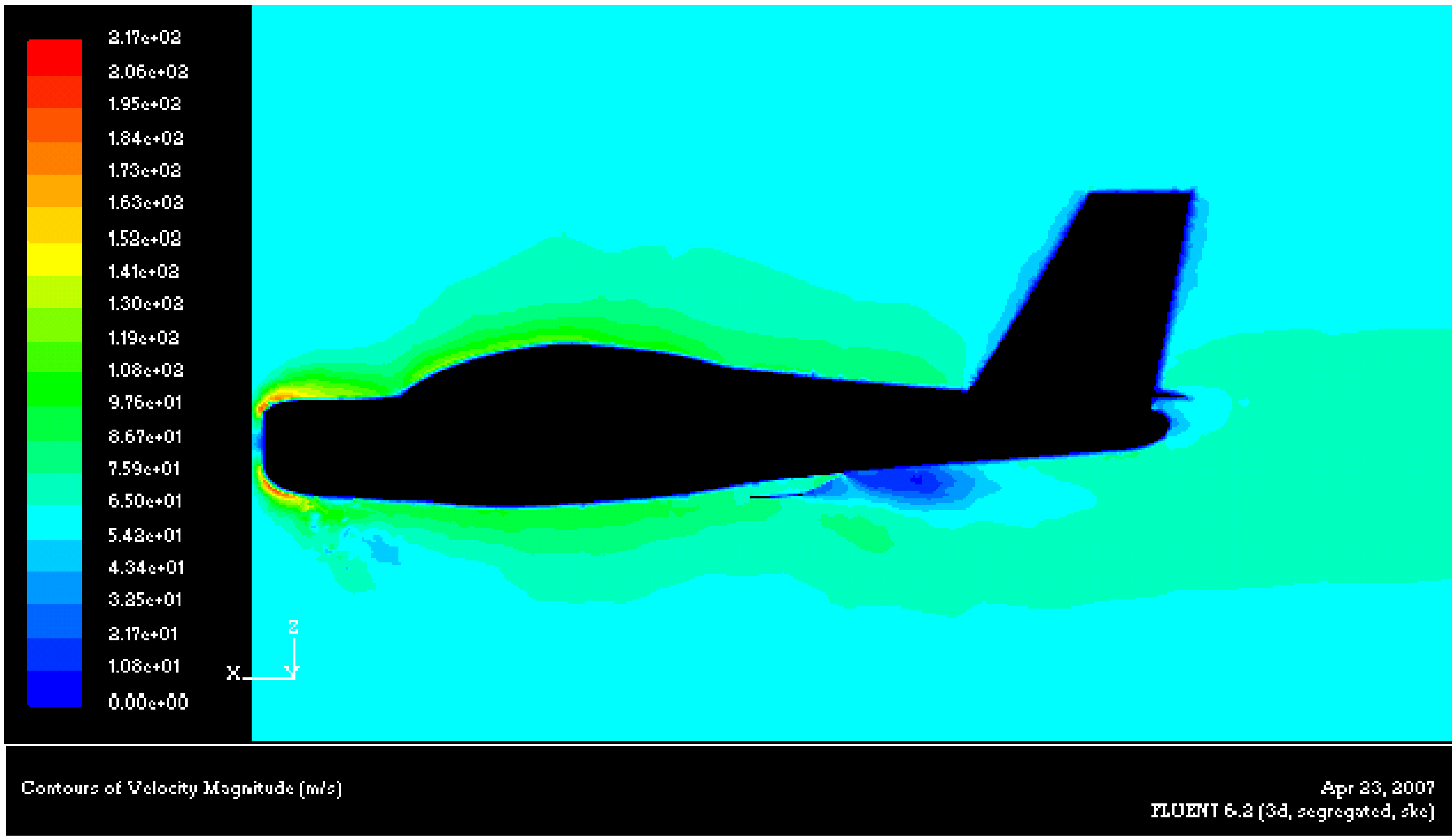
Validation of Prototype Installation



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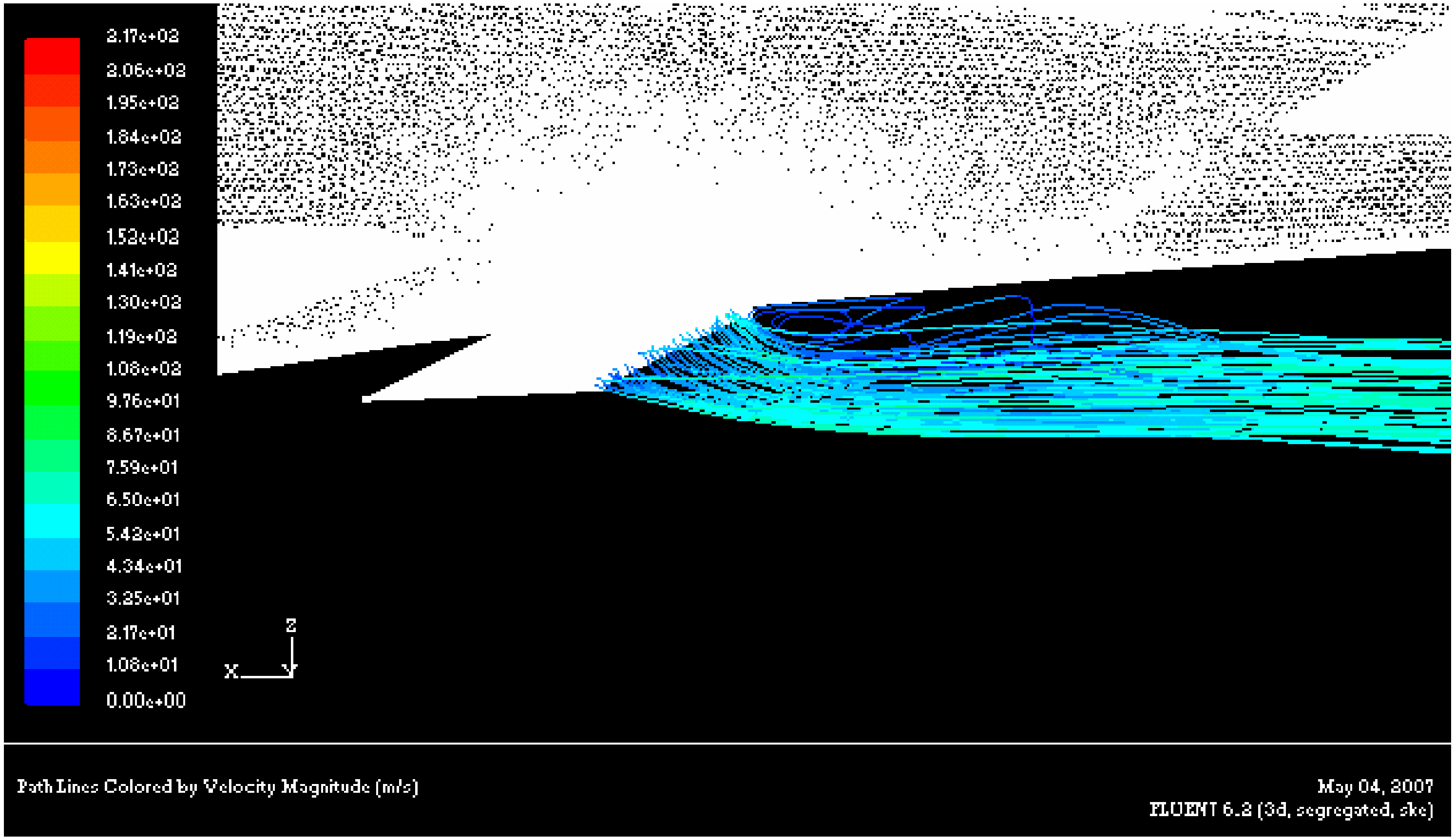
Validation of Prototype Installation



Case study provided by Ilaria Giolo

Validation of Prototype Installation

	Lift [N]	Drag [N]
Aircraft with gas cooler nacelle	8443.0069	10420.232
Aircraft without gas cooler nacelle	8420.0861	10595.29
Force variation	0.27%	1.67%

Case study provided by Ilaria Giolo

Further Readings

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