Artificial Compressibility Method Revisited: Theory of Asymptotic Numerical Method for the Incompressible Navier-Stokes Equations

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1 INTRODUCTION

Various existing numerical methods for the incompressible Navier-Stokes system (INSS) employ Poisson (or Poisson type) solver, which requires additional iteration in each time step. Chorin's artificial compressibility method (ACM) is well-known as the INSS method that avoids this cumbersome operation. ACM employs the artificial continuity equation that governs the time evolution of the pressure. Since the divergence free condition is not satisfied exactly in the time-dependent case, ACM is widely regarded as a numerical method for obtaining steady solutions. On the other hand, the lattice Boltzmann method (LBM) is another Poisson free method for INSS. It deals with the time evolution of velocity distribution function of "artificial" gas molecules and the solution of the time-dependent INSS is obtained from the leading term of the asymptotic solution for a kinetic equation. This asymptotic method becomes a variant of ACM for a special value of the computational parameter, where LBM is called LKS. This implies a potential ability of ACM as an asymptotic method for time-dependent INSS. In the present study, we reinvestigate ACM from this point of view and propose a high order Poisson free method.

2 PROBLEM AND BASIC EQUATIONS

INSS is expressed in the following dimensionless form:

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + \frac{\partial P}{\partial x_i} = \nu \frac{\partial^2 u_i}{\partial x_j^2}, \tag{1}$$

$$\frac{\partial u_i}{\partial x_i} = 0, \tag{2}$$

where u_i and P are the (dimensionless) flow velocity and pressure and ν is a constant corresponding to the inverse of Reynolds number. In order to avoid complexity and elucidate the essential point, we consider the problem in a square or cubic domain with a suitable periodic boundary condition. We assume that the variation of the solution is moderate. Under suitable scaling, this is equivalent to

$$u_i \sim \frac{\partial u_i}{\partial x_j} \sim \frac{\partial u_i}{\partial t} \sim O(1), \qquad P \sim \frac{\partial P}{\partial x_i} \sim \frac{\partial P}{\partial t} \sim O(1). \tag{3}$$

3 THEORY OF ARTIFICIAL COMPRESSIBILITY METHOD

Following Chorin, we replace Eq. (2) by the artificial continuity equation:

$$bk\frac{\partial P}{\partial t} + \frac{\partial u_i}{\partial x_i} = 0, \tag{4}$$

where b and k are positive constants. We assume $b \sim O(1)$ and $k \ll 1$. We will show that the artificial compressibility system (ACS), Eqs. (4) and (1), yields the approximate solution of time-dependent INSS.

3.1 Asymptotic analysis of artificial compressibility system

We expand u_i and P with respect to k:

$$u_i = \tilde{u}_i^{(0)} + \tilde{u}_i^{(1)}k + \tilde{u}_i^{(2)}k^2 + \cdots, \qquad P = \tilde{P}^{(0)} + \tilde{P}^{(1)}k + \tilde{P}^{(2)}k^2 + \cdots. \tag{5}$$

We assume

$$\tilde{u}_i^{(m)} \sim \frac{\partial \tilde{u}_i^{(m)}}{\partial x_j} \sim \frac{\partial \tilde{u}_i^{(m)}}{\partial t} \sim O(1) \qquad \tilde{P}^{(m)} \sim \frac{\partial \tilde{P}^{(m)}}{\partial x_i} \sim \frac{\partial \tilde{P}^{(m)}}{\partial t} \sim O(1) \qquad (m = 0, 1, 2, \cdots), \tag{6}$$

which are compatible with Eq. (3). Substituting the above expansions into ACS and equating the same order terms, we have the following sequence of the equation systems:

$$\frac{\partial \tilde{u}_i^{(0)}}{\partial t} = \mathcal{N}_i(\tilde{u}_k^{(0)}, \tilde{P}^{(0)}; \nu); \qquad \frac{\partial \tilde{u}_i^{(0)}}{\partial x_i} = 0, \tag{7}$$

$$\frac{\partial \tilde{u}_i^{(1)}}{\partial t} = \mathcal{L}_i(\tilde{u}_k^{(1)}, \tilde{P}^{(1)}; \tilde{u}_k^{(0)}, \nu); \qquad \frac{\partial \tilde{u}_i^{(1)}}{\partial x_i} = -b\frac{\partial \tilde{P}^{(0)}}{\partial t}, \tag{8}$$

$$\frac{\partial \tilde{u}_i^{(2)}}{\partial t} = \mathcal{L}_i(\tilde{u}_k^{(2)}, \tilde{P}^{(2)}; \tilde{u}_k^{(0)}, \nu) + \tilde{u}_j^{(1)}\frac{\partial \tilde{u}_i^{(1)}}{\partial x_j}; \qquad \frac{\partial \tilde{u}_i^{(2)}}{\partial x_i} = -b\frac{\partial \tilde{P}^{(1)}}{\partial t}, \tag{9}$$

$$\cdots\cdots$$

where

$$\mathcal{N}_i(u_k, P; \nu) \equiv -u_j\frac{\partial u_i}{\partial x_j} - \frac{\partial P}{\partial x_i} + \nu\frac{\partial^2 u_i}{\partial x_j^2}, \qquad \mathcal{L}_i(u_k, P; v_k, \nu) \equiv -v_j\frac{\partial u_i}{\partial x_j} - u_j\frac{\partial v_i}{\partial x_j} - \frac{\partial P}{\partial x_i} + \nu\frac{\partial^2 u_i}{\partial x_j^2}. \tag{10}$$

INSS appears at the leading order and Oseen-type equation systems follow. Since the Oseen-type equation system for $(\tilde{u}_i^{(1)}, \tilde{P}^{(1)})$ has the inhomogeneous term $-b\partial_t\tilde{P}^{(0)}$, the solution does not vanish even for the homogeneous initial data $(\tilde{u}_i^{(1)}, \tilde{P}^{(1)}) = (0, 0)$ unless $\partial_t P^{(0)} \equiv 0$; ACS can yield the approximate solution of time-dependent INSS with the error of $O(k)$.

3.2 Strategy for realization of incompressible trajectory

We will design an explicit numerical method for ACS. Let the mesh spacing and the time step be ϵ and Δt, respectively. ACS involves the acoustic mode besides the diffusive mode (INSS) shown previously.Then, the time step will be subject to the restriction relevant to acoustic wave, i.e. $\Delta t \lesssim \epsilon/C_s$, besides the one relevant to diffusion, i.e. $\Delta t \lesssim \epsilon^2/\nu$, where C_s is the sound speed in ACS and is equal to $(bk)^{-1/2}$. The former restriction becomes severer and the error of INSS becomes smaller for smaller k. We consider the case of $k = \epsilon^2$ as the compromise, since these two restrictions become comparable in the case of $\nu \sim O(1)$; the acoustic restriction becomes $\Delta t \lesssim \epsilon^2$.

Similar to the previous subsection, we assume that the numerical solution is expressed as the power series of $k(=\epsilon^2 \sim \Delta t)$. The equation systems for $(u_i^{(m)}, P^{(m)})$ $(m = 0, 1, 2, \cdots)$ in the previous subsection are subject to the correction because of the discretization error. When the first order accurate time integration method is employed, the equation system for $m \geq 1$ are altered. Here, the spatial discretization error is not taken into account. If the discretization error is $O(\epsilon^2)$, then it appears in the equation systems for $m \geq 1$. However, the equation system (7) is not altered. The $(u_i^{(m)}, P^{(m)})$ for $m \geq 1$ are the error of numerical solution. Thus, we can construct an explicit scheme for time-dependent INSS which is first order accurate in time and second order accurate in space. The solution of Oseen-type equation system (8) yields the intrinsic error of the asymptotic approach. By making use of the linearity of the solution with respect to b, we can cancel out this part fortunately. We notice that the numerical solution satisfies Eqs. (7) and (8) if the time integration is second order accurate and the spatial discretization is fourth order accurate. These requirements are fulfilled if appropriate computational gadgets, such as the second order Runge-Kutta method and five point central finite-difference formulas, are employed. The intrinsic error is canceled out in a linear combination of two numerical solutions for different values of b under the same resolution. Thus, we can construct the method which is second order accurate in time and fourth order accurate in space. Once the explicit numerical method for the time integration and spatial discretization are specified, the legitimacy of the above strategy of high order Poisson free method is confirmed theoretically by carrying out the asymptotic analysis of the resulting numerical method according to the recipe given in Ref. [1].

We briefly mention the initial data for ACS. A divergence free velocity field and the solution of the corresponding Poisson equation are employed as the initial condition for INSS. However, this is not appropriate when ACS is employed. The divergence free velocity field means that the time derivative of P is zero in ACS and the incompatible initial condition activates the acoustic mode of ACS. In order to launch the solution of ACS along the trajectory of INSS smoothly, special initial data for the error term $(u_i^{(m)}, P^{(m)})$ $(m = 1, 2, \cdots)$ should be chosen. For example, the initial data for $u_i^{(1)}$ should satisfy the second equation in (8), which requires the information of time derivative of pressure for INSS.

4 NUMRICAL VALIDATION

We have confirmed the validity of the above strategy for high order asymptotic computation of INSS in the problem of 2D Taylor-Green test problem. We carried the computation for the case where the exact solution is given by

$$u_1 = -\cos(\frac{\pi x}{3})\sin(\frac{\pi y}{3})\exp(-\frac{2\pi^2 \nu t}{9}), \quad u_2 = \cos(\frac{\pi y}{3})\sin(\frac{\pi x}{3})\exp(-\frac{2\pi^2 \nu t}{9}), \tag{11}$$

$$P = -\frac{1}{4}[\cos(\frac{2\pi x}{3}) + 1]\cos(\frac{2\pi y}{3})\exp(-\frac{4\pi^2 \nu t}{9}). \tag{12}$$

Figures 1 and 2 show the time evolution of L_1 error of the numerical solution for the case of $\nu = 0.2$ ($\epsilon = 1/12$ and $\Delta t = \epsilon^2/8$). It is seen from these figures that the error is nearly proportional to b. The acoustic mode is activated by the initial impact and is seen as small oscillations in these figures. The convergence rate of the numerical solution is shown in Figs. 3 and 4. The symbols $\triangle$ and $\Box$ indicate the results for $b = 4$ and $b = 8$, respectively. The symbol $\bullet$ indicate the result generated as the linear combination of these two cases. The solid line indicates the fourth order convergence rate and the dashed line indicates the second order one. The convergence rate for the velocity is nearly second order. However, clear second order convergence rate is not observed for P. This is considered to be due to the abovementioned oscillations. The cancellation of the leading error of $O(\epsilon^2)$ by combining the two numerical soluions is not demonstrated clearly because of the osillations, although a great improvement of the accuracy is clearly confirmed.

References

[1] M. Junk and Z. Yang, "Asymptotic analysis of finite difference method," Appl. Math. Comput. **158**, 267-301 (2004).

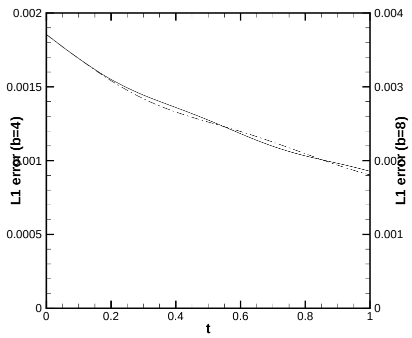

Figure 1: The time evolution of L_1 error for u_1: The solid line indicates the case of $b = 4$ and the dash-dot line indicates the case of $b = 8$.

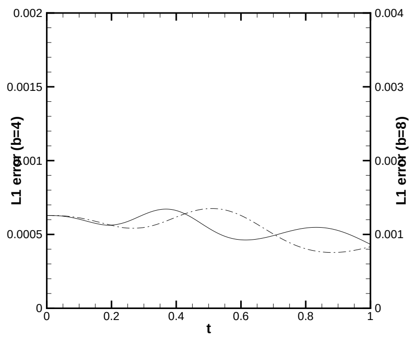

Figure 2: The time evolution of L_1 error for P: The solid line indicates the case of $b = 4$ and the dash-dot line indicates the case of $b = 8$.

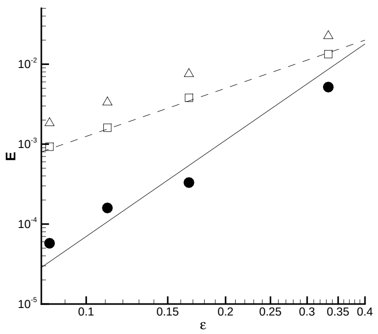

Figure 3: The convergence rate for u_1.

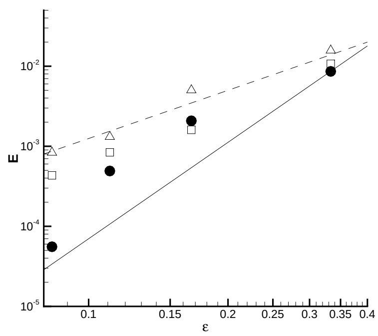

Figure 4: The convergence rate for P.