

Contents

Preface *xv*

Part I Overview

1	Financial Markets: Functions, Institutions, and Traded Assets	<i>1</i>
1.1	What is the purpose of finance?	<i>2</i>
1.2	Traded assets	<i>12</i>
1.2.1	The balance sheet	<i>15</i>
1.2.2	Assets vs. securities	<i>20</i>
1.2.3	Equity	<i>22</i>
1.2.4	Fixed income	<i>24</i>
1.2.5	FOREX markets	<i>27</i>
1.2.6	Derivatives	<i>29</i>
1.3	Market participants and their roles	<i>46</i>
1.3.1	Commercial vs. investment banks	<i>48</i>
1.3.2	Investment funds and insurance companies	<i>49</i>
1.3.3	Dealers and brokers	<i>51</i>
1.3.4	Hedgers, speculators, and arbitrageurs	<i>51</i>
1.4	Market structure and trading strategies	<i>53</i>
1.4.1	Primary and secondary markets	<i>53</i>
1.4.2	Over-the-counter vs. exchange-traded derivatives	<i>53</i>
1.4.3	Auction mechanisms and the limit order book	<i>53</i>
1.4.4	Buying on margin and leverage	<i>55</i>
1.4.5	Short-selling	<i>58</i>
1.5	Market indexes	<i>60</i>
	Problems	<i>63</i>
	Further reading	<i>65</i>
	Bibliography	<i>65</i>
2	Basic Problems in Quantitative Finance	<i>67</i>
2.1	Portfolio optimization	<i>68</i>
2.1.1	Static portfolio optimization: Mean–variance efficiency	<i>70</i>
2.1.2	Dynamic decision-making under uncertainty: A stylized consumption–saving model	<i>75</i>
2.2	Risk measurement and management	<i>80</i>

2.2.1	Sensitivity of asset prices to underlying risk factors	81
2.2.2	Risk measures in a non-normal world: Value-at-risk	84
2.2.3	Risk management: Introductory hedging examples	93
2.2.4	Financial vs. nonfinancial risk factors	100
2.3	The no-arbitrage principle in asset pricing	102
2.3.1	Why do we need asset pricing models?	103
2.3.2	Arbitrage strategies	104
2.3.3	Pricing by no-arbitrage	108
2.3.4	Option pricing in a binomial model	112
2.3.5	The limitations of the no-arbitrage principle	116
2.4	The mathematics of arbitrage	117
2.4.1	Linearity of the pricing functional and law of one price	119
2.4.2	Dominant strategies	120
2.4.3	No-arbitrage principle and risk-neutral measures	125
S2.1	Multiobjective optimization	129
S2.2	Summary of LP duality	133
	Problems	137
	Further reading	139
	Bibliography	139

Part II

Fixed-income assets

3	Elementary Theory of Interest Rates	143
3.1	The time value of money: Shifting money forward in time	146
3.1.1	Simple vs. compounded rates	147
3.1.2	Quoted vs. effective rates: Compounding frequencies	150
3.2	The time value of money: Shifting money backward in time	153
3.2.1	Discount factors and pricing a zero-coupon bond	154
3.2.2	Discount factors vs. interest rates	158
3.3	Nominal vs. real interest rates	161
3.4	The term structure of interest rates	163
3.5	Elementary bond pricing	165
3.5.1	Pricing coupon-bearing bonds	165
3.5.2	From bond prices to term structures, and vice versa	168
3.5.3	What is a risk-free rate, anyway?	171
3.5.4	Yield-to-maturity	174
3.5.5	Interest rate risk	180
3.5.6	Pricing floating rate bonds	188
3.6	A digression: Elementary investment analysis	190
3.6.1	Net present value	191
3.6.2	Internal rate of return	192

3.6.3	Real options	193
3.7	Spot vs. forward interest rates	193
3.7.1	The forward and the spot rate curves	197
3.7.2	Discretely compounded forward rates	197
3.7.3	Forward discount factors	198
3.7.4	The expectation hypothesis	199
3.7.5	A word of caution: Model risk and hidden assumptions	202
S3.1	Proof of Equation (3.42)	203
Problems		203
Further reading		205
Bibliography		205
4	Forward Rate Agreements, Interest Rate Futures, and Vanilla Swaps	207
4.1	LIBOR and EURIBOR rates	208
4.2	Forward rate agreements	209
4.2.1	A hedging view of forward rates	210
4.2.2	FRAs as bond trades	214
4.2.3	A numerical example	215
4.3	Eurodollar futures	216
4.4	Vanilla interest rate swaps	220
4.4.1	Swap valuation: Approach 1	221
4.4.2	Swap valuation: Approach 2	223
4.4.3	The swap curve and the term structure	225
Problems		226
Further reading		226
Bibliography		226
5	Fixed-Income Markets	229
5.1	Day count conventions	230
5.2	Bond markets	231
5.2.1	Bond credit ratings	233
5.2.2	Quoting bond prices	233
5.2.3	Bonds with embedded options	235
5.3	Interest rate derivatives	237
5.3.1	Swap markets	237
5.3.2	Bond futures and options	238
5.4	The repo market and other money market instruments	239
5.5	Securitization	240
Problems		244
Further reading		244
Bibliography		244
6	Interest Rate Risk Management	247
6.1	Duration as a first-order sensitivity measure	248
6.1.1	Duration of fixed-coupon bonds	250

6.1.2	Duration of a floater	254
6.1.3	Dollar duration and interest rate swaps	255
6.2	Further interpretations of duration	257
6.2.1	Duration and investment horizons	258
6.2.2	Duration and yield volatility	260
6.2.3	Duration and quantile-based risk measures	260
6.3	Classical duration-based immunization	261
6.3.1	Cash flow matching	262
6.3.2	Duration matching	263
6.4	Immunization by interest rate derivatives	265
6.4.1	Using interest rate swaps in asset–liability management	266
6.5	A second-order refinement: Convexity	266
6.6	Multifactor models in interest rate risk management	269
	Problems	271
	Further reading	272
	Bibliography	273

Part III Equity portfolios

7	Decision-Making under Uncertainty: The Static Case	277
7.1	Introductory examples	278
7.2	Should we just consider expected values of returns and monetary outcomes?	282
7.2.1	Formalizing static decision-making under uncertainty	283
7.2.2	The flaw of averages	284
7.3	A conceptual tool: The utility function	288
7.3.1	A few standard utility functions	293
7.3.2	Limitations of utility functions	297
7.4	Mean–risk models	299
7.4.1	Coherent risk measures	300
7.4.2	Standard deviation and variance as risk measures	302
7.4.3	Quantile-based risk measures: $V@R$ and $CV@R$	303
7.4.4	Formulation of mean–risk models	309
7.5	Stochastic dominance	310
S7.1	Theorem proofs	314
S7.1.1	Proof of Theorem 7.2	314
S7.1.2	Proof of Theorem 7.4	315
	Problems	315
	Further reading	317
	Bibliography	317
8	Mean–Variance Efficient Portfolios	319
8.1	Risk aversion and capital allocation to risky assets	320

8.1.1	The role of risk aversion	324
8.2	The mean–variance efficient frontier with risky assets	325
8.2.1	Diversification and portfolio risk	325
8.2.2	The efficient frontier in the case of two risky assets	326
8.2.3	The efficient frontier in the case of n risky assets	329
8.3	Mean–variance efficiency with a risk-free asset: The separation property	332
8.4	Maximizing the Sharpe ratio	337
8.4.1	Technical issues in Sharpe ratio maximization	340
8.5	Mean–variance efficiency vs. expected utility	341
8.6	Instability in mean–variance portfolio optimization	343
S8.1	The attainable set for two risky assets is a hyperbola	345
S8.2	Explicit solution of mean–variance optimization in matrix form	346
	Problems	348
	Further reading	349
	Bibliography	349
9	Factor Models	351
9.1	Statistical issues in mean–variance portfolio optimization	352
9.2	The single-index model	353
9.2.1	Estimating a factor model	354
9.2.2	Portfolio optimization within the single-index model	356
9.3	The Treynor–Black model	358
9.3.1	A top-down/bottom-up optimization procedure	362
9.4	Multifactor models	365
9.5	Factor models in practice	367
S9.1	Proof of Equation (9.17)	368
	Problems	369
	Further reading	371
	Bibliography	371
10	Equilibrium Models: CAPM and APT	373
10.1	What is an equilibrium model?	374
10.2	The capital asset pricing model	375
10.2.1	Proof of the CAPM formula	377
10.2.2	Interpreting CAPM	378
10.2.3	CAPM as a pricing formula and its practical relevance	380
10.3	The Black–Litterman portfolio optimization model	381
10.3.1	Black–Litterman model: The role of CAPM and Bayesian Statistics	382
10.3.2	Black–Litterman model: A numerical example	386
10.4	Arbitrage pricing theory	388
10.4.1	The intuition	389
10.4.2	A not-so-rigorous proof of APT	391
10.4.3	APT for Well-Diversified Portfolios	392
10.4.4	APT for Individual Assets	393

10.4.5	Interpreting and using APT	394
10.5	The behavioral critique	398
10.5.1	The efficient market hypothesis	400
10.5.2	The psychology of choice by agents with limited rationality	400
10.5.3	Prospect theory: The aversion to sure loss	401
S10.1	Bayesian statistics	404
S10.1.1	Bayesian estimation	405
S10.1.2	Bayesian learning in coin flipping	407
S10.1.3	The expected value of a normal distribution	408
Problems		411
Further reading		413
Bibliography		413

Part IV Derivatives

11	Modeling Dynamic Uncertainty	417
11.1	Stochastic processes	420
11.1.1	Introductory examples	422
11.1.2	Marginals do not tell the whole story	428
11.1.3	Modeling information: Filtration generated by a stochastic process	430
11.1.4	Markov processes	433
11.1.5	Martingales	436
11.2	Stochastic processes in continuous time	438
11.2.1	A fundamental building block: Standard Wiener process	438
11.2.2	A generalization: Lévy processes	440
11.3	Stochastic differential equations	441
11.3.1	A deterministic differential equation: The bank account process	442
11.3.2	The generalized Wiener process	443
11.3.3	Geometric Brownian motion and Itô processes	445
11.4	Stochastic integration and Itô's lemma	447
11.4.1	A digression: Riemann and Riemann–Stieltjes integrals	447
11.4.2	Stochastic integral in the sense of Itô	448
11.4.3	Itô's lemma	453
11.5	Stochastic processes in financial modeling	457
11.5.1	Geometric Brownian motion	457
11.5.2	Generalizations	460
11.6	Sample path generation	462
11.6.1	Monte Carlo sampling	463
11.6.2	Scenario trees	465
S11.1	Probability spaces, measurability, and information	468

Problems	476
Further reading	478
Bibliography	478
12 Forward and Futures Contracts	481
12.1 Pricing forward contracts on equity and foreign currencies	482
12.1.1 The spot–forward parity theorem	482
12.1.2 The spot–forward parity theorem with dividend income	485
12.1.3 Forward contracts on currencies	487
12.1.4 Forward contracts on commodities or energy: Contango and backwardation	489
12.2 Forward vs. futures contracts	490
12.3 Hedging with linear contracts	493
12.3.1 Quantity-based hedging	493
12.3.2 Basis risk and minimum variance hedging	494
12.3.3 Hedging with index futures	496
12.3.4 Tailing the hedge	499
Problems	501
Further reading	502
Bibliography	502
13 Option Pricing: Complete Markets	505
13.1 Option terminology	506
13.1.1 Vanilla options	507
13.1.2 Exotic options	508
13.2 Model-free price restrictions	510
13.2.1 Bounds on call option prices	511
13.2.2 Bounds on put option prices: Early exercise and continuation regions	514
13.2.3 Parity relationships	517
13.3 Binomial option pricing	519
13.3.1 A hedging argument	520
13.3.2 Lattice calibration	523
13.3.3 Generalization to multiple steps	524
13.3.4 Binomial pricing of American-style options	527
13.4 A continuous-time model: The Black–Scholes–Merton pricing formula	530
13.4.1 The delta-hedging view	532
13.4.2 The risk-neutral view: Feynman–Kač representation theorem	539
13.4.3 Interpreting the factors in the BSM formula	543
13.5 Option price sensitivities: The Greeks	545
13.5.1 Delta and gamma	546
13.5.2 Theta	550
13.5.3 Relationship between delta, gamma, and theta	551

13.5.4	Vega	552
13.6	The role of volatility	553
13.6.1	The implied volatility surface	553
13.6.2	The impact of volatility on barrier options	555
13.7	Options on assets providing income	556
13.7.1	Index options	557
13.7.2	Currency options	558
13.7.3	Futures options	559
13.7.4	The mechanics of futures options	559
13.7.5	A binomial view of futures options	560
13.7.6	A risk-neutral view of futures options	562
13.8	Portfolio strategies based on options	562
13.8.1	Portfolio insurance and the Black Monday of 1987	563
13.8.2	Volatility trading	564
13.8.3	Dynamic vs. Static hedging	566
13.9	Option pricing by numerical methods	569
	Problems	570
	Further reading	575
	Bibliography	576
14	Option Pricing: Incomplete Markets	579
14.1	A PDE approach to incomplete markets	581
14.1.1	Pricing a zero-coupon bond in a driftless world	584
14.2	Pricing by short-rate models	588
14.2.1	The Vasicek short-rate model	589
14.2.2	The Cox–Ingersoll–Ross short-rate model	594
14.3	A martingale approach to incomplete markets	595
14.3.1	An informal approach to martingale equivalent measures	598
14.3.2	Choice of numeraire: The bank account	600
14.3.3	Choice of numeraire: The zero-coupon bond	601
14.3.4	Pricing options with stochastic interest rates: Black’s model	602
14.3.5	Extensions	603
14.4	Issues in model calibration	603
14.4.1	Bias–variance tradeoff and regularized least-squares	604
14.4.2	Financial model calibration	609
	Further reading	612
	Bibliography	612
 Part V		
Advanced optimization models		
15	Optimization Model Building	617
15.1	Classification of optimization models	618

15.2	Linear programming	625
15.2.1	Cash flow matching	627
15.3	Quadratic programming	628
15.3.1	Maximizing the Sharpe ratio	629
15.3.2	Quadratically constrained quadratic programming	631
15.4	Integer programming	632
15.4.1	A MIQP model to minimize TEV under a cardinality constraint	634
15.4.2	Good MILP model building: The role of tight model formulations	636
15.5	Conic optimization	642
15.5.1	Convex cones	644
15.5.2	Second-order cone programming	650
15.5.3	Semidefinite programming	653
15.6	Stochastic optimization	655
15.6.1	Chance-constrained LP models	656
15.6.2	Two-stage stochastic linear programming with recourse	657
15.6.3	Multistage stochastic linear programming with recourse	663
15.6.4	Scenario generation and stability in stochastic programming	670
15.7	Stochastic dynamic programming	675
15.7.1	The dynamic programming principle	676
15.7.2	Solving Bellman's equation: The three curses of dimensionality	679
15.7.3	Application to pricing options with early exercise features	680
15.8	Decision rules for multistage SLPs	682
15.9	Worst-case robust models	686
15.9.1	Uncertain LPs: Polyhedral uncertainty	689
15.9.2	Uncertain LPs: Ellipsoidal uncertainty	690
15.10	Nonlinear programming models in finance	691
15.10.1	Fixed-mix asset allocation	692
	Problems	693
	Further reading	695
	Bibliography	696
16	Optimization Model Solving	699
16.1	Local methods for nonlinear programming	700
16.1.1	Unconstrained nonlinear programming	700
16.1.2	Penalty function methods	703
16.1.3	Lagrange multipliers and constraint qualification conditions	707
16.1.4	Duality theory	713
16.2	Global methods for nonlinear programming	715

16.2.1	Genetic algorithms	716
16.2.2	Particle swarm optimization	717
16.3	Linear programming	719
16.3.1	The simplex method	720
16.3.2	Duality in linear programming	723
16.3.3	Interior-point methods: Primal-dual barrier method for LP	726
16.4	Conic duality and interior-point methods	728
16.4.1	Conic duality	728
16.4.2	Interior-point methods for SOCP and SDP	731
16.5	Branch-and-bound methods for integer programming	732
16.5.1	A matheuristic approach: Fix-and-relax	735
16.6	Optimization software	736
16.6.1	Solvers	737
16.6.2	Interfacing through imperative programming languages	738
16.6.3	Interfacing through non-imperative algebraic languages	738
16.6.4	Additional interfaces	739
Problems		739
Further reading		740
Bibliography		741
Index		743

Preface

This book arises from slides and lecture notes that I have used over the years in my courses *Financial Markets and Instruments* and *Financial Engineering*, which were offered at Politecnico di Torino to graduate students in Mathematical Engineering. Given the audience, the treatment is naturally geared toward a mathematically inclined reader. Nevertheless, the required prerequisites are relatively modest, and any student in engineering, mathematics, and statistics should be well-equipped to tackle the contents of this introductory book.¹ The book should also be of interest to students in economics, as well as junior practitioners with a suitable quantitative background.

We begin with quite elementary concepts, and material is introduced progressively, always paying due attention to the practical side of things. Mathematical modeling is an art of selective simplification, which must be supported by intuition building, as well as by a healthy dose of skepticism. This is the aim of remarks, counterexamples, and financial horror stories that the book is interspersed with. Occasionally, we also touch upon current research topics.

Book structure

The book is organized into five parts.

1. Part One, **Overview**, consists of two chapters. Chapter 1 aims at getting unfamiliar readers acquainted with the role and structure of financial markets, the main classes of traded assets (equity, fixed income, and derivatives), and the main types of market participants, both in terms of institutions (e.g., investment banks and pension funds) and roles (e.g., speculators, hedgers, and arbitrageurs). We try to give a practical flavor that is essential to students of quantitative disciplines, setting the stage for the application of quantitative models. Chapter 2 overviews the basic problems in finance, like asset allocation, pricing, and risk management, which may be tackled by quantitative models. We also introduce the fundamental concepts related to arbitrage theory, including market completeness and risk-neutral measures, in a simple static and discrete setting.
2. Part Two, **Fixed-income assets**, consists of four chapters and introduces the simplest assets depending on interest rates, starting with plain bonds. The fundamental concepts of interest rate modeling, including the term

¹In case of need, the mathematical prerequisites are covered in my other book: *Quantitative Methods: An Introduction for Business Management*. Wiley, 2011.

structure and forward rates, as well as bond pricing, are covered in Chapter 3. The simplest interest rate derivatives (forward rate agreements and vanilla swaps) are covered in Chapter 4, whereas Chapter 5 aims at providing the reader with a flavor of real-life markets, where details like day count and quoting conventions are relevant. Chapter 6 concludes this part by showing how quantitative models may be used to manage interest rate risk. In this part, we do not consider interest rate options, which require a stronger mathematical background and are discussed later.

3. Part Three, **Equity portfolios**, consists of four chapters, where we discuss equity markets and portfolios of stock shares. Actually, this is not the largest financial market, but it is arguably the kind of market that the layman is more familiar with. Chapter 7 is a bit more theoretical and lays down the foundations of static decision-making under uncertainty. By static, we mean that we make one decision and then we wait for its consequences, finger crossed. Multistage decision models are discussed later. In this chapter, we also introduce the basics of risk aversion and risk measurement. Chapter 8 is quite classical and covers traditional mean–variance portfolio optimization. The impact of statistical estimation issues on portfolio management motivates the introduction of factor models, which are the subject of Chapter 9. Finally, in Chapter 10, we discuss equilibrium models in their simplest forms, the capital asset pricing model (CAPM), which is related to a single-index factor model, and arbitrage pricing theory (APT), which is related to a multifactor model. We do not discuss further developments in equilibrium models, but we hint at some criticism based on behavioral finance.
4. Part Four, **Derivatives**, includes four chapters. We discuss dynamic uncertainty models in Chapter 11, which is more challenging than previous chapters, as we have to introduce the necessary foundations of option pricing models, namely, stochastic differential equations and stochastic integrals. Chapter 12 describes simple forward and futures contracts, extending concepts that were introduced in Chapter 4, when dealing with forward and futures interest rates. Chapter 13 covers option pricing in the case of complete markets, including the celebrated and controversial Black–Scholes–Merton formula, whereas Chapter 14 extends the basic concepts to the more realistic setting of incomplete markets.
5. Part Five, **Advanced optimization models**, is probably the less standard part of this book, when compared to typical textbooks on financial markets. We deal with optimization model building, in Chapter 15, and optimization model solving, in Chapter 16. Actually, it is difficult to draw a sharp line between model building and model solving, but it is a fact of life that advanced software is available for solving quite sophisticated models, and the average user does not need a very deep knowledge of the involved algorithms, whereas she must be able to build a model. This is the motivation for separating the two chapters.

Needless to say, the choice of which topics should be included or omitted is debatable and based on authors' personal bias, not to mention the need to keep a book size within a sensible limit. With respect to introductory textbooks on financial markets, there is a deeper treatment of derivative models. On the other hand, more challenging financial engineering textbooks do not cover, e.g., equilibrium models and portfolio optimization. We aim at an intermediate treatment, whose main limitations include the following:

- We only hint at criticism put forward by behavioral finance and do not cover market microstructure and algorithmic trading strategies.
- From a mathematical viewpoint, we pursue an intuitive treatment of financial engineering models, as well as a simplified coverage of the related tools of stochastic calculus. We do not rely on rigorous arguments involving self-financing strategies, martingale representation theorems, or change of probability measures.
- From a financial viewpoint, by far, the most significant omission concerns credit risk and credit derivatives. Counterparty and liquidity risk play a prominent role in post-Lehman Brothers financial markets and, as a consequence of the credit crunch started in 2007, new concepts like CVA, DVA, and FVA have been introduced. This is still a field in flux, and the matter is arguably not quite assessed yet.
- Another major omission is econometric time series models.

Adequate references on these topics are provided for the benefit of the interested readers.

My choices are also influenced by the kind of students this book is mainly aimed at. The coverage of optimization models and methods is deeper than usual, and I try to open readers' critical eye by carefully crafted examples and counterexamples. I try to strike a satisfactory balance between the need to illustrate mathematics in action and the need to understand the real-life context, without which quantitative methods boil down to a solution in search of a problem (or a hammer looking for nails, if you prefer). I also do not disdain just a bit of repetition and redundancy, when it may be convenient to readers who wish to jump from chapter to chapter. More advanced sections, which may be safely skipped by readers, are referred to as *supplements* and their number is marked by an initial "S."

In my Financial Engineering course, I also give some more information on numerical methods. The interested reader might refer to my other books:

- P. Brandimarte, *Numerical Methods in Finance and Economics: A MATLAB-Based Introduction* (2nd ed.), Wiley, 2006
- P. Brandimarte, *Handbook in Monte Carlo Simulation: Applications in Financial Engineering, Risk Management, and Economics*, Wiley, 2014

Acknowledgements

In the past years, I have adopted the following textbooks (or earlier editions) in my courses. I have learned a lot from them, and they have definitely influenced the writing of this book:

- Z. Bodie, A. Kane, and A. Marcus, *Investments* (9th ed.), McGraw-Hill, 2010
- J.C. Hull, *Options, Futures, and Other Derivatives* (8th ed.), Prentice Hall, 2011
- P. Veronesi, *Fixed Income Securities: Valuation, Risk, and Risk Management*, Wiley, 2010

Other specific acknowledgements are given in the text. I apologize in advance for any unintentional omission.

Additional material

Some end-of-chapter problems are included and fully worked solutions will be posted on a web page. My current URL is

- <http://staff.polito.it/paolo.brandimarte/>

A hopefully short list of errata will be posted there as well. One of the many corollaries of Murphy's law states that my URL is going to change shortly after publication of the book. An up-to-date link will be maintained on the Wiley web page:

- <http://www.wiley.com/>

For comments, suggestions, and criticisms, all of which are quite welcome, my e-mail address is

- paolo.brandimarte@polito.it

PAOLO BRANDIMARTE

Turin, September 2017