

Contents

Preface *xiii*

Part I

Overview and Motivation

1	Introduction to Monte Carlo Methods	3
1.1	Historical origin of Monte Carlo simulation	4
1.2	Monte Carlo simulation vs. Monte Carlo sampling	7
1.3	System dynamics and the mechanics of Monte Carlo simulation	10
1.3.1	Discrete-time models	10
1.3.2	Continuous-time models	13
1.3.3	Discrete-event models	16
1.4	Simulation and optimization	22
1.4.1	Nonconvex optimization	23
1.4.2	Stochastic optimization	26
1.4.3	Stochastic dynamic programming	28
1.5	Pitfalls in Monte Carlo simulation	30
1.5.1	Technical issues	31
1.5.2	Philosophical issues	33
1.6	Software tools for Monte Carlo simulation	35
1.7	Prerequisites	37
1.7.1	Mathematical background	37
1.7.2	Financial background	38
1.7.3	Technical background	38
	For further reading	39
	References	39
2	Numerical Integration Methods	41
2.1	Classical quadrature formulas	43
2.1.1	The rectangle rule	44
2.1.2	Interpolatory quadrature formulas	45
2.1.3	An alternative derivation	46
2.2	Gaussian quadrature	48
2.2.1	Theory of Gaussian quadrature: The role of orthogonal polynomials	49
2.2.2	Gaussian quadrature in \mathbb{R}	51
2.3	Extension to higher dimensions: Product rules	53

2.4	Alternative approaches for high-dimensional integration	55
2.4.1	Monte Carlo integration	56
2.4.2	Low-discrepancy sequences	59
2.4.3	Lattice methods	61
2.5	Relationship with moment matching	67
2.5.1	Binomial lattices	67
2.5.2	Scenario generation in stochastic programming	69
2.6	Numerical integration in R	69
	For further reading	71
	References	71

Part II

Input Analysis: Modeling and Estimation

3	Stochastic Modeling in Finance and Economics	75
3.1	Introductory examples	77
3.1.1	Single-period portfolio optimization and modeling returns	78
3.1.2	Consumption–saving with uncertain labor income	81
3.1.3	Continuous-time models for asset prices and interest rates	83
3.2	Some common probability distributions	86
3.2.1	Bernoulli, binomial, and geometric variables	88
3.2.2	Exponential and Poisson distributions	92
3.2.3	Normal and related distributions	98
3.2.4	Beta distribution	105
3.2.5	Gamma distribution	107
3.2.6	Empirical distributions	108
3.3	Multivariate distributions: Covariance and correlation	112
3.3.1	Multivariate distributions	113
3.3.2	Covariance and Pearson’s correlation	118
3.3.3	R functions for covariance and correlation	122
3.3.4	Some typical multivariate distributions	124
3.4	Modeling dependence with copulas	128
3.4.1	Kendall’s tau and Spearman’s rho	134
3.4.2	Tail dependence	136
3.5	Linear regression models: A probabilistic view	137
3.6	Time series models	138
3.6.1	Moving-average processes	142
3.6.2	Autoregressive processes	147
3.6.3	ARMA and ARIMA processes	151
3.6.4	Vector autoregressive models	155
3.6.5	Modeling stochastic volatility	157
3.7	Stochastic differential equations	159
3.7.1	From discrete to continuous time	160

3.7.2	Standard Wiener process	163
3.7.3	Stochastic integration and Itô's lemma	167
3.7.4	Geometric Brownian motion	173
3.7.5	Generalizations	176
3.8	Dimensionality reduction	178
3.8.1	Principal component analysis (PCA)	179
3.8.2	Factor models	189
3.9	Risk-neutral derivative pricing	192
3.9.1	Option pricing in the binomial model	193
3.9.2	A continuous-time model for option pricing: The Black–Scholes–Merton formula	196
3.9.3	Option pricing in incomplete markets	203
	For further reading	205
	References	206
4	Estimation and Fitting	209
4.1	Basic inferential statistics in R	211
4.1.1	Confidence intervals	211
4.1.2	Hypothesis testing	214
4.1.3	Correlation testing	218
4.2	Parameter estimation	219
4.2.1	Features of point estimators	221
4.2.2	The method of moments	222
4.2.3	The method of maximum likelihood	223
4.2.4	Distribution fitting in R	227
4.3	Checking the fit of hypothetical distributions	228
4.3.1	The chi-square test	229
4.3.2	The Kolmogorov–Smirnov test	231
4.3.3	Testing normality	232
4.4	Estimation of linear regression models by ordinary least squares	233
4.5	Fitting time series models	237
4.6	Subjective probability: The Bayesian view	239
4.6.1	Bayesian estimation	241
4.6.2	Bayesian learning and coin flipping	243
	For further reading	248
	References	249

Part III

Sampling and Path Generation

5	Random Variate Generation	253
5.1	The structure of a Monte Carlo simulation	254
5.2	Generating pseudorandom numbers	256
5.2.1	Linear congruential generators	256
5.2.2	Desirable properties of random number generators	260

5.2.3	General structure of random number generators	264
5.2.4	Random number generators in R	266
5.3	The inverse transform method	267
5.4	The acceptance–rejection method	269
5.5	Generating normal variates	276
5.5.1	Sampling the standard normal distribution	276
5.5.2	Sampling a multivariate normal distribution	278
5.6	Other ad hoc methods	282
5.7	Sampling from copulas	283
	For further reading	286
	References	287
6	Sample Path Generation for Continuous-Time Models	289
6.1	Issues in path generation	290
6.1.1	Euler vs. Milstein schemes	293
6.1.2	Predictor-corrector methods	295
6.2	Simulating geometric Brownian motion	297
6.2.1	An application: Pricing a vanilla call option	299
6.2.2	Multidimensional GBM	301
6.2.3	The Brownian bridge	304
6.3	Sample paths of short-term interest rates	308
6.3.1	The Vasicek short-rate model	310
6.3.2	The Cox–Ingersoll–Ross short-rate model	312
6.4	Dealing with stochastic volatility	315
6.5	Dealing with jumps	316
	For further reading	319
	References	320

Part IV Output Analysis and Efficiency Improvement

7	Output Analysis	325
7.1	Pitfalls in output analysis	327
7.1.1	Bias and dependence issues: A financial example	330
7.2	Setting the number of replications	334
7.3	A world beyond averages	335
7.4	Good and bad news	337
	For further reading	338
	References	338
8	Variance Reduction Methods	341
8.1	Antithetic sampling	342
8.2	Common random numbers	348
8.3	Control variates	349
8.4	Conditional Monte Carlo	353

8.5	Stratified sampling	357
8.6	Importance sampling	364
8.6.1	Importance sampling and rare events	370
8.6.2	A digression: Moment and cumulant generating functions	373
8.6.3	Exponential tilting	374
	For further reading	376
	References	377
9	Low-Discrepancy Sequences	379
9.1	Low-discrepancy sequences	380
9.2	Halton sequences	382
9.3	Sobol low-discrepancy sequences	387
9.3.1	Sobol sequences and the algebra of polynomials	389
9.4	Randomized and scrambled low-discrepancy sequences	393
9.5	Sample path generation with low-discrepancy sequences	395
	For further reading	399
	References	400
 Part V Miscellaneous Applications		
10	Optimization	405
10.1	Classification of optimization problems	407
10.2	Optimization model building	421
10.2.1	Mean–variance portfolio optimization	421
10.2.2	Modeling with logical decision variables: Optimal portfolio tracking	422
10.2.3	A scenario-based model for the newsvendor problem	425
10.2.4	Fixed-mix asset allocation	426
10.2.5	Asset pricing	427
10.2.6	Parameter estimation and model calibration	430
10.3	Monte Carlo methods for global optimization	432
10.3.1	Local search and other metaheuristics	433
10.3.2	Simulated annealing	435
10.3.3	Genetic algorithms	439
10.3.4	Particle swarm optimization	441
10.4	Direct search and simulation-based optimization methods	444
10.4.1	Simplex search	445
10.4.2	Metamodeling	446
10.5	Stochastic programming models	448
10.5.1	Two-stage stochastic linear programming with recourse	448
10.5.2	A multistage model for portfolio management	452
10.5.3	Scenario generation and stability in stochastic programming	457
10.6	Stochastic dynamic programming	469

10.6.1	The shortest path problem	470
10.6.2	The functional equation of dynamic programming	473
10.6.3	Infinite-horizon stochastic optimization	477
10.6.4	Stochastic programming with recourse vs. dynamic programming	478
10.7	Numerical dynamic programming	480
10.7.1	Approximating the value function: A deterministic example	480
10.7.2	Value iteration for infinite-horizon problems	484
10.7.3	A numerical approach to consumption–saving	493
10.8	Approximate dynamic programming	506
10.8.1	A basic version of ADP	507
10.8.2	Post-decision state variables in ADP	510
10.8.3	Q -learning for a simple MDP	513
	For further reading	519
	References	520
11	Option Pricing	525
11.1	European-style multidimensional options in the BSM world	526
11.2	European-style path-dependent options in the BSM world	532
11.2.1	Pricing a barrier option	532
11.2.2	Pricing an arithmetic average Asian option	539
11.3	Pricing options with early exercise features	546
11.3.1	Sources of bias in pricing options with early exercise features	548
11.3.2	The scenario tree approach	549
11.3.3	The regression-based approach	552
11.4	A look outside the BSM world: Equity options under the Heston model	560
11.5	Pricing interest rate derivatives	563
11.5.1	Pricing bonds and bond options under the Vasicek model	565
11.5.2	Pricing a zero-coupon bond under the CIR model	567
	For further reading	569
	References	569
12	Sensitivity Estimation	573
12.1	Estimating option greeks by finite differences	575
12.2	Estimating option greeks by pathwise derivatives	581
12.3	Estimating option greeks by the likelihood ratio method	585
	For further reading	589
	References	590
13	Risk Measurement and Management	591
13.1	What is a risk measure?	593
13.2	Quantile-based risk measures: Value-at-risk	595

13.3	Issues in Monte Carlo estimation of $V@R$	601
13.4	Variance reduction methods for $V@R$	607
13.5	Mean–risk models in stochastic programming	613
13.6	Simulating delta hedging strategies	619
13.7	The interplay of financial and nonfinancial risks	625
	For further reading	626
	References	627
14	Markov Chain Monte Carlo and Bayesian Statistics	629
14.1	Acceptance–rejection sampling in Bayesian statistics	630
14.2	An introduction to Markov chains	631
14.3	The Metropolis–Hastings algorithm	636
	14.3.1 The Gibbs sampler	640
14.4	A re-examination of simulated annealing	643
	For further reading	646
	References	647
Index		649

Preface

The aim of this book is to provide a wide class of readers with a low- to intermediate-level treatment of Monte Carlo methods for applications in finance and economics. The target audience consists of students and junior practitioners with a quantitative background, and it includes not only students in economics and finance, but also in mathematics, statistics, and engineering. In fact, this is the kind of audience I typically deal with in my courses. Not all of these readers have a strong background in either statistics, financial economics, or econometrics, which is why I have also included some basic material on stochastic modeling in the early chapters, which is typically skipped in higher level books. Clearly, this is not meant as a substitute for a proper treatment, which can be found in the references listed at the end of each chapter. Some level of mathematical maturity is assumed, but the prerequisites are rather low and boil down to the essentials of probability and statistics, as well as some basic programming skills.¹ Advanced readers may skip the introductory chapters on modeling and estimation, which are also included as a reminder that no Monte Carlo method, however sophisticated, will yield useful results if the input model is flawed. Indeed, the power and flexibility of such methods may lure us into a false sense of security, making us forget some of their inherent limitations.

Option pricing is certainly a relevant application domain for the techniques we discuss in the book, but this is not meant to be a book on financial engineering. I have also included a significant amount of material on optimization in its many guises, as well as a chapter related to computational Bayesian statistics. I have favored a wide scope over a deeply technical treatment, for which there are already some really excellent and more demanding books. Many of them, however, do not quite help the reader to really “feel” what she is learning, as no ready-to-use code is offered. In order to allow anyone to run the code, play with it, and hopefully come up with some variations on the theme, I have chosen to develop code in R. Readers familiar with my previous book written in MATLAB might wonder whether I have changed my mind. I did not: I never use R in research or consulting, but I use it a lot for teaching. When I started writing the book, I was less than impressed by the lack of an adequate development environment, and some design choices of the language itself left me a bit puzzled. As an example, the `*` operator in MATLAB multiplies matrices row by column; whenever you want to work elementwise, you use the `.` operator, which has a clear and uniform meaning when applied to other operators. On the

¹In case of need, the mathematical prerequisites are covered in my other book: *Quantitative Methods: An Introduction for Business Management*. Wiley, 2011.

contrary, the operator `*` works elementwise in R, and row-by-column matrix product is accomplished by the somewhat baroque operator `%*%`. Furthermore, having to desperately google every time you have to understand a command, because documentation is a bit poor and you have to make your way in a mess of packages, may be quite frustrating at times. I have also found that some optimization functions are less accurate and less clever in dealing with limit cases than the corresponding MATLAB functions. Having said that, while working on the book, I have started to appreciate R much more. Also my teaching experience with R has certainly been fun and rewarding. A free tool with such a potential as R is certainly most welcome, and R developers must be praised for offering all of this. Hopefully, the reader will find R code useful as a starting point for further experimentation. I did *not* assemble R code into a package, as this would be extremely misleading: I had no plan to develop an integrated and reliable set of functions. I just use R code to illustrate ideas in concrete terms and to encourage active learning. When appropriate, I have pointed out some programming practices that may help in reducing the computational burden, but as a general rule I have tried to emphasize clarity over efficiency. I have also avoided writing an introduction to R programming, as there are many freely available tutorials (and a few good books²). A reader with some programming experience in any language should be able to make her way through the code, which has been commented on when necessary. My assumption is that a reader, when stumbling upon an unknown function, will take advantage of the online help and the example I provide in order to understand its use and potentiality. Typically, R library functions are equipped with optional parameters that can be put to good use, but for the sake of conciseness I have refrained from a full description of function inputs.

Book structure

The book is organized in five parts.

1. Part I, Overview and Motivation, consists of two chapters. Chapter 1 provides an introduction to Monte Carlo methods and applications. The different classes of dynamic models that are encountered in simulation are outlined, and due emphasis is placed on pitfalls and limitations of Monte Carlo methods. Chapter 2 deals with numerical integration methods. Numerical integration is quite relevant, as it provided most of the historical motivation to develop Monte Carlo methods; furthermore, there are cases in which one is much better off using good quadrature formulas than throwing random numbers around. Finally, framing Monte Carlo methods within numerical integration provides the necessary discipline to understand and properly use low-discrepancy sequences, sometimes referred to as quasi-Monte Carlo methods.

²Unfortunately, I have also run into very bad books using R; hopefully, this one will not contribute to the list.

2. Part II, Input Analysis: Modeling and Estimation, is specifically aimed at students and newcomers, as it includes two introductory chapters dealing with stochastic model building (Chapter 3) and model fitting (Chapter 4). Essentially, in this part of the book we are concerned with the modeling of inputs of a Monte Carlo simulation. Many advanced books on Monte Carlo methods for finance skip and take for granted these concepts. I have preferred to offer a limited treatment for the sake of unfamiliar readers, such as students in engineering or practitioners without an econometrics background. Needless to say, space does not allow for a deep treatment, but I believe that it is important to build at least a framework for further study. In order to make this part useful to intermediate readers, too, I have taken each topic as an excuse for a further illustration of R functionalities. Furthermore, some more advanced sections may be useful to students in economics and finance as well, such as those on stochastic calculus, copulas, and Bayesian statistics.
3. Part III, Sampling and Path Generation, is more technical and consists of two chapters. In Chapter 5 we deal with pseudorandom number and variate generation. While it is certainly true that in common practice one takes advantage of reliable generators provided by software tools like R, and there is no need for an overly deep treatment, some basic knowledge is needed in order to select generators and to manage simulation runs properly. We also outline scenario generation using copulas. In Chapter 6 we deal with sample path generation for continuous-time models based on stochastic differential equations. This is an essential tool for any financial engineer and is at the heart of many derivative pricing methods. It is important to point out that this is also relevant for risk managers, insurers, and some economists as well.
4. Part IV, Output Analysis and Efficiency Improvement, looks at the final step of the simulation process. Monte Carlo methods are extremely powerful and flexible; yet, their output may not be quite reliable, and an unreasonable computational effort may be called for, unless suitable countermeasures are taken. Chapter 7 covers very simple, and possibly overlooked, concepts related to confidence intervals. Counterexamples are used to point out the danger of forgetting some underlying assumptions. Chapter 8 deals with variance reduction strategies that are essential in many financial engineering and risk management applications; indeed, the techniques illustrated here are applied in later chapters, too. Chapter 9 deals with low-discrepancy sequences, which are sometimes gathered under the quasi-Monte Carlo nickname. Actually, there is nothing stochastic in low-discrepancy sequences, which should be regarded as deterministic numerical integration strategies. For a certain range of problem dimensionality, they are a good alternative to pseudorandom sampling.
5. Part IV, Miscellaneous Applications, includes five more or less interrelated chapters dealing with:

- The interplay between optimization and Monte Carlo methods, including stochastic methods for deterministic global optimization, scenario generation for stochastic programming with recourse, and stochastic dynamic programming (Chapter 10)
- Option pricing, with an emphasis on variance reduction methods (Chapter 11)
- Obtaining sensitivity of performance measures with Monte Carlo simulation (Chapter 12)
- Risk measurement and management, with an emphasis on value-at-risk and related risk measures for financial risk management (Chapter 13)
- Markov chain Monte Carlo (MCMC) methods, which are relevant for different applications, most notably computational Bayesian statistics (Chapter 14)

There are some logical precedences among these final chapters, but they need not be read in a strict sequential order. Chapter 14 is independent of the others, and the only link is represented by the possibility of regarding simulated annealing, a stochastic approach for both global and combinatorial optimization, as an MCMC method. Stochastic dynamic programming, dealt with in Chapter 10, is needed to understand American-style option pricing in Chapter 11. Measuring option price sensitivity is used as a motivating example in Chapter 12, but the methods outlined there have a much more general applicability, as they can also be used within optimization procedures. Finally, there is certainly a natural link between option pricing and the risk management examples discussed in Chapter 13.

Supplements and R code

The code has been organized in two files per chapter. The first one contains all of the function definitions and the reference to required packages, if any; this file should be sourced before running the scripts, which are included as chunks of code in a second file. An archive including all of the R code will be posted on a webpage. My current URL is

- <http://staff.polito.it/paolo.brandimarte/>

A hopefully short list of errata will be posted there as well. One of the many corollaries of Murphy's law states that my URL is going to change shortly after publication of the book. An up-to-date link will be maintained on the Wiley webpage:

- <http://www.wiley.com/>

For comments, suggestions, and criticisms, all of which are quite welcome, my e-mail address is

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Turin, February 2014