Multiscale Methods for Crowd Dynamics
Individuality vs. Collectivity

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Individuality vs. Collectivity
Microscopic (Particle-Based) Models

“Social force” model (Helbing et al., 1995)

\[
\begin{align*}
\dot{x}_i &= v_i \\
\dot{v}_i &= \frac{v_{0,i} - v_i}{\tau} - \sum_{j=1}^{N} \nabla U_{ij}(x_j - x_i) + \ldots
\end{align*}
\]

“Contact handling” model (Maury and Venel, 2007)

\[
\dot{x}_i = \mathcal{P}_C(x_i)(V_{\text{des}}(x_i))
\]

- $V_{\text{des}} : \mathbb{R}^d \to \mathbb{R}^d$ pedestrian desired velocity
- $\mathcal{P}_C(x_i)$ projection operator on the space of admissible velocities
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“Contact handling” model (Maury and Venel, 2007)

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\dot{x}_i = \mathcal{P}_C(X)(V_{\text{des}}(x_i))
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- $V_{\text{des}}: \mathbb{R}^d \to \mathbb{R}^d$ pedestrian desired velocity
- $\mathcal{P}_C(X)$ projection operator on the space of admissible velocities
Macroscopic (Density-Based) Models


\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho V(\rho) \nu) = 0
\]

- \( V(\rho) \) speed-density relationship
- \( \nu \) preferred direction of movement


\[
\begin{cases}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0 \\
\frac{\partial}{\partial t} (\rho v) + \nabla \cdot (\rho v \otimes v) = \frac{\rho V(\rho) \nu - \rho v}{\tau} - \nabla P(\rho)
\end{cases}
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Macroscopic (Density-Based) Models


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\end{align*}
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Get inspiration from a simple first order particle model:

\[
\dot{x}_i = V_{\text{des}}(x_i) + \sum_{x_j \in S_{R,\alpha}(x_i)} K(x_j - x_i)
\]

Label pedestrians by means of their initial position:

\[
X_t : \mathbb{R}^d \to \mathbb{R}^d \quad \text{(flow map)}
\]

\[
x = X_t(\xi) : \text{position at time } t > 0 \text{ of the walker who initially was in } \xi \in \mathbb{R}^d
\]

Fix a walker \(\xi\) and rewrite the particle model using the flow map:

\[
\dot{X}_t(\xi) = V_{\text{des}}(X_t(\xi)) + \int_{X_t^{-1}(S_{R,\alpha}(X_t(\xi)))} K(X_t(\eta) - X_t(\xi)) \, d\mu_0(\eta)
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Get inspiration from a simple first order particle model:

\[ \dot{x}_i = V_{\text{des}}(x_i) + \sum_{x_j \in S_{R,\alpha}(x_i)} K(x_j - x_i) \]

- Transport \( \mu_0 \) by means of \( X_t \), i.e., \( \mu_t := X_t \# \mu_0 \), to discover:

\[
\begin{aligned}
\frac{\partial \mu}{\partial t} + \nabla \cdot (\mu v[\mu]) &= 0 \\
v[\mu_t](x) &= V_{\text{des}}(x) + \int_{S_{R,\alpha}(x)} K(y - x) \, d\mu_t(y) \\
\end{aligned}
\]

x \in \mathbb{R}^d, \ t > 0

- Description compatible with both a discrete and a continuous view of the crowd:

\[
\begin{aligned}
\text{discrete: } \mu_0 &= \sum_{i=1}^{N} \delta_{\xi_i} \\
\text{continuous: } \mu_0 &= \rho_0 \mathcal{L}^d
\end{aligned}
\]
Get inspiration from a simple first order particle model:

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\dot{x}_i = V_{\text{des}}(x_i) + \sum_{x_j \in S_{R,\alpha}(x_i)} K(x_j - x_i)
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- Description compatible with both a discrete and a continuous view of the crowd:

  - **Discrete:** \(\mu_0 = \sum_{i=1}^{N} \delta_{\xi_i}\)
  - **Continuous:** \(\mu_0 = \rho_0 \mathcal{L}^d\)
Comparison between discrete and continuous models in terms of statistical distributions of pedestrians:

\[
\mu_{0}^{\text{discr}} = \frac{1}{N} \sum_{i=1}^{N} \delta_{\xi_{i}}, \quad \mu_{0}^{\text{cont}} = \frac{1}{\mathcal{L}^{d}(\Omega)} \mathcal{L}^{d}
\]

\[
\Omega \leadsto W_{1}(\mu_{0}^{\text{discr}}, \mu_{0}^{\text{cont}}) \leq \frac{\sqrt{d}}{2} \Delta_{h} + \frac{\text{diam}(\Omega)}{\mathcal{L}^{d}(\Omega)} \mathcal{L}^{d}(\Omega \setminus K_{N})
\]

\[
N \to \infty \quad 0
\]

Continuous dependence estimate (for smooth \( V_{\text{des}} \) and \( K \)):

\[
W_{1}(\mu_{t}^{\text{discr}}, \mu_{t}^{\text{cont}}) \leq C W_{1}(\mu_{0}^{\text{discr}}, \mu_{0}^{\text{cont}}) \quad \forall t \in (0, T]
\]

hence \( \mu_{t}^{\text{discr}} \equiv \mu_{t}^{\text{cont}} \) for all \( t \) in the limit \( N \to \infty \).
Comparison between discrete and continuous models in terms of statistical distributions of pedestrians:

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\mu_0^{\text{discr}} = \frac{1}{N} \sum_{i=1}^{N} \delta_{\xi_i}, \quad \mu_0^{\text{cont}} = \frac{1}{L^d(\Omega)} L^d
\]

\[
\Omega \xrightarrow{K_N} W_1(\mu_0^{\text{discr}}, \mu_0^{\text{cont}}) \leq \frac{\sqrt{d}}{2} h + \frac{\text{diam}(\Omega)}{L^d(\Omega)} L^d(\Omega \setminus K_N)
\]

\[
N \to \infty \quad \Rightarrow \quad 0
\]

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Comparison between discrete and continuous models in terms of statistical distributions of pedestrians:

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\mu_{0\text{discr}} = \frac{1}{N} \sum_{i=1}^{N} \delta_{\xi_i}, \quad \mu_{0\text{cont}} = \frac{1}{L^d(\Omega)} L^d
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\xrightarrow{N \to \infty} 0
\]

Continuous dependence estimate (for smooth \(V_{\text{des}}\) and \(K\)):

\[
W_1(\mu_{t\text{discr}}, \mu_{t\text{cont}}) \leq \mathcal{C} W_1(\mu_{0\text{discr}}, \mu_{0\text{cont}}) \quad \forall t \in (0, T]
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\text{hence} \quad \mu_{t\text{discr}} \equiv \mu_{t\text{cont}} \quad \text{for all } t \text{ in the limit } N \to \infty.
\]
Does the limit $N \to \infty$ really make sense for crowds?

- Often it does not: pedestrians in a crowd are not as many as $10^{23}$ gas molecules.
- For finite $N$ physical mass distributions matter.
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