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ON THE MATHEMATICAL THEORY OF VEHICULAR TRAFFIC FLOW I. FLUID DYNAMIC AND KINETIC MODELLING

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This review reports the existing literature on traffic flow modelling in the framework of a critical overview which aims to indicate research perspectives. The contents mainly refer to modelling by fluid dynamic and kinetic equations and are arranged in three parts. The first part refers to methodological aspects of mathematical modelling and to the interpretation of experimental results. The second part is devoted to modelling and deals both with methodological aspects and with the description of some specific models. The third part reports about an overview on applications and research perspectives.

Keywords: Traffic flow; hydrodynamics; kinetic theory; nonlinear sciences.

AMS Subject Classification: 76P05

1. Introduction

The fast growing number of vehicles on networks of roads, either highways or urban streets, and the related economical and social implications, e.g. pollution and energy control, prevention of car crashes, etc., constantly motivates an intense research activity in the field of traffic flow modelling. Both applied mathematicians and engineers are involved and several interesting results have been obtained despite the great complexity of the above system, which appears to be rather difficult to be constrained into a mathematical framework.

The objective of mathematical research consists first in deriving suitable models to describe the evolution in time and space of the flow conditions: car density and velocity. Then in solving mathematical problems generated by the application of models to real traffic flow conditions. The output may hopefully be useful for engineers involved in traffic flow control and optimization. Indeed, mathematical models may be regarded as tools to be used toward complex optimization programs addressed to improve fluidity of traffic flow, to reduce pollution due to emission of exhaust gases, and possibly to control the number of car crashes.

Mathematical modelling as usual, see Ref. 8, can be developed with different methods corresponding to different scales of the phenomenological observation of the system. Specifically, **microscopic modelling** corresponds to model the dynamics of each single vehicle under the action of the surrounding vehicles. **Statistical description**, in a framework close to the one of the kinetic theory of gases, consists of the derivation of an evolution equation for the distribution function on the position and velocity of the vehicle along the road. **Macroscopic description**, analogous to the one of fluid dynamics, refers to the derivation, on the basis of conservation equations and material models, of an evolution equation for the mass density, linear momentum and energy, regarded as a macroscopic observable quantity of the flow of vehicles assumed to be continuous.

Different types of equations correspond to the above classes of models. Microscopic models are stated in terms of large systems of ordinary differential or difference equations; macroscopic models in terms of partial differential equations and kinetic models by integro-differential equations similar to the Boltzmann equation. A common feature of all mathematical models is the presence of various nonlinear terms both in the evolution equation and in the statement of related mathematical problems.

Some books and expository papers provide a useful background. The interested reader is referred to the books by Prigogine and Herman,[59] mainly concerned with kinetic models, and by Leutzbach,[48] mainly devoted to microscopic and hydrodynamic modelling. It is worth mentioning that the book by Prigogine and Herman, although devoted to kinetic modelling, suggests the development of various experiments and anticipates their interpretation. So far their analysis is a relevant reference point for all subsequent research activity in the field. Useful suggestions which may hopefully generate critical analysis and attempt improvements can be recovered in their book.

In addition to the above books some papers concerning general topics in the field have appeared: the survey papers by Klar, Küne and Wegener,[41] by Klar and Wegener,[43] and by Helbing *et al.*,[32] mainly refer to kinetic type modelling although a concise introduction on microscopic and macroscopic modelling is reported. Moreover, Bellomo, Marasco and Romano[10] report the various models concerning the drivers' behavior in traffic flow and their influence on macroscopic models and about the related statement of problems. The recent review by Helbing[33] reports a large literature not only in the fields of mathematics, but also in technological and engineering sciences. The great merit of the above review consists of having shown how a critical analysis of several phenomena which appear in traffic flow experimental observation can be properly interpreted and transferred into mathematical analytic terms. The above literature appears to be the background for the contents of this review paper which reports about the existing literature within a critical overview which aims to indicate research perspectives.

The existing literature is vast and characterized by various contributions covering modelling aspects, statement of problems, qualitative analysis and simulations related to applications. This scientific production is certainly related to economical and social motivations. On the other hand, due to the high complexity of the phenomena related to traffic flow, modelling has not yet reached a satisfactory level so that further analysis and improvements may be necessary in order to reach the standard needed by applications to real traffic flow problems. Therefore a review paper cannot be limited by reporting about the state-of-the-art. The development of a critical analysis followed by indication of research perspectives is one of the main issues of this paper which mainly refers to modelling by fluid dynamic and kinetic equations.

Therefore the contents of this paper concern the above topics which have to be regarded as a relatively small subset of the whole range of mathematical topics related to traffic flow modelling. Referring to such subset, the critical analysis will be addressed to identify improvements of the existing models and possibly development of new mathematical structures which may be able to take properly account of the complexity of the system we are dealing with. A crucial aspect in the modelling process is that the derivation of models has to take into account not only the mechanistic behavior of the vehicles, but also the psycho-mechanic interaction between drivers and vehicles.

The first section of this paper deals with an introduction to modelling of traffic flow including a preliminary analysis of the complexity of the physical phenomena described. This section also provides an outline of the aims of the paper and of its organization. The contents which follow are organized through seven additional sections. In detail, the first part, Secs. 2 and 3, refers to some preliminary aspects of mathematical modelling. Section 2 reports the above-mentioned observation scales and the related selection of the variables for the description of the system. Section 3 reports the experimental results available in the literature.

Sections 4–7 refer to modelling methods and to the derivation of some specific models. Specifically, Sec. 4 deals with the description of the general fluid dynamical framework: mass, linear momentum and energy conservation equations, that may technically generate specific traffic flow models. Section 5 deals with a survey, and critical analysis, of the various models which are available in the literature. This section also indicates how the said model may hopefully be improved and generalized to various flow conditions of practical interest. An analogous report is developed for kinetic models: Sec. 6 deals with mathematics aspects related to the derivation of kinetic type models, while Sec. 7 deals with the description of some specific models.

The last section deals with a critical analysis and perspectives, such as multilane modelling, lanes with modification of viability conditions, networks of roads; with an analysis of the links among microscopic and macroscopic description; finally the section ends with an indication of research perspectives mainly toward the design of new models.

As already mentioned, this paper is mainly concerned with modelling. An analogous report is in progress on the development and application of mathematical methods for the solution of initial and/or boundary value problems. Although the contents refers to macroscopic and kinetic type description, it is worth anticipating part of the critical analysis proposed in the last section on the deficiency of the present state of the art on the modelling of the complex system we are dealing with. In other words, it is not naively claimed that the flow of vehicles can be identified as a continuous fluid as it is needed by the macroscopic description, nor that it is a rarefied flow as it is required by the Boltzmann-like description. Indeed models which will be described in what follows have to be regarded as an approximation of physical reality. Research activity, as documented in Ref. 27, has to be addressed to the development of new modelling methods. This paper aims at contributing towards this objective.

2. Scaling and Representation of Granular Flow

The modelling of real physical systems starts with the selection of the observation and representation scales. Then, within each scale, the variables which are charged to represent the physical system in the model have to be defined. For various technical reasons, it is useful to scale all variables with respect to suitable reference quantities so that all independent and dependent variables take value in the domain $[0, 1]$ and are dimensionless.[8]

Selecting the correct observation and representation scale for traffic flow phenomena is not a simple task. As a matter of fact, the denomination **granular flow** is used to indicate that the usual continuity assumption applied in hydrodynamics has to be handled carefully considering that distances between vehicles may not be negligible with respect to the length of the road.

Bearing this in mind, consider the one-dimensional flow of vehicles along a road with length ℓ with one or more lanes labelled by the superscript j, with $j = 1, \ldots, r$.

In order to define dimensionless quantities, one has to identify characteristic time T and length ℓ, as well as maximum density n_M and maximum average velocity v_M. Specifically:

n_M is the maximum density of vehicles corresponding to bump-to-bump traffic jam;
v_M is the maximum admissible mean velocity which may be reached by vehicles in the empty road.

It is spontaneous to assume $v_M T = \ell$, which means that T is the time necessary to cover the whole road length at the maximum mean velocity. After the above preliminaries, we are now in position to define dimensionless independent and dependent variables. This choice, as we shall see, can be related to the structure of the mathematical model.

- $t = t_r/T$, the dimensionless time variable referred to the characteristic time T, where t_r is the real time;

- $x = x_r/\ell$, the dimensionless space variable referred to the characteristic length of the road ℓ, where x_r is the real dimensional space.

The dimensionless dependent variables are:

- $u = n/n_M$, the dimensionless density referred to the maximum density n_M of vehicles;
- $V = V_R/v_M$, the dimensionless velocity referred to the maximum mean velocity v_M, where V_R is the real velocity of the single vehicle;
- q, the dimensionless linear mean flux referred to the maximum admissible mean flux q_M;

It must be pointed out that a fast isolated vehicle can reach velocities larger that v_M. In particular a limit velocity can be defined

$$V_\ell = (1 + \mu)v_M \,, \quad \mu > 0 \,, \tag{2.1}$$

such that no vehicle can reach a velocity larger than V_ℓ. Of course both V_M and μ may depend on the characteristics of the lane, say a country lane or a highway, as well as to the type of vehicles, say a slow car, a fast car, a lorry, etc.

All the above variables can assume different characterization according to the modelling scales which can be adopted for the observation and modelling. In particular, one may consider the following types of descriptions.

- **Microscopic description**: all vehicles are individually identified. Position and velocity of each vehicle define the state of the system as dependent variables of the time variable.
- **Kinetic description**: the state of the system is still identified by position and velocity of the vehicles however their identification does not refer to each vehicle but to a suitable probability distribution.
- **Macroscopic description**: the state is described by locally averaged quantities, i.e. density, mass velocity and energy, regarded as dependent variables of time and space

In the following subsections, the selection of the dependent variables for each type of representation will be considered. This procedure can also be useful toward the interpretation of experimental results. As already mentioned, this paper essentially refers to hydrodynamic and kinetic modelling, microscopic interactions are analyzed only in view of the kinetic description. Motivations of this choice are explained and critically analyzed in the last section of this paper. In fact, the above choice is technically: car flow cannot be regarded either as continuous flow nor as a rarefied gas. So far, a deep insight into the kinetic and macroscopic description needs to be critically viewed and taken as starting point to look for new approaches to modelling as it will be discussed in the last section of this paper.

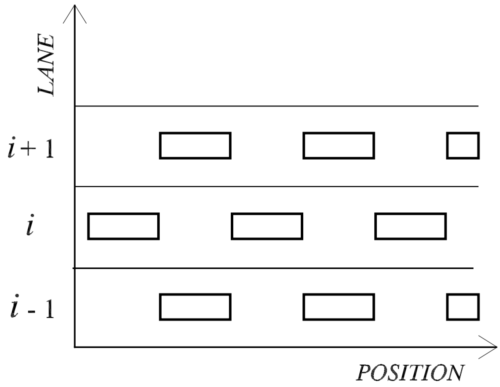

Fig. 2.1. Multilane flow.

2.1. *Microscopic representation*

When all vehicles are individually identified, the state of the whole system is defined, for each lane, by dimensionless position and velocity of the vehicles; they can be regarded, neglecting their dimensions, as single points

$$x_i^j = x_i^j(t)\,, \quad V_i^j = V_i^j(t)\,, \quad i = 1, \ldots, N\,, \quad j = 1, \ldots, r\,, \tag{2.2}$$

where the subscript refers to the vehicle and the superscript to the lane. It is worthwhile to stress here that, when dealing with microscopic modelling, capital letters represent dimensionless quantities referred to individual *particles.*

The knowledge of the above quantities may provide, by suitable averaging processes, gross quantities such as density and mass velocity. However, this is a delicate problem related to the fact that the real discrete system, made up of single vehicles, has been approximated by a continuous flow. In principle, one can average the above physical quantities either at fixed time over a certain space domain or at fixed space over a certain time range. For instance the number density is given, for each lane, by the number of vehicles which at the time t are in the tract $[x - \Delta_x, x + \Delta_x]$, say

$$u(t; x) \cong \frac{N(t)}{2\Delta_x n_M}\,. \tag{2.3}$$

A similar reasoning can be applied for the mean velocity

$$v(t; x) \cong \frac{1}{N(t) v_M} \sum_{i=1}^{N(t)} V_i(t)\,, \tag{2.4}$$

where $V_i(t)$ denotes the velocity of the i-vehicle at time t. Of course, the choice of the space interval is a critical problem and fluctuations may be generated by different choices.

Models developed at a microscopic scale are generally described by ordinary differential equations. Then, similarly to the Newtonian mechanics for systems of particles, one has to solve a large system of equations. Mean quantities are then obtained by the above averaging process.

2.2. *Statistical (kinetic) description*

The state of the whole system is defined, for each lane, by the statistical distribution of position and velocity of the vehicles. Specifically the following distribution over the dimensionless microscopic state is considered

$$f^j = f^j(t, x, V)\,, \tag{2.5}$$

where $f^j\,dxdV$ is the number of vehicles which at time t and in lane j are in the phase domain $[x, x+dx] \times [V, V+dV]$.

Macroscopic observable quantities can be obtained, under suitable integrability assumptions, as momenta of the distribution f, normalized with respect to the maximum density n_M so that all the derived variables such as density, flux etc. will be given in a dimensionless form. Specifically, the dimensionless **local density** in the j-lane is given by

$$u^j(t, x) = \int_0^{1+\mu} f^j(t, x, V)\,dV\,. \tag{2.6}$$

The **total number of vehicles** at the time t is given by

$$N(t) = n_M \sum_{j=1}^{r} \int_0^1 u^j(t, x)\,dx = n_M \sum_{j=1}^{r} \int_0^{1+\mu} \int_0^1 (t, x, V)\,dV dx\,. \tag{2.7}$$

In the same way, the **mean velocity** can be computed as follows:

$$v^j(t, x) = E[V^j](t, x) = \frac{q^j(t, x)}{u^j(t, x)} = \frac{1}{u^j(t, x)} \int_0^{1+\mu} V^j f^j(t, x, V)\,dV\,, \tag{2.8}$$

and the **speed variance**

$$\sigma^j(t, x) = \frac{p^j(t, x)}{u^j(t, x)} = \frac{1}{u^j(t, x)} \int_0^{1+\mu} [V^j - v^j(t, x)]^2 f^j(t, x, V)\,dV\,, \tag{2.9}$$

where q^j and p^j are, respectively, the **flow** and the **speed pressure** in the j-lane.

Also in this case, one can obtain quantities averaged over all lanes. For instance, the lane-averaged mean velocity is

$$v(t, x) = \frac{1}{r} \sum_{j=1}^{r} V^j(t, x)\,. \tag{2.10}$$

Similarly to the classical Boltzmann equation, kinetic models are described by integro-differential equations. The solution of mathematical problems, generally initial and/or boundary value problems, provides the evolution of the above distribution function. The above integrations give averaged quantities.

2.3. *Macroscopic description*

The description refers directly to averaged quantities regarded as dependent variables with respect to time and space; mathematical models are stated in terms of evolution equations for the above variables. The models will be obtained by conservation equations corresponding to mass, linear momentum and energy. As already mentioned, the domain of definition of the mean velocity v is smaller than the domain of the velocity which can be reached by an isolated vehicle V. Therefore models refer, for each lane, to the variables

$$u^j = u^j(t,x) \in [0,1]\,, \quad v^j = v^j(t,x) \in [0,1]\,, \quad j = 1,\ldots,r\,, \tag{2.11}$$

as well as to additional variables such as the mean energy, which will be defined later.

Generally models are described by partial differential equations. The solution of mathematical problems, generally initial and/or boundary value problems, provides the evolution of the macroscopic quantities.

3. Complexity of Traffic Flow and Experiments

Mathematical modelling needs the support of appropriate experiments which may contribute both to the design of specific models and to their validation. Various results are available in the literature. However, the complexity of the system studied, does not generally allow an immediate utilization of experimental results without their proper interpretation through suitable phenomenological models.

Experimental data are the main tools available in order to analyze the traffic phenomena. As documented in the books by Prigogine and Herman[59] and Leutzbach,[48] most of the experiments refer to macroscopic quantities; flux and mean velocity of traffic as function of the vehicular density.

In details, the flow-density curve is called the **fundamental diagram** and is measured in *steady state conditions*: this means that the quantities characterizing the system vary sufficiently slowly with respect to space and time; this situation would be better denoted, using the language of kinetic theory, as *steady and spatially homogeneous state*. The fundamental diagram is valid for the specific road where the experiments is performed: this means that it may show qualitatively different behavior in different situations, like change of road, time and weather. However, its qualitative shape has some common features: for low densities, the flow grows quasi-linearly with the density; then it begins to grow less rapidly, goes through a maximum, and finally falls to zero at the *jam* density.

The mean speed-density curve, called **velocity diagram**, is related to the fundamental diagram. For low density the mean speed is almost constant, equal to the *maximum mean free speed*, since for dilute traffic the drivers behave independently of each other, but try to reach the largest value consistent with their usual way of driving the vehicle. For higher density the average speed begins to decrease with increasing density as a result of mutual interactions of drivers. For *jam* density, the mean velocity reaches the value equal to zero.

The stochastic nature of traffic flow leads to fluctuations in the measured data. In the same experimental conditions, data exhibit large fluctuations near the presumed location of the maximum flow volume, and even larger fluctuations at higher densities in correspondence to the transition from free or partially constrained flow, to constrained flow: this instability somehow reminds the well-known distinction between laminar and turbulent flow in classical hydrodynamics. A very important factor is the length of the time interval over which data are collected: the shorter the interval of measurement, the more effective is the impact of slow vehicles and the stochastic element of traffic flow on the experimental results.

It may appear strange that the data show large fluctuations due to a stochastic component when the density is large, as it is easy to verify that the vehicles should move at the speed imposed by the flow and cannot choose their speed according to their individual will. On the other hand, it has to be notice that a queue presents the well-known discontinuous situation such as *stop-and-go*. In this way one may justify the fluctuation of experimental data and can state that in a constrained flow the stochasticity is not pertinent to the single but to the whole flow of vehicle. Larger time intervals of measurements smooth fluctuations by a sort of averaging process over time.

The behavior at low density may be largely affected by weather conditions and effective speed control. In other words, if the speed control is effective and the weather conditions are good, then all vehicles tend to reach the maximum admissible velocity: this means that fluctuations are small. Otherwise fluctuations may be large even at low density.

Critical analysis on the complexity of the system and on the organization and interpretation of experiments can be recovered in various papers, e.g. Refs. 34–39, 56 and 61. Summarizing some of the critical aspects, the following items may be indicated:

(i) experiments are generally developed in steady state conditions. On the other hand, modelling needs information in unsteady conditions.
(ii) measurements corresponding to repeated experiments provide different results with fluctuations around a certain mean value or most probable value. The output of experimental results can be regarded as a random variable with non-negligible variance.
(iii) experiments related to gross quantities, such as density and mass velocity, are obtained, as is explained in Sec. 2, by an averaging process, either in space or in time, of microscopic measurements. This procedure unavoidably generates errors.

The deterministic interpretation of the above-mentioned experimental representation consists of looking for their simple analytic description, which may be used, as one can see, also for designing suitable evolution models. For instance, the following formula

$$v_e = (1 - u^{1+a})^{1+b}\,, \quad q_e = u(1 - u^{1+a})^{1+b}\,, \tag{3.1}$$

with $a, b \geq 0$, is due to Kḧune and Rödinger.[40] Usually the relatively simpler expression $v_e = 1 - u$, corresponding to $a = b = 0$, is used.

Various authors have suggested alternative expressions for the above deterministic best fitting of experimental data. However, discussing about technical improvements of Eq. (3.1), it is worthwhile, in view of developments of mathematical models, to present some new ideas toward the statistical representation which can be recovered in the dissertation[24] and in Ref. 15. In particular, a critical analysis is proposed in Ref. 15 concerning the variability of experimental evidence with local conditions. Specifically, two formula are proposed in Ref. 15 and are shown to give a relatively more satisfactory, with respect to (3.1), agreement with experiments. The first one gives a clousure relation for the equilibrium velocity which is identified with the most probable velocity v_p:

$$v_p = \exp\left\{-\alpha\left(\frac{u}{1-u}\right)^2\right\}, \quad u \in [0, 1[\,, \quad \alpha > 0\,, \tag{3.2}$$

while the second one takes into account that below a certain critical density u_{c}, the mean velocity keeps a constant (maximum) value;

$$v_p = H(z) + K(z)\exp\left\{-\alpha\left(\frac{u}{1-u}\right)^2\right\}, \tag{3.3}$$

where $z = u - u_{\mathrm{c}}$ and

$$z \leq 0 \Rightarrow H(z) = 1\,, \quad K(z) = 0\,; \quad z > 0 \Rightarrow H(z) = 0\,, \quad K(z) = 1\,. \tag{3.4}$$

Models (3.2) and (3.3) show that the mean velocity at low density keeps a constant value (or $v_p(u)$ starts with flat tangent). This is reasonable for a large highway where v_p is large. For relatively small roads this phenomenon may not appear so that one may not have to modify, as shown in the forthcoming paper,[16] the structure of the argument of the exponential.

Referring to this matter, the analysis developed in Ref. 16 appears to be particularly interesting for the applications. Indeed, it identifies three phenomena which are relevant in the description of traffic flow phenomena:

(i) decay of the mean velocity with local density;
(ii) transition from free flow at the highest speed to the decaying velocity regime;
(iii) effect of the density gradients on the drivers' reaction.

Then the identification of parameters is related to each of the above phenomena. Specifically, α to (i), u_{c} to (ii), while the effect of density gradients is related to a parameter η which will be defined later in Sec. 5. In particular, η refers to the modelling of the fact that the driver **feels** a density which is smaller or larger than the real one in the presence of a negative or positive, respectively, density gradient. As shown in Ref. 16, all the above parameters can be identified by suitable experiments.

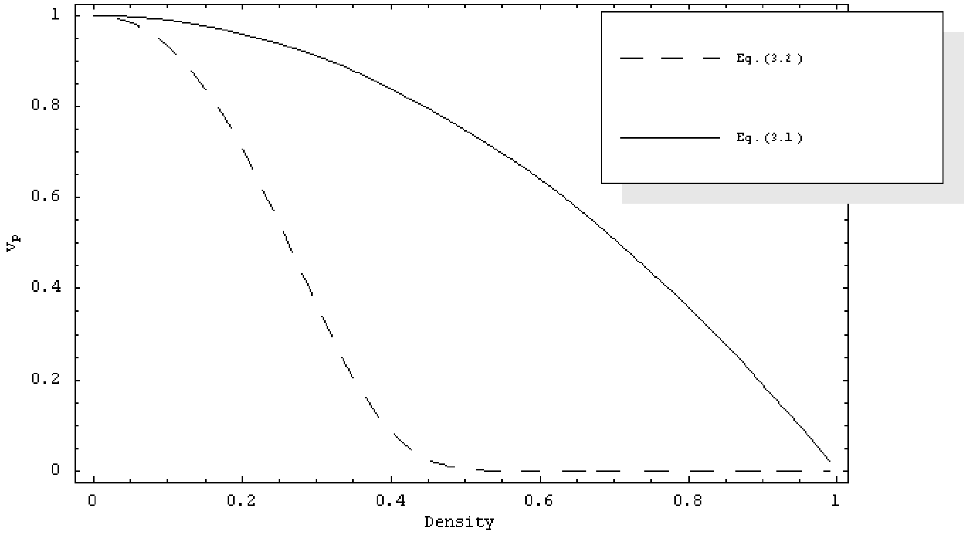

Fig. 3.1. The velocity diagram.

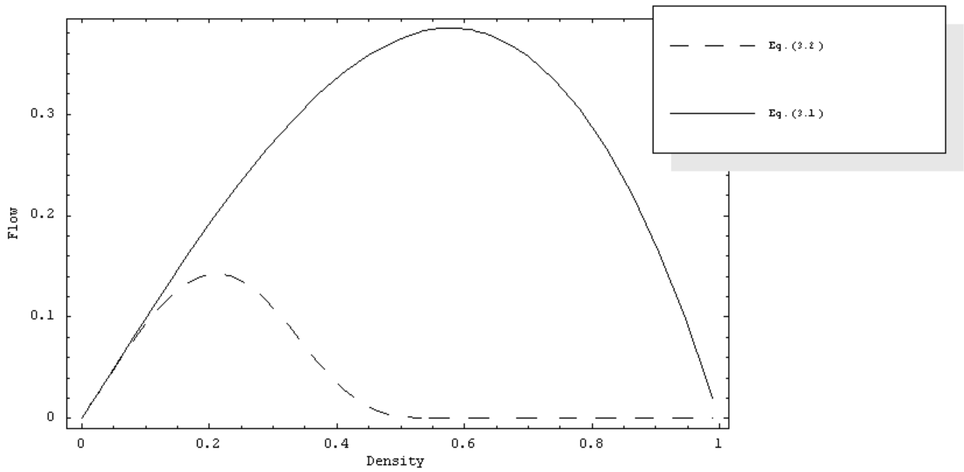

Fig. 3.2. The fundamental diagram.

Specifically Fig. 3.1 shows the *velocity diagram* corresponding to Eq. (3.1), with $a = 1$, $b = 0$ as suggested in Ref. 23, compared with the representation delivered by Eqs. (3.3) with $\alpha = 5.50$. The same comparison is shown for the *fundamental diagram* reported in Fig. 3.2. For both comparisons, substantial qualitative and quantitative differences can be observed. This means that experiments realized in different environments may lead to qualitatively different behaviors.

This paper does not aim at discussing the validity of experiments, but rather their experimental interpretation. Therefore we simply observe that one should attempt to find a universal formula to describe all observed phenomena such that only one parameter is related to each phenomena. This is the case of models (3.2) and (3.3), while model (3.1) has the disadvantage of the presence of two parameters, so that it may happen (and it does effectively) that different couples of a and b lead to the same best fitting of experimental data.

Moreover, considering that stochastic fluctuations appear to be an important feature of the phenomenon, their interpretation is still a relevant objective, to be exploited, as we shall see, toward the modelling of traffic flow phenomena. This aspect has been recently studied so that, still following Refs. 15 and 24, it is worth analyzing a conceivable stochastic interpretation of experimental data. The idea simply consists of assuming that the equilibrium velocity is a random variable linked to a suitable distribution function $f_e = f_e(V;\cdot)$ conditioned, through some parameters, by the local density, the mean velocity and the local speed variance, and so related, respectively, to the zeroth, first and second-order momenta of the distribution function. This, for a certain lane, means the following:

$$u_e = \int_0^{1+\mu} f_e(V;\cdot)\, dV\,, \tag{3.5}$$

$$v_e(u) = \frac{1}{u}\int_0^{1+\mu} V f_e(V;\cdot)\, dV \tag{3.6}$$

and

$$\sigma_e(u) = \frac{1}{u}\int_0^{1+\mu} [V - v_e]^2 f_e(V;\cdot)\, dV\,. \tag{3.7}$$

Of course the knowledge of the momenta depends on the effective possibility of their measurement. Other alternatives are possible. For instance, to keep low the number of parameters, the knowledge of the mean velocity as a function of the density may be replaced by the measurement of the most probable velocity, say $v_p = v_p(u)$, which is reached by vehicles in a certain road. This possibly simplifies the measurement as one has to verify the accumulation of velocities rather than measure a mean value by averaging processes. Moreover, f_e should satisfy the following properties:

(i) f_e tends to a delta function over $V = 0$ when the density u tends to the maximum admissible value $u = 1$. Indeed, for complete jam conditions all vehicles are obliged to stop.
(ii) f_e decays rapidly to zero for velocities larger than $V_\ell = 1+\mu$. In other words, the probability of finding a vehicle with a velocity larger than V_ℓ must be negligible or possibly equal to zero.

A parametric identification of the equilibrium distribution is proposed in Ref. 15. This means to guess a certain form, characterized by a small number of parameters, of the distribution function, and then to identify the above parameters by exploiting the knowledge of the momenta or of the most probable value. In particular, similarly to the classical kinetic theory of gases, an exponential decay of the energy can be assumed

$$f_e(V;u,\gamma) = \nu(u;\gamma)\frac{1}{v_p(u)}V^\gamma \exp\left\{-\gamma\frac{V^2}{2v_p^2(u)}\right\}, \quad \gamma > 2\,, \tag{3.8}$$

defined for $V \in [0, 1+\mu]$, while for larger values, $f_e = 0$.

The terms γ and ν can be identified by exploiting experimental information. The simplest situation is when γ is assumed to be constant. The term ν can be determined by zeroth order moment. The parameter γ can be identified through the variance. In more details, the variance σ, which depends on u for various values of the parameter γ, and according to the above phenomenological interpretation, is equal to zero for $u = 1$ and monotonically increases with decreasing density.

It is clear that the above interpretation is not universal for all traffic flow situation. As already mentioned, on highways with effective speed control maintained at low values referred to the performances of the vehicles, one may then expect that the variance tends to zero for u tending to zero.

4. Macroscopic Framework

Macroscopic models of traffic flow can be derived by methods which show some analogy with those used in classical continuum mechanics, e.g. Wilmansky,[66] and in particular in fluid mechanics, see Chorin and Marsden[18] or Truesdell and Rajagopal.[64] This section deals with the description of the fluid dynamical framework for designing specific traffic flow models.

The phenomena of interest in continuum mechanics are macroscopic. One does not look at dynamics of individual molecules constituting the fluid but at the gross behavior of many of them. A **point** or a **particle** in the continuum is a small portion of the whole body, negligible with respect to its size, but very large compared to the molecular length. Indeed, the above framework can be accepted for a dense fluid, but it has to be considered only a crude approximation of physic reality for traffic flow. Then, simply looking for an approximation of real traffic flow we assume from now on that we are dealing with a fluid, the physical state will be described by properties of the fluid particles and not by the physical state of all the microscopic molecules. The macroscopic fields describing the state, as, for instance, the density field $u(x,t)$, the velocity field $v(x,t)$, the energy field $e(x,t)$ can be physically interpreted by means of averages of suitable microscopic quantities as we have seen in Sec. 2.

Leaving to Sec. 7 the analysis of the link between the microscopic and the macroscopic descriptions, we assume here that the significant macroscopic dependent variables, such as local density u, velocity v and energy density e, are (sufficiently regular) functions of $n+1$ independent variables x, t ($n = 1, 2, 3$ being the dimension of the physical space where the phenomena occur). In the sequel, we shall consider only one-dimensional space variables, for $n = 1$. A key point in the theory of continuum mechanics is that these fields must obey to suitable *balance laws* that, in their general setting, assume the form

$$
\begin{cases}
\dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}(uv) = 0\,, \\[2mm]
\dfrac{\partial v}{\partial t} + v\dfrac{\partial v}{\partial x} = f[u,v,e]\,, \\[2mm]
\dfrac{\partial e}{\partial t} + v\dfrac{\partial e}{\partial x} = g[u,v,e]\,,
\end{cases}
\tag{4.1}
$$

where f defines the acceleration referred to each particle and g is an energy production term. Square brackets are used to indicate that the model of f and g may be a functional of the arguments. In practice it may not be simply a function of the variables, but also of their first order derivatives. Specifically, $(4.1)_1$ mathematically translates the conservation of mass, $(4.1)_2$ the conservation of linear momentum and $(4.1)_3$ that of the energy density. It is worth to notice, using the language of continuum mechanics, that the domain in which the system (4.1) is set is of **spatial** (Eulerian) nature. The above equations refer to dimensionless variables and, in particular, time has been related to $T = \ell/v_M$.

Remark 4.1. The word **acceleration** is used, when dealing with traffic flow models, to avoid the use of the term **force** for a system where the mass cannot be properly defined.

Remark 4.2. From the mathematical point of view, (4.1) is a system of partial differential equations, which has to be implemented with suitable initial and boundary data. The data measurement, of course, is affected by errors and fluctuations. Analogously, the measurement of the **forcing** terms f and g is subject to errors. For this reason, it is important to analyze the sensitivity of the solution to the mathematical problem with respect to perturbations in the data and in the production terms. In practice, the measurement of the data is relevant in the same definition of the variables of interest. For example, in the traffic flow modelling, a typical method to define the local density u in a given control point is to measure the flux at two points, upstream and downstream with respect to the control point, and evaluate the number of vehicles in the zone. A way to overcome the problem of sensitivity of the above procedure to errors and fluctuations is to derive an appropriate evolution equation for the flux. We shall analyze this aspect in the next section.

In order to have solvability of system (4.1) (at least in principle), it is necessary to specify how the production terms f and g depend on their argument. These functional dependencies, in analogy with continuum mechanics, may be called **constitutive assumptions**. Different constitutive assumptions lead to different models, describing different physical situations. It comes out that, according to the specific constitutive assumption, different models can be derived that involve only some of the three equations in (4.1). In the same way, the traffic flow models that can be designed in the macroscopic framework can be classified as follows:

- **Scalar models** which are obtained by mass conservation equation only with a closure $v = v_e[u]$ obtained by a phenomenological model describing the driver's psycho-mechanic adjustment of the velocity to a suitable local velocity v_e which, in the absence of space gradients, corresponds to the one given by the velocity representation of steady flow we have seen in Sec. 3.
- **Models with an acceleration** which are obtained by mass and linear momentum conservation equations with the addition of a phenomenological relation describing the psycho-mechanic action $f = f[u, v]$ on the vehicles.
- **Higher order models** which also include the additional energy conservation equation, with a suitably defined energy density e.

In the next section we shall describe in some details specific traffic flow models according to the above classifications. Here we present the general fluid mechanics framework leading to the different type of models.

We start to make some remarks on Eq. $(4.1)_1$. The equation of mass conservation is of hyperbolic nature, with (real) characteristics corresponding to particle trajectories. This means, in particular, that it is able to propagate a discontinuous initial data (a shock wave). Such a fact has to be taken in account when building realistic scalar traffic flow models.

Models with acceleration, that make use of Eqs. $(4.1)_{1,2}$, seem to be more related to a correct fluid dynamics framework, where the introduction of higher order relations typically leads to a more accurate description of the shocks' structure. As we mentioned, the choice of the acceleration term $f[u, v]$ specifies the range of validity of the fluid dynamical model. For instance, if one assumes (in dimensionless variables)

$$f[u, v] \equiv f[u] = -\frac{1}{u}\frac{\partial p}{\partial x}\,, \tag{4.2}$$

then, the following

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}(uv) = 0\,, \\ \dfrac{\partial v}{\partial t} + v\dfrac{\partial v}{\partial x} = -\dfrac{1}{u}\dfrac{\partial p}{\partial x}\,, \end{cases} \tag{4.3}$$

is obtained, where p is the pressure.

In order for system (4.3) to be closed it is necessary to specify the pressure as a function of the density. A possible relation is the equation of ideal gases: $p = cu$, where c is a constant (in isothermal conditions). Equation (4.3) represents the equations of motion of an inviscid, compressible fluid in one space dimension. If, instead of (4.2), we assume the following:

$$f[u, v] = -\frac{1}{u}\frac{\partial p}{\partial x} + \frac{2}{3uR_e}\frac{\partial^2 v}{\partial x^2}\,, \tag{4.4}$$

we have

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}(uv) = 0\,, \\ \dfrac{\partial v}{\partial t} + v\dfrac{\partial v}{\partial x} = -\dfrac{1}{u}\dfrac{\partial p}{\partial x} + \dfrac{2}{3uR_e}\dfrac{\partial^2 v}{\partial x^2}\,. \end{cases} \tag{4.5}$$

Here $R_e = V_0 L u_0/\mu$ with V_0, L and u_0 reference speed, length and density respectively, and $\mu > 0$ a material parameter called viscosity. R_e is a positive constant called Reynolds number, that gives a dimensionless measure of the (inverse of) viscosity. Equation (4.5) is the equation of motion of a viscous, compressible fluid in one spatial dimension. As is obvious, the presence or the absence of the viscous term leads to a change in the mathematical structure of the equations, with great consequences in the properties of the models'. Changing the relation (4.2) leads to different models with acceleration (also called **second-order models**), rather strictly related with hydrodynamic models.

Higher order models, that make use of the whole set of Eqs. (4.1), remind the mathematical description of a compressible, thermoconductive fluid. They require the specification of a constitutive relation both for the acceleration $f[u, v, e]$ and for the "energy" production $g[u, v, e]$. Here we only mention the difficulty one has to face in defining an energy for a traffic flow, while we stress that such models, introducing more parameters to be identified, are hard to be handed and compared with the experimental observation. In the next section we shall see in which way, and to which extent, a fluid dynamics framework can be employed to build traffic flow models.

As already mentioned all the above examples have to be accepted as a tutorial introduction, which may show contradictions related to the flow if vehicles is not an ideal (nor a real) gas. Specific models have to be designed after a detailed analysis and interpretation of effective flow conditions obtained by experimental data.

5. Hydrodynamic Models

As already mentioned, the framework described above can be exploited to design specific models. Before dealing with the review of the models proposed in the literature, it is worth discussing the intrinsic limits of hydrodynamic models, that is the main points where a modellization of vehicular traffic differs with respect to a real fluid one (cf. Refs. 20 and 27).

First, for any order model, it is important to remark that a vehicle is not a particle but rather a system linking driver and mechanics, so that one has to take into account the reaction of the driver, who may be aggressive, timid, prompt etc. independently of the motion. Actually, this fundamental criticism also applies to kinetic type models.

Then, when analyzing the properties of specific models, it has to be recognized that if they are able to describe the anisotropic nature of traffic flow, i.e. a car mostly responds to frontal stimuli, whereas a fluid particle responds to stimuli from the front and from behind.

Again, if a particular model allows for shock discontinuities (as in scalar models of Lighthill and Whitham[49]), it is worthwhile to consider that the width of a traffic shock only involves a few vehicles, while a shock in a fluid flow encompasses thousands of particles.

Finally, it has to be considered a rather serious task of parameters identification. When building models in analogy with fluid dynamics one encounters the problem of how to identify the analogous of fluid material parameters, such as viscosity, and how to measure them. In this respect, the validity of second or higher order models, which are introduced to overcome some of the above quoted difficulties, should not be overemphasized, due to the fact that increasing the number of equations increases the number of parameters to be identified.

Bearing the above criticisms in mind, we start describing and critically analyzing various models suggested in the literature. Models will be described for one lane traffic flow conditions. However, all of them can be generalized to a multilane description such that the vehicles are allowed to move laterally from one lane to the other.

5.1. *Scalar models*

Scalar models are obtained by mass conservation only. Considering that this equation involves both u and v, a self-consistent model can be obtained if a phenomenological relation can be proposed to link v to u and its derivatives. Models available in the literature are developed assuming that the driver adapts instantaneously the velocity of the vehicle to a local equilibrium velocity which can be experimentally observed in the absence of gradients and depends on the local density u. As we have seen, experiments are represented by the **fundamental diagram**. Analytic expressions of the equilibrium velocity have been discussed in Sec. 3. The simple case is defined by Eq. (3.1) with $a = b = 0$. In this case, one simply has

$$v_e = (1 - u)\,. \tag{5.1}$$

In order to avoid heavy notations, all calculations will be developed, in what follows using the above relation (5.1) and leaving to the interested reader technical improvements. However, the reader should be aware, after Sec. 3, that different expressions have to be used in order to take into account the experimental information, so that technical calculations may result to be heavier than those reported in what follows. Some models will be reviewed and critically analyzed.

Model S.1. The evolution model for the dimensionless density u is obtained by mass conservation linked to a phenomenological relation of the type $v = v_e(u)$, where v_e is an equilibrium velocity, function of the local density u, reached instantaneously by the driver (cf. Ref. 60). The formal structure of the model is

$$\frac{\partial u}{\partial t} + \left[v_e(u) + u\frac{\partial v_e}{\partial u}(u)\right]\frac{\partial u}{\partial x} = 0\,, \tag{5.2}$$

which using (5.1) for the expression of the equilibrium velocity gives the following model:

$$\frac{\partial u}{\partial t} + (1-2u)\frac{\partial u}{\partial x} = 0\,. \tag{5.3}$$

The above model generates, when u reaches the value $1/2$, unrealistic shockwave description. Therefore Lightill, Witham and Payne[49] suggested the addition of a linear diffusion term. In analogy with the usual shocks smoothing procedure of fluid dynamics, this is obtained assuming that v_e depends not only on u but also on its spatial derivative $\partial u/\partial x$ as in the relation

$$v_e = 1 - u - \frac{\varepsilon}{u}\frac{\partial u}{\partial x}\,, \tag{5.4}$$

where ε plays the role of a viscosity. The model is as follows:

Model S.2. The evolution model for the dimensionless density u is obtained by mass conservation linked to a phenomenological relation $v = v_e(u)$ of type (5.4) yielding, on the right-hand side of the equation, a linear diffusion term:

$$\frac{\partial u}{\partial t} + (1-2u)\frac{\partial u}{\partial x} = \varepsilon\frac{\partial^2 u}{\partial x^2}\,. \tag{5.5}$$

A fundamental criticism relative to Eq. (5.5) concerns its inability to capture the anisotropic nature of traffic flow. In particular, Eq. (5.5) predicts negative velocities when considering the time evolution of the rear of a stopped queue without any arriving traffic. A recent paper by De Angelis[23] remarks that linear diffusion is not a realistic assumption considering that the variable u takes values from vacuum conditions $u = 0$ to totally filled roads $u = 1$. A nonlinear diffusion term is then necessary in order to take into account that diffusion cannot occur in the above two conditions. In the spirit of fluid dynamics, this can be done assuming, instead of (5.4), a different form of the closure relation between speed and density, such as

$$v_e = 1 - u - \varepsilon\frac{k(u)}{u}\frac{\partial u}{\partial x}\,, \tag{5.6}$$

where $k(u)$ has to be suitably chosen in such a way that $k(u) = 0$ when $u = 0$ and $u = 1$. De Angelis' model reads as follows:

Model S.3. The evolution model for the dimensionless density u is obtained by mass conservation linked to a phenomenological relation $v = v_e(u)$ of type (5.6) with $k(u) = u(1-u)$ yielding, on the right-hand side of the equation, a nonlinear diffusion term

$$\frac{\partial u}{\partial t} + (1-2u)\frac{\partial u}{\partial x} = \varepsilon u(1-u)\frac{\partial^2 u}{\partial x^2} + \varepsilon(1-2u)\left(\frac{\partial u}{\partial x}\right)^2\,. \tag{5.7}$$

The paper by De Angelis,[23] technically modified by Bonzani,[13] introduces the interesting concept of **apparent local density** related to the fact that the driver does not measure exactly the local density, but simply feels it. Specifically, the driver feels a density u^* which is larger than the real one if the local density gradient is positive (trend to jam conditions), while it is smaller than the real one if the gradient is negative (trend to vacuum). In addition, the above multiplicative effect increases with decreasing density. A conceivable expression of the apparent density is the following:

$$u^* = u \left[1 + \eta(1-u)\frac{\partial u}{\partial x} \right], \tag{5.8}$$

where η is a positive parameter. This means that the equilibrium velocity which is felt by the driver, again using (5.1), is given by

$$v_e[u] = 1 - u^* = (1-u)\left[1 - \eta u \frac{\partial u}{\partial x} \right]. \tag{5.9}$$

The solution of evolution problems must be subject to constraints (which may be used to select admissible values of the parameter η)

$$u^* \geq 0 \Rightarrow 1 + \eta(1-u)\frac{\partial u}{\partial x} \geq 0\,, \quad \text{and} \quad u^* \leq 1 \Rightarrow u\left[1 + \eta(1-u)\frac{\partial u}{\partial x} \right] \leq 1\,, \tag{5.10}$$

which yields the following constraints for η:

$$\eta \geq \left((u-1)\frac{\partial u}{\partial x} \right)^{-1}. \tag{5.11}$$

The use of the driver velocity model leads to evolution models characterized by a parabolic term smoothing the solution which is similar to the artificial diffusion term. The assumption that the term η constant may be reasonably accepted far from tollgates. The model is as follows:

Model S.4. The evolution model for the dimensionless density u is obtained by mass conservation linked to a phenomenological relation of the type $v = v_e[u]$, such that the driver adapts instantaneously the velocity of the vehicle to an equilibrium velocity as is described by Eq. (5.9). Technical calculations yield

$$\frac{\partial u}{\partial t} + (1-2u)\frac{\partial u}{\partial x} = \eta u^2(1-u)\frac{\partial^2 u}{\partial x^2} + \eta u(2-3u)\left(\frac{\partial u}{\partial x}\right)^2. \tag{5.12}$$

In order to critically analyze the above models, it is useful reporting some observations proposed by Bonzani, where the equivalence is analyzed among Models S.2, S.3 and S.4. Specifically, the following equivalence have been shown in Ref. 15

$$k = 1 \Leftrightarrow u^* = u\left(1 + \varepsilon \frac{1}{u^2}\frac{\partial u}{\partial x} \right) \tag{5.13}$$

and

$$k = u^2(1-u) \Leftrightarrow u^* = u\left(1 + \varepsilon(1-u)\frac{\partial u}{\partial x} \right). \tag{5.14}$$

An immediate consequence is that the model with linear diffusion has to be regarded unrealistic. In fact the comparison shows that the driver may feel, when u is close to zero, unrealistically high density in the presence of small gradients. Therefore, Eq. (5.13) shows that a relatively more consistent model is obtained in the case of nonlinear diffusion.

All the above models use the density as dependent variable. On the other hand, Marasco[53] observed that the expression of the equilibrium flow, which is given with reference to Eq. (5.9), given by

$$q_e[u] = u(1-u)\left[1 - \eta u \frac{\partial u}{\partial x}\right], \tag{5.15}$$

can be used, jointly to Eq. (5.12), to obtain the following equivalent statement of Model S.4, which also uses the flow as dependent variable:

Model S.5. The evolution model for the dimensionless density u and flow q is obtained by mass conservation linked to a phenomenological relation of the type $q = q_e[u]$, such that the driver adapts instantaneously the flow of the vehicle to an equilibrium flow as it is described by Eq. (5.15). The model is the following:

$$\begin{cases} \dfrac{\partial u}{\partial t} = -\dfrac{\partial q}{\partial x}, \\ \dfrac{\partial q}{\partial t} = \left[(1-2u) + \eta u(2-3u)\dfrac{\partial q}{\partial x}\right]\dfrac{\partial q}{\partial x} + \eta u(1-u)\dfrac{\partial^2 q}{\partial x^2}. \end{cases} \tag{5.16}$$

The above model can be used for the applications exploiting the fact that flow measurements at the boundary are technically more reliable that density measurements. Specifically, when dealing with network of roads, it is natural to use boundary conditions for the flux rather than for the density.

One of the problems related to all the above models which use the concept of apparent density is the identification of the term η, which may be obtained by experiments observing the behaviors of drivers in real flow conditions. An interesting analysis is developed in Ref. 14, where it is observed that if η spans in the domain $[0.1, 0.2]$ the solution to initial–boundary value problems for large time intervals and over the whole length of the road minimize a suitable quadratic functional. This functional is related to the minimization of acceleration and braking behaviors and to maximization of the mean velocity. Indeed, it is shown that an ideal driver should operate within the above domain. Additional analysis based on experimental data in steady non-uniform flow confirms the above identification.

5.2. *Models with acceleration*

Models with an acceleration can be obtained in the framework of conservation equations for mass and linear momentum, even though some of them are usually not written in a conservation law form. All models belonging to the above class need a suitable phenomenological expression of the resulting acceleration of a vehicle due to the action of the surrounding vehicles.

Model A.1. The acceleration of a vehicle consists of two contributions: the first corresponds to a trend to equilibrium and the second to the action of the density gradient

$$f[u,v] = f_1[u,v] + f_2[u,v] = c(v_e(u) - v) - \frac{1}{u}\frac{\partial p}{\partial x} = c(v_e(u) - v) - \frac{p'(u)}{u}\frac{\partial u}{\partial x}\,, \tag{5.17}$$

where c is a constant representing the inverse of the relaxation time of v toward the equilibrium velocity v_e, which is a given function of the local density. The use of $v_e = 1 - u$ generates the following model:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}(vu) = 0\,, \\ \dfrac{\partial v}{\partial t} + v\dfrac{\partial v}{\partial x} = c[(1-u) - v] - \dfrac{p'(u)}{u}\dfrac{\partial u}{\partial x}\,. \end{cases} \tag{5.18}$$

The last term on right-hand side of $(5.18)_2$, called **anticipation term**, takes into account the awareness of the driver for the traffic conditions ahead. It is worth to observe the analogy of (5.17) with (4.2), i.e. of (5.18) with (4.3). Model A.1 corresponds to the motion of an inviscid, compressible fluid where a relaxation to equilibrium speed is explicitly introduced in the constitutive relation. The **pressure** $p(u)$ has to be prescribed as a function of the density u. A possible choice (Kerner & Konhäuser[34]), is $p(u) = \bar{c}u$, with $\bar{c}$ a positive constant. Exactly as in fluid dynamics, the above model generates, similarly to Model S.1, undesired shock waves so that Payne and Whitham,[60] suggested the addition of a linear velocity diffusion term, corresponding to a viscous dissipation in the fluid dynamical framework. The model is as follows:

Model A.2. The acceleration of a vehicle is given by three contributions: the first corresponds to a trend to equilibrium, the second to the action of the density gradient as in Model A.1, and the third to a dissipative velocity diffusion

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}(vu) = 0\,, \\ \dfrac{\partial v}{\partial t} + v\dfrac{\partial v}{\partial x} = c(v_e - v) - \dfrac{p'(u)}{u}\dfrac{\partial u}{\partial x} + \dfrac{\varepsilon}{u}\dfrac{\partial^2 v}{\partial x^2}\,. \end{cases} \tag{5.19}$$

Fundamental criticisms to Model A.2 are contained in the paper by Daganzo.[20] In particular, it does not satisfy the basic requirement that a vehicle behaves anisotropically, responding mainly to frontal stimuli. A simple way to recognize this fact is to observe that (5.19) is a second-order, nonlinear hyperbolic system, whose characteristic speeds are $v \pm \sqrt{p'(u)}$. Therefore, there is information that may travel faster than the speed v of vehicles. To clarify the situation, consider a car travelling with velocity v, whose driver sees in front of him/her a higher traffic density (density increasing with respect to x) moving faster than v (density decreasing with respect to $x - vt$). In this condition, Model A.1 predicts that the driver would brake, while any reasonable driver would accelerate since he/she sees a frontal denser traffic travelling faster than him/her.

It is worthwhile to observe that Model A.2 shows similar flaws even in the simpler case of no relaxation and no diffusion, corresponding to Eq. (5.19) with $c = \varepsilon = 0$. In this case, in fact, the well-known Riemann problem, that is Eq. (5.19) with the above prescriptions and initial data

$$v_0 = 0\,, \quad u_0 = \begin{cases} U_- & x < 0 \\ U_+ & x > 0 \end{cases}$$

will produce solutions with negative speed, and this paradox is not fixed adding to the equations relaxation and/or diffusion terms (at least for a certain time interval).

A possibility to overcome these undesirable aspects of second-order models has been suggested by Aw and Rascle,[4] at least in the case when $c = \varepsilon = 0$. They start assuming a number of condition needed in order for the model to be meaningful, among which we recall the following:

(i) the model equations must form a hyperbolic system;
(ii) in any solution corresponding to arbitrary, bounded, non-negative Riemann data, the density u and the velocity v must remain bounded and non-negative;
(iii) no information (i.e. no wave) can travel faster than v.

They heuristically suggest the following:

Model A.3. The acceleration of a vehicle is given by

$$f[u, v] = up'(u)\frac{\partial v}{\partial x}\,, \tag{5.20}$$

so that the equations of the model reads

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}(vu) = 0\,, \\ \dfrac{\partial v}{\partial t} + v\dfrac{\partial v}{\partial x} = up'(u)\dfrac{\partial v}{\partial x}\,. \end{cases} \tag{5.21}$$

The **pressure** term is a smooth, increasing function of the density, satisfying the sole requirements

$$p(u) \sim u^\gamma \quad \text{close to } u = 0 \quad \text{and} \quad (up(u))'' > 0\,. \tag{5.22}$$

It is interesting to observe, see Colombo,[19] that, by setting $U \equiv (u, v)$, system (5.21) can be rewritten as follows:

$$\frac{\partial U}{\partial t} + A(U)\frac{\partial U}{\partial x} = 0\,, \tag{5.23}$$

where the matrix

$$A(U) = \begin{pmatrix} v & u \\ 0 & v - up'(u) \end{pmatrix}$$

admits the eigenvalues:

$$\lambda_1 = v - up'(u) \leq \lambda_2 = v\,. \tag{5.24}$$

This means that (5.21) is strictly hyperbolic except at vacuum conditions, where $\lambda_1 = \lambda_2$. In addition, due to the nature of the eigenvalues (5.24) (λ_1 is genuinely nonlinear, because of assumptions (5.22), while λ_2 is linearly degenerate), Model A.3 admits either shock waves (or rarefactions), or contact discontinuities. The latter correspond to the larger, linearly degenerate eigenvalue λ_2, for their speed is always equal to $\lambda_2 = v$, that is, *all waves travel at most with speed* v. Finally, Aw & Rascle[4] show that their model also satisfies point (ii) above.

A drawback persists, however, in Model A.3. In fact, it is possible to show that the maximal speed the vehicles can reach on an empty road *depends on the initial data*, which is clearly wrong. Before looking at the possible ways to fix this problem, let us observe that Model A.3 can be written in a conservative form as

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}(vu) = 0\,, \\[2mm] \dfrac{\partial q}{\partial t} + \dfrac{\partial}{\partial x}(vq) = 0\,, \end{cases} \tag{5.25}$$

where $q = vu + up(u)$ plays the role of the momentum as in fluid dynamics. On this, Colombo[19] introduces the following:

Model A.4. The evolution equations for the model are given by

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}(vu) = 0\,, \\[2mm] \dfrac{\partial q}{\partial t} + \dfrac{\partial}{\partial x}(v(q - q_*)) = 0\,. \end{cases} \tag{5.26}$$

In Model A.4, the density $u \in [0, u_M]$ and

$$v = \left(\frac{1}{u} - \frac{1}{u_M}\right) q\,, \tag{5.27}$$

where the maximal (jam) density u_M and q_* are characteristic of the road.[a] The main improvements of Model A.4 are summarized as follows:

(i) whenever a queue is formed, the maximal density is reached;
(ii) below u_M, no car may stop;
(iii) at high car density and speed, braking produces shock waves while accelerating produces rarefaction waves. The opposite occurs at low car density and speed;
(iv) a maximal speed v_M exists, characteristic of the road, as an outcome of the model.

[a] Here we leave the dimensional form for the maximum speed and density since we wish to stress their relevance in the present model.

It is worthwhile to observe that, in both Models A.3 and A.4, from the mathematical point of view the well-posedness is lost close to vacuum conditions, when $u = 0$. In particular, in that condition the solution does not depend continuously on the data. This fact seems to be quite reasonable, for instabilities might appear at very low car density where, among other things, a deterministic relation between car density and car speed, such as (5.27), reasonably fails to hold. For this reason, Colombo introduces the concept of *admissible initial data* as those data which are of bounded variation and far from vacuum, and finds that the initial value problem for Model A.4 is globally well posed, i.e. admits a unique solution that depends continuously on the data. In a recent paper, Li[50] furnishes a L^1 well-posedness theory (for initial data $u_0 > 0$) for a model with anticipation and relaxation due to Zhang,[67] which is equivalent to (5.18) with a pressure dependent on the equilibrium velocity v_e; $p(u) = (uv_e(u))^2$.

In particular, he shows the convergence of solutions to and the uniqueness of the zero relaxation limit ($c = 0$) in the L^1 topology.

The above models are based on the assumption that the driver feels that effective velocity and not an artificial one as for the case of the scalar models. Again following De Angelis[23] and Bellomo *et al.*,[9] the following model can be proposed:

Model A.5. The acceleration of a vehicle is given by two contributions: the first corresponds to a trend to equilibrium and the second to the action of the density gradient as in Model A.1.

$$f[u,v] = f_1[u,v] + f_2[u,v] = c_1(v_e - v) - \frac{c_2}{u}\frac{\partial u}{\partial x}\,, \tag{5.28}$$

where c_1 and c_2 are suitable constants. However, the driver feels an apparent local density $u_*[u,v]$ which is larger than the real one if the local density and velocity gradients are positive, while it is smaller than the real one if the gradient is negative (trend to vacuum). In addition, the above effect increases with decreasing density

$$u_*[u,v] = u\left[1 + \eta_1(1-u)\frac{\partial u}{\partial x} + \eta_2(1-u)\frac{\partial v}{\partial x}\right], \tag{5.29}$$

where η_1 and η_2 are positive parameters. The model is such that the equilibrium velocity is now

$$v_e[u,v] = 1 - u_*[u,v] = 1 - u - \eta_1 u(1-u)\frac{\partial u}{\partial x} + \eta_2 u(1-u)\frac{\partial v}{\partial x}\,. \tag{5.30}$$

The expression of the acceleration is then obtained as follows:

$$\begin{aligned} f &= c_1\left[1 - u - \eta_1 u(1-u)\frac{\partial u}{\partial x} + \eta_2 u(1-u)\frac{\partial v}{\partial x} - v\right] \\ &\quad - c_2\left[u + \eta_1 u(1-u)\frac{\partial u}{\partial x} + \eta_2 u(1-u)\frac{\partial v}{\partial x}\right]^{-1}\left\{\frac{\partial u}{\partial x} - \eta_1(1-2u)\left(\frac{\partial u}{\partial x}\right)^2\right. \\ &\quad \left. - \eta_1 u(1-u)\frac{\partial^2 u}{\partial x^2} - \eta_2(1-2u)\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} - \eta_2 u(1-u)\frac{\partial^2 v}{\partial x^2}\right\}. \end{aligned} \tag{5.31}$$

Relatively simpler expressions are obtained neglecting the velocity gradients, i.e. for $\eta_2 = 0$ and $\eta_1 = \eta$. In this case one obtains

$$f = c_1 \left[1 - u - \eta u(1-u)\frac{\partial u}{\partial x} - v\right] - c_2 \left[u + \eta u(1-u)\frac{\partial u}{\partial x}\right]^{-1}$$
$$\times \left\{\frac{\partial u}{\partial x} - \eta(1-2u)\left(\frac{\partial u}{\partial x}\right)^2 - \eta u(1-u)\frac{\partial^2 u}{\partial x^2}\right\}. \tag{5.32}$$

The model proposed by Aw and Rascle[4] needs, as we have seen, a different phenomenological law. However, also in this case, one may refer the equivalent pressure to the local apparent density. For instance, one can use $p = cu_*$ instead of $p = cu$.

It is also clear that in this case the artificial diffusion term (which is nonlinear when ϵ is taken as a function of u) induces, similarly to the case of scalar models, some energy dissipation which are not physical. On the other hand, one may use ideas similar to those of Sec. 3 to model the above term using u_* in the expression of f. Using (5.32), simple calculations lead to the following model:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}(vu) = 0\,, \\[2mm] \dfrac{\partial v}{\partial t} + v\dfrac{\partial v}{\partial x} = c_1\left[1 - u - \eta_1 u(1-u)\dfrac{\partial u}{\partial x} - v\right] - c_2\left[u + \eta_1 u(1-u)\dfrac{\partial u}{\partial x}\right]^{-1} \\[2mm] \qquad\qquad \times \left\{\dfrac{\partial u}{\partial x} + c_2\eta_1(1-2u)\left(\dfrac{\partial u}{\partial x}\right)^2 - c_2\eta_1 u(1-u)\dfrac{\partial^2 u}{\partial x^2}\right\}. \end{cases} \tag{5.33}$$

5.3. *Higher order models*

In this subsection we give some notes on a higher-order model due to Helbing.[30] In this model, the whole set of Eqs. (4.1) is considered, namely:

Model A.6. The acceleration $f[u, v, e]$ and the **energy** production $g[u, v, e]$ of a vehicle are given by three contributions each: the first comes from a trend to an equilibrium, the second depends on the density and velocity gradients respectively, and the third expresses (linear) diffusion

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}(vu) = 0\,, \\[2mm] \dfrac{\partial v}{\partial t} + v\dfrac{\partial v}{\partial x} = c_1(v_e - v) - \dfrac{1}{u}\dfrac{\partial p}{\partial x} + \dfrac{c_2}{u}\dfrac{\partial^2 \Theta}{\partial x^2}\,, \\[2mm] \dfrac{\partial \Theta}{\partial t} + v\dfrac{\partial \Theta}{\partial x} = -\dfrac{2p}{u}\dfrac{\partial v}{\partial x} + 2c_1(\Theta_e - \Theta) + \dfrac{c_3}{u}\dfrac{\partial^2 \Theta}{\partial x^2}\,. \end{cases} \tag{5.34}$$

As usual in the fluid dynamical framework, Eqs. (5.34) have to be implemented with an **equation of state** relating the **pressure** p, the density u and the **temperature** Θ, such as: $p = u\Theta$. Moreover, v_e and Θ_e represent the equilibrium speed and temperature, and they must be prescribed as functions of u; finally, c_1, c_2 and c_3 are constant parameters.

We have so far used the term **temperature** for the additional variable Θ in order to render the similarity of Model A.6 with the non-isothermal equations of motion of a viscous fluid. Actually, it has to be interpreted as vehicular velocity variance on the basis of kinetic considerations.

Helbing's model, though able to overcome some of the drawbacks of second-order models of Aw and Rascle's type such as the possible development of car densities higher than the maximal density (postulated in that type of models), on the other hand, involves a number of constant and function parameters, whose identification might be a problem harder that those the model tries to fix. We believe that the previously described method of apparent density could be used to improve the model.

6. Mathematical Frameworks for Kinetic Modelling

Kinetic modelling is an alternative to fluid dynamical modelling. It consists of deriving an evolution equation for the statistical distribution over the microscopic state of vehicles on the road. The derivation follows lines similar to those of the kinetic theory of gases, e.g. Cercignani, Illner, and Pulvirenti,[17] or Bellomo,[5] and leads to an integro-differential equation similar to the Boltzmann equation.

The derivation of the model is based on the modelling of the microscopic interactions among vehicles (or between pairs of vehicles) and uses suitable balance relations, in the phase-space volume $dxdV$, by equating the total derivative of the distribution function to the inlet minus the outlet of vehicles in the said volume. According to the general gas kinetic theory, it is assumed that the factorization of the joint probability related to the two vehicles. As in kinetic theory, different ways of modelling local interactions generate different types of evolution equations.

Models which are available in the literature have occasionally been derived by heuristic arguments. Therefore the development of proper methodology is necessary. This section provides, symmetrically to Sec. 4, a description of some conceivable frameworks which can be used toward modelling. In particular, the following ones will be concisely described:

(i) phenomenological kinetic models,
(ii) models based on local binary interactions,
(iii) models based on averaged binary interactions,
(iv) mean field models.

Mathematical models may be derived within the above frameworks by combinations of different of them. Actually not all of them have been exploited, therefore the description given in this section provides a background toward modelling which is broader than the one available in the literature. The various frameworks, which will still be described in the sequel, should support, in the author's opinion, the development of new models. So far this section has to be regarded as a methodological one: the description and critical analysis of some specific models will follow in the next section.

Some models proposed in the literature are briefly reviewed and critically analyzed. The framework which follows is proposed in Ref. 25 with the aim of designing an environment toward the development of models derived according to suitable generalizations of the Boltzmann and Vlasov equations. The interest in this type of modellization is documented in the collection of surveys edited in Refs. 7 and 11.

6.1. *Boltzmann phenomenological models*

Boltzmann phenomenological models are designed according to the following structure:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + V\frac{\partial f}{\partial x} + F(t,x)\frac{\partial f}{\partial V} = Q[f;u]\,, \tag{6.1}$$

where $f = f(t,x,V)$ is the distribution function of the **test** vehicle, F is the acceleration applied to it by the outer environment, supposed to be independent of velocity, and Q, which is derived by phenomenological arguments, depends on f and on local gross quantities, generally on the vehicular density.

A simple way to model the term Q consists of describing a trend to equilibrium analogous to the BGK model (see Cercignani[17]) in kinetic theory

$$Q = Q(f;u) = c_r(u)(f_e(V;\cdot) - f(t,x,V))\,, \tag{6.2}$$

where the rate of convergence c_r and the equilibrium distribution function $f_e(V;\cdot)$ may be assumed to depend on the velocity and on some macroscopic quantities, e.g. the local density.

Generally traffic flow models assume $F = 0$. However, an acceleration term may be imposed by the outer environment, e.g. signaling to accelerate or decelerate. Moreover, if the modelling assumes that F depends on f, say $F = F[f]$, then the formal structure of the model modifies as follows:

$$\frac{\partial f}{\partial t} + V\frac{\partial f}{\partial x} + \frac{\partial(fF[f])}{\partial V} = Q(f;u)\,. \tag{6.3}$$

The frameworks proposed in Ref. 25, suppose that F may be induced by the surrounding vehicles and depend on the density u^* felt by the driver rather than the real one.

6.2. *Boltzmann models with binary interactions*

Boltzmann-like models with binary interaction are found on suitable microscopic modelling which assumes binary interaction between the **test** and the **field** vehicles. Interactions are localized either in the point x of the field vehicle or at a fixed distance on its front.

For both types of interactions, the formal structure of the evolution equation writes as follows:

$$\frac{\partial f}{\partial t} + V\frac{\partial f}{\partial x} + F(t,x)\frac{\partial f}{\partial V} = J[f] = G[f] - L[f]\,, \tag{6.4}$$

where G and L represent the inflow (**gain**) and outflow (**loss**) of vehicles in the control volume of the phase space, generally integral operators on f.

If interactions are localized at the point x of both field and test vehicles, then the following formal structure can be proposed:

$$G[f] = \int_0^{1+\mu}\int_0^{1+\mu} C_e(V^*,W^*)A(V^*,W^*;V)f(t,x,V^*)f(t,x,W^*)\,dV^*dW^* \tag{6.5}$$

and

$$L[f] = f(t,x,V)\int_0^{1+\mu} C_e(V,W^*)f(t,x,W^*)dW^*\,, \tag{6.6}$$

where C_e and A model, respectively, the encounter rate between the test vehicle with velocity V and the field vehicle with velocity W (or V^* and W^*); and $A(V^*,W^*;V)$ models the probability density that a vehicle with velocity V^* interacting with a vehicle with velocity W^* ends up in the velocity V. The density A must be equal to zero for $V \geq 1+\mu$.

The gain term G models the rate of increase of the distribution function due to vehicles which are, at time t, in the space position x with velocity V as effect of pair interactions or, if defined, of proper desire. The loss term L models the rate of loss in the distribution function of vehicles in x with velocity V due to transition to another state.

Interactions may be assumed localized at a fixed distance d from the test vehicles, similarly to the Enskog equations.[5] The technical modification of the structure is immediate. For $F = f[f]$, an acceleration term similar to the one reported in Eq. (4.3) has to be applied. Actually, the mathematical structure defined in Eqs. (6.4)–(6.6) is the one proposed in Ref. 2 to deal with a large variety of generalized kinetic models in applied sciences.

6.3. *Boltzmann models with averaged binary interactions*

Modelling **averaged binary interactions** consists of assuming that the driver of the test vehicle in x has a visibility zone $[x, x+c_v]$ on the front, while interactions are weighted, within the above zone, by a suitable weight function $\varphi = \varphi(x,y)$, with $y \in [x, x+c_v]$ such that

$$y\uparrow \Rightarrow \varphi\downarrow\,, \qquad \int_x^{x+c_v} \varphi(x,y)\,dy = 1\,.$$

Then, the structure defined in (6.4) can be used with

$$G[f] = \int_x^{x+c_v}\int_0^{1+\mu}\int_0^{1+\mu} \varphi(x,y)C_e(V^*,W^*)A(V^*,W^*;V)f(t,x,V^*) \times f(t,y,W^*)\,dV^*dW^*dy \tag{6.7}$$

and

$$L[f] = f(t,x,V) \int_x^{x+c_v} \int_0^{1+\mu} \varphi(x,y) C_e(V,W^*) f(t,y,W^*)\, dW^*\, dy\,. \tag{6.8}$$

It is immediate to show that assuming $\varphi(y) = \delta(y-x)$, where δ denotes Dirac's delta function, or $\varphi(y) = \delta(y-x+d)$, Eq. (6.7), gives localized interaction models.

6.4. *Mean field kinetic models*

Mean field models are derived under quite different ideas. Similarly to Vlasov type models, see e.g. Dobrushin,[26] it is defined a mean field action on the **test** vehicles due to the **field** vehicles, possibly more than one. The structure of the evolution equation is as follows:

$$\frac{\partial f}{\partial t} + V\frac{\partial f}{\partial x} + \frac{\partial}{\partial V}(F[f]f) = 0\,, \tag{6.9}$$

where $F[f]$ is the acceleration, depending on the field of the surrounding vehicles felt by the test vehicle. The mean field description gives $F[f]$ by means of a suitable interaction potential which generates the action

$$F[f](t,x,V) = \int_x^{x+c_v} \int_0^{1+\mu} \int_0^{1+\mu} C_e(V,W^*) \mathcal{F}(x,y,V,W^*) f(t,y,W^*)\, dy\, dW^*\,, \tag{6.10}$$

where $\mathcal{F}$ is the positional acceleration applied by the vehicle in y with velocity W^* to the one in x with velocity V (the weight function $\varphi(x,y)$ is already included in the definition of the positional acceleration).

7. Kinetic Models

This section, following the same style of presentation of the hydrodynamic description proposed in Secs. 4 and 5, deals with a description of some of the models which can be recovered in the literature. Completeness is not claimed. On the other hand, it may be interesting to see how various models can be inserted into the above frameworks, which can also be possibly exploited toward the derivation of new models. This methodological approach[25] has been developed to improve the previous derivation methods by a critical analysis of the correct frameworks to be properly exploited toward modelling.

Kinetic modelling in a Boltzmann framework applied to traffic flow was first initiated by Prigogine and Hermann.[59] Then, various contributions have been proposed by several authors starting from the critical analysis and substantial improvement proposed by Paveri Fontana[57] toward the most recent developments proposed, among others, by Nelson,[54,55] Klar and Wegener[41,44,65] and Sopasakis.[62,63] A concise description of some kinetic models is presented here. In contrast to the above frameworks the domain of integration of the velocity variable will not be precisely defined. In fact most of the models do not define precisely constraints to the interaction terms. Moreover, one can see that only some of the above mathematical structures are fully exploited toward modelling. This aspect may be interpreted as a hint to design new models.

Prigogine's model describes the system traffic according to the scheme

$$\frac{\partial f}{\partial t} + V\frac{\partial f}{\partial x} = J_P[f]\,, \tag{7.1}$$

where the operator J_P is the sum of two terms

$$J_P[f] = J_r[f] + J_i[f]\,, \tag{7.2}$$

which describe the rate of change of f due to two different contributes: the relaxation term J_r, due to the behavior of the drivers of changing spontaneously speed to reach a **desired** velocity, and the (slowing down) interaction term J_i, due to the mechanics of the interactions between vehicles with different velocities. In particular, the term J_r is related to the fact that each driver, whatever its speed, has a program in terms of a desired velocity. Let $f_d = f_d(V)$ denote the **desired-velocity distribution function**, meaning that $f_d(V)\,dxdV$ gives the number of vehicles that, at time t and position in $x \in [x, x + dx]$, desire to reach a velocity between V and $V + dV$. The driver's desire also consists of reaching this velocity within a certain relaxation time T_r, related to the normalized density and equal for each driver. This term is an analogous to a BGK term, where the desired distribution function f_d is modelled by phenomenological intuitions.

Prigogine's relaxation term is defined by

$$J_r[f](t,x,V) = \frac{1}{T_r[f]}(f_d(V) - f(t,x,V))\,, \tag{7.3}$$

with

$$T_r[f](t,x) = \tau\frac{u(t,x)}{1-u(t,x)}\,, \tag{7.4}$$

where τ is a constant. The relaxation time is smaller, the smaller the density; instead for density approaching to the bump-to-bump condition, $u \to 1$, this term grows indefinitely.

The term J_i is due to the interaction between a test (trailing) vehicle and its heading (field) vehicle. It accounts for the changes in $f(t,x,V)$ caused by a braking of the test vehicle due to an **interaction** with the heading vehicle: it contains a gain term when the test vehicle has velocity $W > V$, a loss term when the heading vehicle has velocity $W < V$. Moreover, J_i is proportional to the probability P that a fast car passes a slower one; of course this probability depends on the traffic conditions and so on the normalized density. Prigogine's interaction term is defined, in analogy to a loss term in a binary localized Boltzmann framework, by

$$J_i[f](t,x,V) = (1 - P[f])f(t,x,V)\int_0^{1+\mu}(W-V)f(t,x,W)\,dW\,, \tag{7.5}$$

where

$$P[f](t,x) = 1 - u(t,x)\,. \tag{7.6}$$

Referring to the classical Boltzmann equation and to generalized Boltzmann models,[6] the above traffic flow model is such that the term J_r can be classified as a phenomenological binary interaction kinetic model, while the term J_i is derived without taking into account the microscopic interactions between vehicles.

It is worth mentioning, as it has been noted in Ref. 57, that the relaxation term $J_r[f]$ becomes meaningless when the vehicle density u tends to zero. Besides, the modelling should also take into account the fact that vehicles may be involved in high concentration traffic (the interesting situation, from the viewpoint of modelling, of traffic jams), so that the diluted gas assumption, typical of the Boltzmann equation, has to be put in question. Then, this model shows some contradiction for low density, with a meaningless relaxation time, and for high density, with the difficulty in using the Boltzmann framework. Besides, the desired distribution should be a Lagrangian quantity following the evolution of the system, depending on the local condition of traffic, instead of being assigned *a priori*.

The first substantial modification of the model to avoid the above-mentioned problem was proposed by **Paveri Fontana**.[57] The desired velocity V_d is assumed to be an independent variable of the problem, and a generalized (one) vehicle distribution function $g = g(t,x,V;V_d)$ is introduced to describe the distribution of vehicles at (t,x) with speed V and desired speed V_d. Hence the distribution f_d concerning the desired speed and the distribution f concerning the actual speed are given by

$$f_d(t,x,V_d) = \int g(t,x,V;V_d)\,dV \tag{7.7}$$

and

$$f(t,x,V) = \int g(t,x,V;V_d)\,dV_d\,. \tag{7.8}$$

The evolution equation, which now refers to the generalized distribution function g, is again determined by equating the transport term on g to the sum of the slowing down interaction term and the relaxation term similarly to (7.1), (7.2). However, now, the operators apply to g, and the passing probability P is assumed to be also a function of a certain critical density u_c.

The relaxation toward a certain program of velocities is related to vehicles acceleration. This is taken into account by means of a relaxation time T_r which is a function of the passing probability P, and hence of the density. Paveri Fontana's relaxation term J_r is defined by

$$J_r[g](t,x,V;V_d) = -\frac{\partial}{\partial V}\left(\frac{V_d - V}{T_r[f]}\,g(t,x,V;V_d)\right), \tag{7.9}$$

where

$$T_r[f](t,x) = \tau\frac{1 - P[f](t,x)}{P[f](t,x)} = \tau\frac{u(t,x)}{u_c - u(t,x)} \tag{7.10}$$

and

$$P[f](t,x) = (1 - u(t,x)/u_{\mathrm{c}})H(u_{\mathrm{c}} - u(t,x))\,. \tag{7.11}$$

Here H is the Heaviside function.

Paveri Fontana's interaction term J_i is defined by

$$\begin{aligned} J_i[g](t,x,V;V_d) &= (1 - P[f])f(t,x,V) \int_{W \geq V} (W - V) g(t,x,W;V_d)\,dW \\ &\quad - (1 - P[f])g(t,x,V;V_d) \int_0^V (V - W) f(t,x,W)\,dW\,. \end{aligned} \tag{7.12}$$

To compare the Paveri Fontana's evolution model, given by Eqs. (7.7)–(7.12), with the Prigogine's model, one has to integrate it with respect to V and V_d. This yields

$$\begin{aligned} &\frac{\partial f}{\partial t} + V\frac{\partial f}{\partial x} + \frac{\partial}{\partial V}\left[\frac{1}{T}\left(\int V_d g(t,x,V;V_d)\,dV_d - Vf(t,x,V)\right)\right] \\ &\quad = (1 - P[f])f(t,x,V)\int (W - V) f(t,x,W)\,dW \end{aligned} \tag{7.13}$$

and

$$\frac{\partial f_d}{\partial t} + \frac{\partial}{\partial x}\left(\int Vg(t,x,V;V_d)\,dV\right) = 0\,. \tag{7.14}$$

This equation for f_d shows that the desired distribution depends on the time evolution of the system. It overcomes the above-mentioned weak point of the Prigogine's model; however the complexity of the model has increased with the increased dimensionality of the model, with one more speed dimension. Moreover, as in the case of Prigogine's model, the collision operator takes into account the driver's behavior in a phenomenological way.

Another kind of development of the Prigogine's model was proposed, among various authors, by **Sopasakis**.[62] He derives an equilibrium solution of the Prigogine nonlinear kinetic equation able to describe the fluctuations of the experimental data in the high density region, the so-called unstable flow region. When the vehicular density is higher than a certain critical density, as already underlined in Sec. 3, the flow is no longer smooth but small fluctuation can lead to large variation of the flow. Sopasakis proposes a suitable equilibrium solution and consequently a closed form model of traffic flow able to describe this instability.

Recently various kinetic models, have been developed, based upon a detailed microscopic description of the short range pair interactions. There will be briefly outlined some new approaches proposed by various authors, e.g. Nelson,[54] Klar and Wegner.[42,43,65]

The collision operator is modelled by analyzing each driver short-range reactions to neighborhood vehicles rather than interpreting his overall behavior. The interactions are strictly pairwise: the test vehicle only reacts to what happens in its immediate headings. However, modelling microscopic interactions is, as mentioned in the previous sections, certainly not a simple task. It requires detailed analysis of vehicle dynamics and driver's reactions and to find suitable expressions for the post-interaction velocities which, in the microscopic modelling, are directly related to the pre-interaction ones.

Furthermore, the above equations, cast into a Boltzmann framework, do not allow perturbations to propagate backwards in negative x direction. This is obviously in contrast to real traffic flow observations. The above type of kinetic equations is therefore only applicable for dilute traffic flow without backwards propagating information. To describe the behavior of dense traffic with a kinetic equation, and to obtain consistent derivation of macroscopic equation, it is necessary to include the effects of the finite distance between the vehicles, in a way similar to that followed in deriving Enskog equation.[42] Nevertheless, the Enskog equation is still a diluted gas equation.

Further developments are due to **Nelson** who proposes in Ref. 54 a model based on a **table of the truth** of outgoing velocities in response to each possible incoming circumstance, a transition probability density $A(V^*, V)$ is *a priori* sketched. The set of possible values for the test vehicle's outgoing velocity after the interaction, *id est* change-in-speed event, with the heading (field) vehicle is restricted to a finite set $\{0, V, V_H, V_M = 1\}$, where V_H denotes the heading vehicle velocity.

Fundamental to any interaction process is the **minimal headway** distance ξ which triggers the occurrence of the interaction, and is assumed to be a strictly increasing function only of the first single heading vehicle speed $\xi(V)$. A suitable **headway probability density** $p(h|t, x)$ is introduced to account for the probability that the headway distance may be less than a certain h. The transition probability A is then constructed depending upon the various possible values of h with respect to the value $\xi(V)$.

Considering a test vehicle which (instantaneously) changes its speed from V^* to V due to an interaction with its heading vehicle at velocity W^*, the explicit model proposed by Nelson is

$$\frac{\partial f}{\partial t} + V\frac{\partial f}{\partial x} = \delta(t)J_s[f] + J_l[f]\,, \tag{7.15}$$

where the delta function $\delta(t)$ accounts for the short times response $J_s[f]$ and $J_l[f]$ the long term one. In details, omitting the (t, x) dependence, the short times term is, if $q_1 = (1 - p(\xi(1)))$ denotes the minimal probability of non passing,

$$\begin{aligned} J_s[f](V) = \frac{1}{u}\Bigg[&- f(V)\left(\int_0^V p(\xi(V'))f(V')\,dV' + uq\right) \\ &+ \delta(V-1)q_1\left(\int uf(W^*)\,dW^* + H(V-1)f(1)q_1\right) \\ &+ \delta(V)\left(\int f(V^*)\int_0^{V^*} p(\xi(W^*))f(W^*)\,dW^*\,dV^*\right)\Bigg]. \end{aligned} \tag{7.16}$$

The long times term is given by

$$J_l[f](V) = f(V)\left(\int (p(\xi(V')) - p(\xi(V)))|V' - V| f(V')\, dV'\right). \tag{7.17}$$

The above model essentially exploits a binary interaction framework, although some dense gas effects are taken into account. Indeed, modelling flow conditions from dense to rarefied flow is one of the crucial problems which applied mathematics have to tackle with, as it will be discussed in the last section of this paper.

Starting from Nelson's model, **Klar and Wegener** in Ref. 65, using essentially the same mathematical structure of gain and loss term that is familiar in the Boltzmann theory, i.e. a binary interaction framework, introduce an **outgoing velocity probability density function** $A(V^*, W^*; V)$, which, consistently to Sec. 6.2, is related to the event that the test vehicle instantaneously changes its speed from V^* to V because of an interaction with its heading vehicle with velocity W^*. In fact, they consider several possible **headway thresholds** $h_i(V^*, W^*)$, $i = 1, \ldots, r$, and hence several possible functions A_i corresponding to various different kinds of interactions.

Interactions are assumed to happen only when the headway distance **crosses** any of the thresholds values h_i's. The explicit model of Klar and Wegener writes as follows:

$$\frac{\partial f}{\partial t} + V\frac{\partial f}{\partial x} = G[f] - L[f]\,, \tag{7.18}$$

wherein, omitting the argument, the gain and loss terms are specialized as follows:

$$G[f](V) = \sum_{i=1}^{r} \int f(V^*) A_i(V^*, W^*; V)|V^* - W^*| g(h_i(V^*, W^*), W^*; V^*)\, dV^* dW^*\,, \tag{7.19}$$

$$L[f](V) = f(V) \sum_{i=1}^{r} \int |V - W^*| g(h_i(V, W^*), W^*; V)\, dW^*\,. \tag{7.20}$$

Of course the integration domain is the set of all the velocity pairs that allow ith interaction.

The function g denotes the number density of the field vehicles that are at headway h, and velocity W^*, from a test vehicle at (t, x, V^*); i.e.

$$f_2(t, x, V^*, x + h, W^*) = g(h, W^*; t, x, V^*) f(t, x, V^*)\,, \tag{7.21}$$

where f_2 denotes the pair distribution function.

On the function $g = g(h, W^*; t, x, V^*)$ the same special assumptions are then stated, similar to those already used in Nelson model, and necessary to obtain a closed equation for f as it is documented in Ref. 43.

Subsequently, and following the same lines, Klar and Wegener in Ref. 42 introduce an Enskog-like approach and re-propose the interaction terms (7.19), (7.20) as follows:

$$G[f](V) = \sum_{i=1}^{r} \int |V^* - W^*| A_i(V^*, W^*; V) \times f_2(t, x, V^*, x + h_i(V^*, W^*), W^*)\, dV^* dW^* \tag{7.22}$$

and

$$L[f](V) = \sum_{i=1}^{r} \int |V - W^*| f_2(t, x, V, x + h_i(V, W^*), W^*)\, dW^*\,, \tag{7.23}$$

where $f_2(t, x, V^*, y, W^*)$ denotes the two-vehicle distribution function, a test vehicle at (x, V^*) and a heading vehicle at (y, W^*).

All the above models have been derived by assuming that all vehicles are free to move without external actions of any kind, either human or mechanical. The distribution function is modified only by pair interactions.

The description given in this section defined a report of the state-of-the-art available in the literature. As we have seen, only the frameworks described in Secs. 6.1 and 6.2 have been properly exploited, while those of Secs. 6.3 and 6.4 have not. This aspect deserves a critical analysis which will be proposed in the next section.

8. Critical Analysis and Perspectives

This paper has given a general review of the state-of-the-art related to hydrodynamic and kinetic models of traffic flow on roads. This final section deals with a critical analysis addressed to indicate conceivable research perspectives. The selection of the topics proposed in what follows is related to the personal bias of the authors. Certainly alternative selections can be proposed. Therefore, the reader should look at this final section without claiming completeness, but as a hint to stimulate new research initiatives. In details, the following topics will be dealt with:

(i) applications to the analysis of real traffic flow conditions;
(ii) analysis of the links between microscopic and macroscopic descriptions;
(iii) perspectives toward the design of new models.

Before dealing with the above topics, it is worth mentioning that applications and simulations need the development of specific algorithms for the solution of initial–boundary value problems; this topic will be better explained in another review paper, still working in progress, which is devoted to a numerical analysis of these kinds of models. *Ad hoc* algorithms have to be used to treat traffic flow models: finite differences and finite volumes schemes for macroscopic models, e.g. Daganzo,[21,22] Lebaque[46]; collocation methods[9] for hydrodynamic models; particle methods for kinetic equations.[43] Generally, the crucial point consists of taking care, in the application of computational methods, of hyperbolicity related to the conservation laws and source terms which may be generated by inlet or outlet of vehicles, e.g. among others, Refs. 28, 29 and 58. A critical analysis of models still has to be developed having in mind computational schemes applied toward simulations.

8.1. *Application to real flow conditions*

All the models described in the preceding sections, either macroscopic or kinetic, refer to one lane flow or to an approximation of multilane flows by one lane models.

Therefore, one may look at them as suitable background models for the application to real flow conditions: multilane flow, flow with networks and/or in the presence of traffic lights or tollgates, etc. Moreover, roads may go through tracts where the number of lanes locally reduce. Generally this is a cause of traffic jams. Hence, one can develop models which specifically take into account the above feature: multilane models naturally do it.

An outline of modelling of a multilane flow will first be given with reference to the class of scalar models dealt with in Sec. 5.1. Generalization to other types of models is technical. The objective consists of finding an evolution equation for the densities

$$u_i = u_i(t,x)\,, \quad i = 1, \dots, n\,, \tag{8.1}$$

corresponding to each lane. The mean velocities in each lane will be denoted by $v_i = v_i(t,x)$, being referred to the maximum indicative velocity in each lane

$$v_{iM} = \alpha_i v_M\,, \quad 0 < \alpha_1 < \cdots < \alpha_i < \cdots \alpha_n = 1\,. \tag{8.2}$$

Formally, the evolution equation in each lane writes

$$\frac{\partial u_i}{\partial t} = \mathcal{F}[u_i] + \varphi_i^+[u_{i-1}, u_i, u_{i+1}] - \varphi_i^-[u_{i-1}, u_i, u_{i+1}]\,, \tag{8.3}$$

where the operator $\mathcal{F}[u_i]$ corresponds to the specific model which is dealt with. Moreover, $\varphi_i^+ = \varphi_{(i-1)i} + \varphi_{(i+1)i}$ is the flux from the lower and upper lanes into the ith lane due to relatively fast and slow vehicles respectively; and $\varphi_i^- = \varphi_{i(i-1)} + \varphi_{i(i+1)}$ is the flux from the ith lane into the lower lane due to relatively slow and fast vehicles. Boundary conditions imply that

$$\varphi_{(n+1)n} = \varphi_{n(n+1)} = \varphi_{01} = \varphi_{10} = 0\,. \tag{8.4}$$

A detailed phenomenological model of the above terms leads to the desired evolution system. The simplest model can be based on assumptions on linear dependence on the velocity jumps. The above ideas can also be applied to higher order models. Similarly one can deal with flow of vehicles on roads where the number of lanes suddenly reduces or increases. Actually, one lane modelling can also be used for the above dynamics. For instance, if in a tract $[x_1, x_2]$, the road becomes narrow, one has to impose a velocity upper bound

$$x \in [x_1, x_2] : \quad \frac{\partial u}{\partial t} = \alpha \mathcal{F}[u]\,, \quad \alpha < 1\,; \quad x \notin [x_1, x_2] : \quad \frac{\partial u}{\partial t} = \mathcal{F}[u]\,. \tag{8.5}$$

Kinetic models can be dealt with in a similar way. The formal evolution equation writes

$$\frac{\partial f_i}{\partial t} = \mathbf{K}[f_i] + \varphi_i^+[f_{i-1}, f_i, f_{i+1}] - \varphi_i^-[f_{i-1}, f_i, f_{i+1}]\,, \tag{8.6}$$

where the operator $\mathbf{K}[f_i]$ corresponds to the specific model which is dealt with.

The above approach is justified by the fact that the behavior in parallel lanes is similar as documented in Refs. 36–39. Specific models have been proposed by Helbing[31] and Lo Schiavo.[51]

Dealing with traffic flow in the presence of tollgates or traffic lights means modelling the driver's behavior when the above devices are close to the vehicle. As shown in Ref. 10 a sharp analysis of the sensitivity of drivers to traffic flow conditions (which may even involve modelling of physiological adaptation[1]) may substantially affect the structure of the model and/or the simulations. A first step in modelling the above features is proposed by Marasco,[53] who refers the presence of tollgates to a space dependence of the parameter η. In other words, the driver approaching or leaving a tollgate modifies her/his behavior becoming less or more sensitive to local gradients. The analysis of Ref. 53 refers to scalar models, developing such a topic to higher order hydrodynamic models or to the kinetic description is an interesting research perspective.

Dealing with networks of roads means kinking the evolution model for a single tract to the inlet or outlet of vehicles in the connection nodes.[3,47] Conservation of mass gives suitable algebraic relations for the flux in the nodes. This means that models of the type presented in Sec. 5.1, e.g. Model S.4, can be naturally used for boundary conditions based on flux measurements, while models for the density may need nonlinear boundary conditions such as those dealt with in Ref. 9. Kinetic equations may need moment boundary conditions similarly to the kinetic theory of gases.[52]

8.2. *From kinetic modelling to hydrodynamics*

As already mentioned, kinetic models provide a description of traffic flow which retains features both of microscopic and macroscopic models. In principle, kinetic models should be derived from microscopic models and, macroscopic models should be derivable from kinetic ones.

To derive a macroscopic equation from the kinetic model one proceeds similarly to the gas kinetic case, by taking the moments of the kinetic equation

$$\frac{\partial f}{\partial t} + V\frac{\partial f}{\partial x} = J[f]\,, \tag{8.7}$$

with a function $\phi(V)$ and integrating it with respect to V; for simplicity it is considered a single lane road. This leads to the balance equations

$$\frac{\partial}{\partial t}\int_0^{1+\mu} \phi(V) f\, dV + \frac{\partial}{\partial x}\int_0^{1+\mu} V\phi(V) f\, dV + \int_0^{1+\mu} \phi(V) J[f]\, dV = 0\,. \tag{8.8}$$

With $\phi(V) = 1$ the collision operator, as usual referring to a Boltzmann framework, vanishes if integrated over the velocity V. We get the usual continuity equation,

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(uv) = 0\,, \tag{8.9}$$

considering that v indicates the average value of V. With $\phi(V) = V$, the linear momentum equation becomes

$$\frac{\partial}{\partial t}(uv) + \frac{\partial}{\partial x}(p + uv^2) = \int_0^{1+\mu} V^2 J[f]\, dV = 0\,, \tag{8.10}$$

where p is the aforementioned speed pressure of traffic, see (2.9),

$$p = \int_0^{1+\mu} (V - v)^2 f\, dV = \int_0^{1+\mu} V^2 f\, dV - uv^2\,. \tag{8.11}$$

In general, using $\phi(V) = V^k$ with $k = 0, 1, \ldots$ one obtains an infinite hierarchy of equations. The above system is not closed and suitable closure relations must be outlined. Gas dynamics suggest many possibility to achieve this closure, e.g. Ref. 45, besides, directly applied to traffic models, one can recover details of the previous procedure and suggestions of other possible closure relation in Ref. 43. For instance, if the hierarchy is stopped with $\phi = V$, one has to give a phenomenological closure relation for the pressure term.

A possible choice is to use the phenomenological equilibrium distribution function obtained, as suggested in Sec. 3, by the experimental data: $f_e(V;\cdot)$. In this way one can approximate the traffic pressure p by its equilibrium value, obtained by substituting to the mean value of velocity V the mean equilibrium velocity V_e. Of course, the collision operator is approximated by replacing $f_e(v;u)$ for $f(t,x,V)$.

It is worthwhile to point out that when the model takes into account the finite dimension of the vehicles, through an Enskog approach,[43] to evaluate the integrals of the collision operator it is necessary to separate the Enskog interaction operator into a local interaction term and a deviation from the local term.

8.3. *Perspectives in modelling*

In order to provide some perspective ideas toward development of new models, it is worthwhile to develop a critical analysis of the contents of the preceding sections.

One may critically observe that modelling traffic flow, both by fluid mechanics or by kinetic equations, still leaves several problems open. Some of them are the following:

(i) The dynamics of a vehicle is not simply determined by pair interactions, but by the action of all vehicles surrounding the test vehicle within a certain action domain.

(ii) The dynamics of a vehicle is determined not by real flow conditions, but by those conditions which are effectively felt by the driver.

(iii) The Boltzmann equation, with pair interactions, is a model for a diluted gas such that the distance between particles is relatively large. On the other hand, flow conditions which are interesting to be modelled are dense vehicles conditions (with jams and cluster formation).

(iv) The mathematical statement of boundary conditions for kinetic models requires conditions which are difficult to be properly stated, definitively different from the one required by hydrodynamic equations. This is particularly important if one has to model networks of roads etc.

According to the above criticism, it is plain that models which are available in the literature cannot effectively overcome the problems raised in items (i)–(iv). In principle technical improvements of hydrodynamic models can be obtained by increasing the order of the model, e.g. models with acceleration and energy. On the other hand, the examples in Sec. 5 show that increasing the order of the model also increases the number of parameters to be assessed. Experiments organized toward such a scope hardly provide reliable results. The above difficulty may imply that higher order models are not practical for engineering applications. Moreover, referring to the critical analysis,[27] the hydrodynamic approach fails if we assume the traffic density as a state variable. The principal drawback is that the length scales of interest are such that there are only a small number of vehicles, and the outlet of just one vehicle changes significantly the value of the density. In this way, there is in sufficient number of cars to justify the continuum assumption and, at last, the definition of density itself.

Similar reasons may be raised in the case of kinetic models. Experimental data may improve the modelling of the microscopic interactions between vehicles (also called **the table of truth**). However, the main problems still remain that kinetic equations need rarefied flow assumptions which are not consistent with physical situations for which modelling appears to be interesting; the ratio of the length scale of a car to its following distance is less than the mean free path of gas molecules and thus the dilute assumption seem to be unreasonable.

To overcome this criticism, it is worthwhile to cite a new type of framework recently proposed in Ref. 27 which can open new research perspectives. This paper states the need of a methodology to obtain a macroscopic description of a large scale system, like the traffic problem, starting from an awareness of microscopic description. The traffic flow is treated as a large collection of vehicles and is mathematically modelled by a finite dimensional discrete dynamical system.

The complexity of traffic flow modelling and the possibility of constraining the various related phenomena into mathematical terms is well documented in Ref. 33. Still the main objective, from the mathematical point of view, consists of deriving suitable evolution equations for the description of the whole behavior of the system rather than particular aspects exploiting parameters which can be effectively measured.

Without forgetting the above critical analysis and the new interesting perspectives, this paper will be concluded with the indication of some ideas toward the development of new models. The contents of Sec. 3 will be a useful background. Some of the ideas given in what follows are already object of research activities, while others may hopefully attract applied mathematicians.

- Reference 15 has already suggested to use the equilibrium velocity distribution function (3.5) to modify the desired distribution function proposed in the Prigogine's model. This idea can be systematically applied to other types of models. The above distribution can be exploited to recover macroscopic hydrodynamics from kinetic equations. The experimental results proposed in the above paper have been exploited in Ref. 12 to obtain a new class of models derived by closing the mass conservation equation by a stochastic closure obtained by the velocity field regarded as a random variable linked to a probability density conditioned by the local density and the density gradients. This model, as the one proposed in Ref. 27, has to be regarded as an alternative to the traditional hydrodynamic and kinetic description.
- Various mathematical structures for kinetic models have been indicated in Sec. 6. The structures indicated in Secs. 6.3 and 6.4 have not yet been object of studies. Possibly, new interesting models can be developed properly exploiting the above framework.
- Generalized kinetic (Boltzmann) models with internal structure have been recently proposed in Ref. 2. The internal microscopic structure is introduced in order to model non-mechanical quantities related to the interacting objects. Indeed one may look at vehicles as mechanical objects, while drivers should be related to the internal structure with suitable modification of the mechanical interactions.
- Modelling may look at equations able to retain some features both of hydrodynamic and kinetic equations. For instance considering, as mentioned above, the velocity as a random variable to close the mass conservation equation. Similar ideas may be developed to close the momentum conservation equation.

The reader can recognize that the above suggestions cannot be regarded as simple technical modifications and possibly improvements of the existing models. Each suggestion need a deep analysis to be properly developed. Hopefully it may lead to a substantial revision of the models existing in the literature.

Models have then been related to applications by a proper statement of mathematical problems. This means implementing initial and/or boundary conditions. Well-posedness and qualitative analysis have to be related to the above problems rather than simply to the Cauchy problem in unbounded domains. Development of computational schemes, again related to the above problems may provide simulations of traffic flow phenomena.

A technical difficulty is relating the statement of the problems to the quantities which can be effectively measured. Again we refer to Ref. 33, where it is critically analyzed how time and space averaging to measure physical quantities in traffic flow may provide different results. An account to the above problems is given in

Ref. 9 where computational schemes are adapted to deal with nonlinear boundary conditions induced by measurement problems. There, the analysis was related to collocation interpolation methods. On the other hand, the same reasoning can be developed for alternative, certainly more efficient, discretization schemes for equations with hyperbolic terms.[28] On the other hand, an overall analysis for kinetic models is also not available in the literature and certainly deserves the attention of applied mathematics.

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