Newton Binomial Formulas in Schubert Calculus

Letterio Gatto

Politecnico di Torino

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Joint work with

Joint work with

Jorge Cordovez

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POLITO, Italy

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be the complex Grassmann variety parameterizing k-planes in $\mathbb{C}^n$. It also parameterizes $(k-1)$-dimensional projective linear subvarieties of $\mathbb{P}^{n-1}$.

Intersection Theory of $G(k, n)$ – I

The Chow group $A_*(G(k,n))$ is a $\mathbb{Z}$-module freely generated by:

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is the *fundamental class* (the class of all k-planes of $\mathbb{C}^n$);

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is an isomorphism (Poincaré isomorphism), making $A_*(G(k,n))$ into a free $A^*(G(k,n))$-module of rank 1, generated by the fundamental class $[G(k,n)] \in A_*(G(k,n))$.

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It coincides with $c_h(\mathcal{Q}_k)$, where $\mathcal{Q}_k$ is the universal quotient bundle over $G(k,n)$.

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Well known results ensure that the $\mathbb{Z}$-algebra $A^*(G(k,n))$ is generated precisely by $\sigma := (\sigma_0, \sigma_1, \sigma_2, \ldots)$

Schubert Calculus I

Schubert Calculus on $G(k, n)$ is concerned with the problem of determining the constant structure $\{(LR)_{IJ}^{K}\} \in \mathbb{Z}$ of the algebra $A^*(G(k, n))$, namely the integers occurring in the expansion:

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and the expansion of

$$\sigma_h \cap \Omega_I$$

as a $\mathbb{Z}$-linear combination of $\{\Omega_J\}$, for each $h \geq 0$ and each $I \in \mathcal{I}_n^k$. Such a product is ruled by *Pieri's formula*.

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When $I \in \mathcal{I}_n^k$,

ω_I is the *degree of the Schubert variety* $\Omega_I(E^\bullet)$,

where $E^\bullet$ any complete flag of $\mathbb{C}^n$. It was already computed by Schubert.

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For each $I \in (\mathbb{N}^*)^k$, let

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Then $\bigwedge^k M_n$ is a free $\mathbb{Z}$-module generated by

$$\{\epsilon^I \,|\, I \in \mathcal{I}_n^k\}.$$

Schubert Calculus on a Grassmann Algebra – II

It turns out that $\bigwedge^k M_n$ is a graded module via *weight*:

$$\bigwedge^k M_n = \bigoplus_{w \geq 0} (\bigwedge^k M_n)_w,$$

where

$$(\bigwedge^k M_n)_w = \bigoplus_{wt(I)=w} \mathbb{Z} \cdot \epsilon^I = \bigoplus_{wt(I)=w} \mathbb{Z} \cdot \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}$$

Schubert Calculus on a Grassmann Algebra – II

It turns out that $\bigwedge^k M_n$ is a graded module via *weight*:

$$\bigwedge^k M_n = \bigoplus_{w \geq 0} (\bigwedge^k M_n)_w,$$

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$$(\bigwedge^k M_n)_w = \bigoplus_{wt(I)=w} \mathbb{Z} \cdot \epsilon^I = \bigoplus_{wt(I)=w} \mathbb{Z} \cdot \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}$$

Then the exterior algebra $\bigwedge M_n$ is a bi-graded $\mathbb{Z}$-module:

$$\bigwedge M_n := \bigoplus_{k \geq 0, w \geq 0} (\bigwedge^k M_n)_w$$

A piece of Notation

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Define:

$$\int_n \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} =$$

$$= \begin{cases} 1 & \textit{if} \quad (i_1, \ldots, i_k) \quad \textit{is an even permutation of} \quad (n-k+1, \ldots, n) \\ -1 & \textit{if} \quad (i_1, \ldots, i_k) \quad \textit{is an odd permutation of} \quad (n-k+1, \ldots, n) \\ 0 & \textit{otherwise} \end{cases}$$

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Extend $\int$ by $\mathbb{Z}$-linearity, getting $\int : \bigwedge^k M_n \to \mathbb{Z}$

Schubert Calculus on a Grassmann Algebra – III

Let

$$(\bigwedge M_n)[[t]]$$

be the algebra of formal power series with coefficients in $\bigwedge M_n$.

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There exists one and only one algebra homomorphism

$$D_t := \sum_{i\geq 0} D_i t^i : \bigwedge M_n \longrightarrow \bigwedge M_n[[t]]$$

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—, Asian J. Math., **9**, n.3, 2005, 315–322 or —, Santiago, SCGA, Canad. Math. Bull., to appear)

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such that

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Then, for each $h \geq 0$, $D_h \in End_A(\bigwedge M_n)$ is an endomorphism of the bi-graded $\mathbb{Z}$-algebra $\bigwedge M_n$, homogeneous of bi-degree $(0, h)$, i.e.:

$$D_h(\bigwedge^k M_n)_w \subseteq (\bigwedge^k M_n)_{w+h}.$$

The equation

$$D_t(\alpha \wedge \beta) = D_t\alpha \wedge D_t\beta$$

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is equivalent to:

$$D_h(\mathbf{p} \wedge \mathbf{q}) = \sum_{\substack{h_1 + h_2 = h \\ h_1, h_2 \geq 0}} D_{h_1}\mathbf{p} \wedge D_{h_2}\mathbf{q}$$

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for each $h \geq 0$ and each $\mathbf{p}, \mathbf{q} \in \bigwedge M_n$, which is the h^{th} order Leibniz rule ($h \geq 0$).

For example:

Schubert Calculus on a Grassmann Algebra – IV

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For example:

$$D_2(\mathbf{p} \wedge \mathbf{q}) = D_2\mathbf{p} \wedge \mathbf{q} +$$

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Schubert Calculus on a Grassmann Algebra – IV

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One more example:

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Schubert Calculus on a Grassmann Algebra – IV

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Schubert Calculus on a Grassmann Algebra – IV

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$$= \epsilon^3 \wedge \epsilon^2 + \epsilon^2 \wedge \epsilon^3 + \epsilon^1 \wedge \epsilon^4 = \epsilon^1 \wedge \epsilon^4$$

But

$$\sigma_1^4 \cap [G(2,4)] = D_1^4(\epsilon^1 \wedge \epsilon^4)$$

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$$= D_1(D_1(\epsilon^2 \wedge \epsilon^3 + \epsilon^1 \wedge \epsilon^4)) = D_1(\epsilon^3 \wedge \epsilon^3 + \epsilon^2 \wedge \epsilon^4 + \epsilon^2 \wedge \epsilon^4 + \epsilon^1 \wedge \epsilon^5) =$$

But

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$$= D_1^3(\epsilon^2 \wedge \epsilon^2 + \epsilon^1 \wedge \epsilon^3) = D_1^2(D_1(\epsilon^1 \wedge \epsilon^3)) = D_1^2(\epsilon^2 \wedge \epsilon^3 + \epsilon^1 \wedge \epsilon^4) =$$

$$= D_1(D_1(\epsilon^2 \wedge \epsilon^3 + \epsilon^1 \wedge \epsilon^4)) = D_1(\epsilon^3 \wedge \epsilon^3 + \epsilon^2 \wedge \epsilon^4 + \epsilon^2 \wedge \epsilon^4 + \epsilon^1 \wedge \epsilon^5) =$$

$$= D_1(2\epsilon^2 \wedge \epsilon^4) =$$

But

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and

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Lines through four Lines

But

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$$= D_1(D_1(\epsilon^2 \wedge \epsilon^3 + \epsilon^1 \wedge \epsilon^4)) = D_1(\epsilon^3 \wedge \epsilon^3 + \epsilon^2 \wedge \epsilon^4 + \epsilon^2 \wedge \epsilon^4 + \epsilon^1 \wedge \epsilon^5) =$$

$$= D_1(2\epsilon^2 \wedge \epsilon^4) = 2D_1(\epsilon^2 \wedge \epsilon^4) = 2 \cdot \epsilon^3 \wedge \epsilon^4.$$

Let $\mathcal{A}^*(\bigwedge M_n)$ be the commutative subalgebra of $End_{\mathbb{Z}}(\bigwedge M_n)$ generated by $D := (D_1, D_2, \ldots)$.

Each element of it is a polynomial expression in $D := (D_1, D_2, \ldots)$.

Schubert Calculus on a Grassmann Algebra – V

Let $\mathcal{A}^*(\bigwedge M_n)$ be the commutative subalgebra of $End_{\mathbb{Z}}(\bigwedge M_n)$ generated by $D := (D_1, D_2, \ldots)$.

Each element of it is a polynomial expression in $D := (D_1, D_2, \ldots)$. Consider the natural evaluation map:

$$\begin{cases} \mathrm{ev}_{\epsilon^1\wedge\ldots\wedge\epsilon^k} & : & \mathcal{A}^*(\bigwedge M_n) & \longrightarrow & \bigwedge^k M_n \\ & & P(D) & \longmapsto & P(D)\epsilon^1 \wedge \ldots \wedge \epsilon^k \end{cases}$$

Let $\mathcal{A}^*(\bigwedge M_n)$ be the commutative subalgebra of $End_{\mathbb{Z}}(\bigwedge M_n)$ generated by $D := (D_1, D_2, \ldots)$.

Each element of it is a polynomial expression in $D := (D_1, D_2, \ldots)$. Consider the natural evaluation map:

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Schubert Calculus on a Grassmann Algebra – V

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Let

$$\mathcal{A}^*(\bigwedge^k M_n) := \frac{\mathcal{A}^*(\bigwedge M_n)}{\ker(\mathrm{ev}_{\epsilon^1\wedge\ldots\wedge\epsilon^k})}$$

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Consider the obvious $\mathbb{Z}$-modules isomorphisms:

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The diagram establishes a dictionary.

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For example, for all $I \in (\mathbb{N}^*)^k$, one has:

$$\omega_I := \int D_1^{k(n-k)-wt(I)}\epsilon^I = \int_n D_1^{k(n-k)-wt(I)}\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}$$

Newton's Binomial Formulas

Do we gain anything from our dictionary?

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Newton's type binomial formulas!

1st Newton's formulas $(a + b)^n$

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Challenge:

disprove the claim!

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$$f_{a,b,c,d} := \int_{G(4,n+4)} \sigma_1^a \sigma_2^b \overline{\sigma}_2^c \overline{\sigma}_3^d \cap [G(4, n+4)].$$

Generalization of Ranestad's Question

Is it possible to find any kind of formula for $f_{a,b,c,d}$?

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(Vainsencher, Økland – private communication).

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where

$$I(b_1, b_2, b_3, b_4, b_5, \ell)$$

is the following ordered 6-tuple of positive integers

$$(1 + b_1, 2 + b_2 + 2b_5, 3 + b_3 + \ell, 4 + b_2 + b_3 + 2b_4 - \ell).$$

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...we wrote a trivial (!) CoCoA (version 4.7) code to compute the list varying n,

(http://calvino.polito.it/~gatto/English/research/SCGA/cgscodes.htm)

and with an Apple iBook G4, 1.2GHz

$\longleftarrow$ my ibook

we got the following list in just a couple of hours:

Happy Birthday Israel – I

n	♯(summands)	$HS_n := \int \sigma_2^{2n} \cap [G(4, 4+n)]$	execution time
0	1	1	0,082s
1	13	0	0,242s
2	56	1	0,614s
3	142	5	1,449s
4	331	126	3,340s
5	641	3396	6,434s
6	1191	114675	12,081s
7	1981	4430712	20,053s
8	3221	190720530	32,755s
9	4866	8942188632	50,085s
10	7256	449551230102	1m 20s
11	10268	23948593282950	2m 55s
12	14418	1339757254689348	2m 44s
13	19466	78153481093195800	4m 02s
14	26156	4727142898098368085	5m 2s
15	34086	295116442188446065635	9m 9s
16	44286	18945322608397492982250	10m 46s
17	56141	1246718376589846006057200	11m 52s
18	71031	83878801924226511500933250	16m 37s
19	88071	5756860011979383129907915050	19m 55s
20	109061	402290757162008042628235950300	25m 53s
21	132783	28575935656515287427874861725000	34m 37s
22	161533	2060372706082551084572192852992530	01h 07m

. . . dots, dots, dots . . .

However,

However,

a few days after writing our list

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Jan Magnus Økland sent us another one up to $n = 40$.

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He got HS_{40} in eight ours. . .

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. . . stopping the computation of HS_{41} (after some long while)...

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. . . stopping the computation of HS_{41} (after some long while)...

. . . but when he tried to use (on the same machine) our formula with a slightly modified version of our CoCoA code. . .

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. . . stopping the computation of HS_{41} (after some long while)...

. . . but when he tried to use (on the same machine) our formula with a slightly modified version of our CoCoA code. . .

. . . he got, in just HALF AN HOUR:

Happy Birthday Israel – II

Happy Birthday Israel – II

$$HS_{42} =$$

Happy Birthday Israel – II

$$HS_{42} =$$

$$= 201517182255943002813954873119143476157329393137457696988123090973997900$$

The proof

The (beginning of the) proof

The (beginning of the) proof

$$HS_n = \sum_{\substack{b_i \geq 0 \\ b_1 + \ldots + b_5 = 2n \\ 0 \leq \ell \leq b_2 + b_3}} \frac{(2n)!}{b_1! \cdot \ldots \cdot b_5!} \binom{b_2 + b_3}{\ell} \omega_{I(b_1,\ldots,b_5;\ell)}$$

The (beginning of the) proof

By our dictionary

The (beginning of the) proof

By our dictionary

$$\begin{array}{ccc} \mathcal{A}^*(G(k,n)) \otimes_{\mathbb{Z}} A_*(G(k,n)) & \longrightarrow & A_*(G(k,n)) \\ \iota_k \otimes \jmath_k \Big\downarrow & & \Big\downarrow \jmath_k \\ \mathcal{A}^*(\bigwedge^k M_n) \otimes_{\mathbb{Z}} \bigwedge^k M_n & \longrightarrow & \bigwedge^k M_n \end{array}$$

The (beginning of the) proof

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one has:

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$$\int_{G(4,n+4)} \sigma_2^{2n} \cap [G(4, n+4)] = \int_n D_2^{2n}(\epsilon^1 \wedge \epsilon^2 \wedge \epsilon^3 \wedge \epsilon^4)$$

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Then we apply our Newton's formula to:

$$D_h^m(\epsilon^i \wedge \alpha) = \sum_{j=0}^{m} \binom{m}{j} D_{h-1}^j(\epsilon^{i+j} \wedge D_h^{m-j}\alpha)$$

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and then once again,

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and then once again, and then...

A very important Question:

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WHO CARES?

A very important Question:

WHO CARES?

We don't!

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In fact we may produce many similar formulas!

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For example:

A very important Question:

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For example:

$$f_{a,b,c,d} = \sum_{\substack{\beta \in p_5(b) \\ \gamma \in p_4(c) \\ \delta \in p_4(d) \\ 0 \leq l \leq \beta(2)+\beta(3) \\ 0 \leq m \leq \gamma(1)+\gamma(2)}} \frac{b! \cdot c! \cdot d!}{\beta! \cdot \gamma! \cdot \delta!} \binom{\beta(2)+\beta(3)}{l} \binom{\gamma(1)+\gamma(2)}{m} \omega_{I(\beta,\gamma,\delta;l,m)}$$

A very important Question:

WHO CARES?

We don't!

In fact we may produce many similar formulas!

For example:

$$f_{a,b,c,d} = \sum_{\substack{\beta \in p_5(b) \\ \gamma \in p_4(c) \\ \delta \in p_4(d) \\ 0 \le l \le \beta(2)+\beta(3) \\ 0 \le m \le \gamma(1)+\gamma(2)}} \frac{b!\cdot c!\cdot d!}{\beta!\cdot\gamma!\cdot\delta!}\binom{\beta(2)+\beta(3)}{l}\binom{\gamma(1)+\gamma(2)}{m}\omega_{I(\beta,\gamma,\delta;l,m)}$$

for some $I(\beta,\gamma,\delta;l,m)$ suitably defined and explicitly computed.

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in the present form:

Newton Binomial Formulas in Schubert Calculus*

Jorge Cordovez, Letterio Gatto, Taíse Santiago†

To Israel Vainsencher on occasion of his 60th birthday

Abstract

We prove Newton's binomial formulas for Schubert Calculus to determine numbers of base point free linear series on the projective line with prescribed ramification divisor supported at given distinct points.

1 Introduction

Let $G(k,n)$ be the complex grassmannian variety parametrizing k-dimensional subspaces of $\mathbb{C}^n$. In [5] (see also [6] and [17]), the intersection theory on $G(k,n)$ (Schubert calculus) is rephrased via a natural derivation on the exterior algebra of a free $\mathbb{Z}$-module of rank n. Classical *Pieri's* and *Giambelli's* formulas are recovered, respectively, from *Leibniz's rule* and *integration by parts* inherited from such a derivation. The generalization of [5] to the intersection theory on Grassmann bundles is achieved in [8], by suitably translating previous important work by Laksov and Thorup ([13], [14]) regarding the existence of a (unique) canonical *symmetric structure* on the exterior algebra of a polynomial ring.

It is natural to ask if the aforementioned derivation formalism for Schubert calculus is worthy or if it is nothing more than a mere translation of an old theory into a more or less new language. Indeed, a couple of years ago, K. Ranestad asked us to test our methods to compute (and possibly to find a formula for) the total number, with multiplicities, of non projectively equivalent rational space curves of degree $n+3$ having flexes at given $2n$ distinct points. Any such a curve is the image of the morphism $\phi : \mathbb{P}^1 \to \mathbb{P}^3$ induced by a very ample g^3_{n+3} on $\mathbb{P}^1$, such that the ramification sequence (in the sense of [12], Section 1.2) at each of its ramification point is $(1,2,4,5)$. Results by Eisenbud and Harris [3] ensure that such a number is finite and equal to the degree of a suitable product of Schubert cycles – see Section 5.9 for details. To compute it, we rely on the two main results of this paper, Theorem 3.5 and Theorem 3.6, regarding

**Key words and phrases*: Schubert Calculus on a Grassmann algebra, Newton's binomial formulas in Schubert calculus, enumerative geometry of linear series on the projective line; *2000 MSC:* 14M15, 14N15, 15A75.

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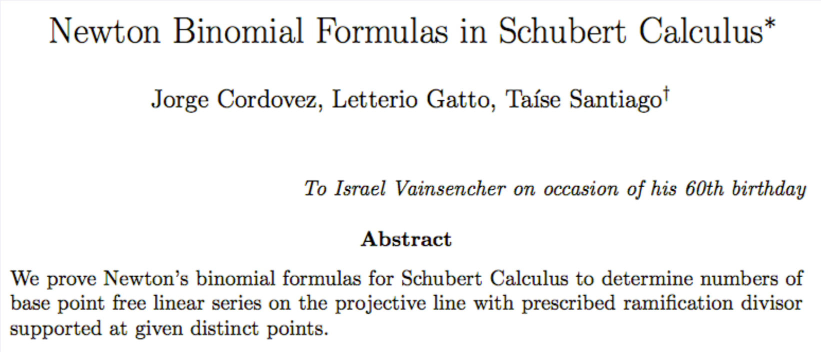

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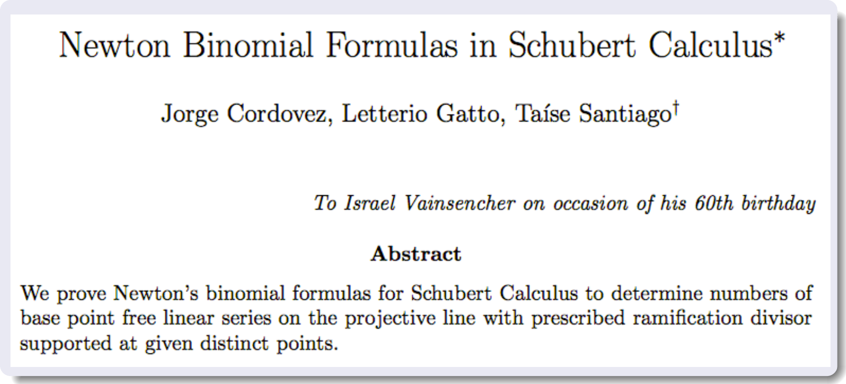

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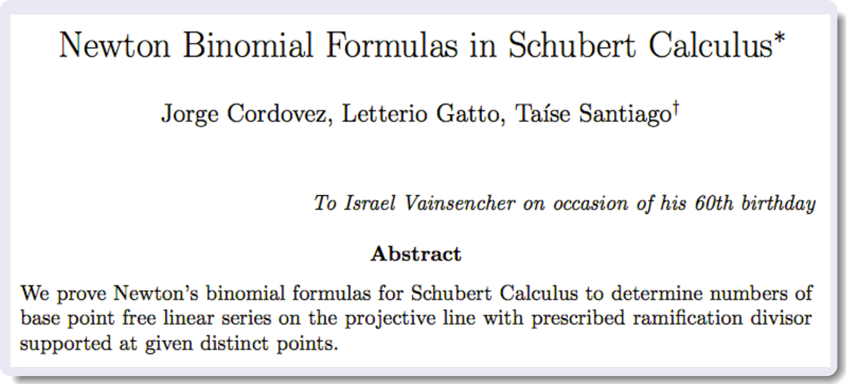
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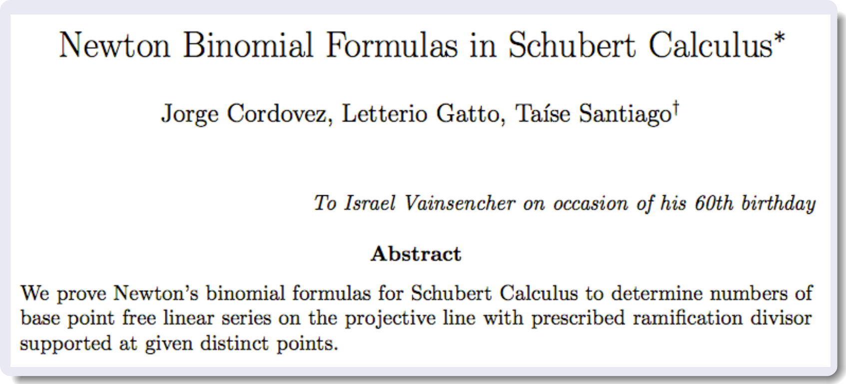
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