Introduction

The story told in these lecture notes begins with a rather simple observation: all the solutions of a linear ODE with constant complex coefficients are analytic, i.e. they can be expressed in terms of convergent power series. It is then natural to suspect that the corresponding theory can be carried out in a purely formal way, working in rings of formal power series with coefficients in an arbitrary $\mathbb{Q}$-algebra. This is indeed the case and, pursuing the task, one easily obtains a simple, economical and elegant theory which offers both practical advantages and a novel perspective for interpreting other mathematical phenomena.

One of the most relevant features of the theory is that it comes with a *universal basis of solutions* for linear homogeneous ODEs of order, say, $r+1$ (Chapter 3). The elements of the universal basis are formal power series with coefficients in the polynomial ring $E_r := \mathbb{Q}[e_1, \ldots, e_{r+1}]$, where the indeterminates $e_1, \ldots, e_{r+1}$ are the coefficients of the equation. For the reader convenience, basics on formal power series, exterior algebra and the philosophy of generating functions, through the well known example of generating functions, are collected in Chapter 1 in order to keep the exposition as self contained as possible.

A linear ODE of order $r+1$ with coefficients $a_1, \ldots, a_{r+1}$ taken in any $\mathbb{Q}$-algebra A (for instance $A = \mathbb{R}$ or $A = \mathbb{C}$) induces on A a natural structure of E_r-algebra, and the module of solutions to the equation is nothing else than the module of universal solutions after extending the coefficients. In down–to–earth, yet suggestive, terms this amounts to solve all the linear ODEs at once, and once and for all.

The idea of solving linear ODEs using power series, of course, is not new, and is taught in any standard calculus textbook – see e.g. [2, pp. 169–172]. The subject of the present notes is also obviously related with linear recurrence sequences, see e.g [1, Section 212]. The cutting–edge aspect is the implementation of it, which is based on a purely algebraic language and some combinatorics inspired by the theory of symmetric functions. In the present context, in particular, the knowledge of the roots of the characteristic polynomial is no longer necessary for solving a linear ODE. As a matter of fact, standard bases of solutions constructed via the exponential of the roots of the characteristic polynomials are not as canonical as the aforementioned universal ones – see Example 3.2.6. The latter, in addition, reveal themselves especially useful for computing the exponential of a square matrix without reducing it to Jordan normal form (Chapter 5), thus completing an observation made by Putzer [32] in 1966 (see also [2, p. 205] and relatively more recently by Leonard [28] (1996) and Liz [26] (1998).

The motivations for investigating, jointly with I. Scherbak, the combinatorics behind the *universal ODE*, come from Schubert calculus for Grassmannians, which can be thought of as the generalization of the classical Bézout theorem[4], widely known for projective spaces, to more general Grassmann varieties $G(r, \mathbb{P}^d)$ that parameterize r-dimensional linear subvarieties of the d-dimensional projective space. In [11, 12, 14] Schubert calculus was dealt with in terms of *derivations on a Grassmann algebra*. The formalism indicates a kinship with *generalized Wronskians* (Chapter 4), associated to a basis of solutions of an ordinary ODE, and their derivatives (see also [13]). The main result of [15] is a kind of Giambelli-Jacobi-Trudy formula for *generalized Wronskians* (Section 4.4). It shows that, from a formal point of view, the celebrated Pieri's formula that governs Schubert Calculus is nothing but Leibniz's rule for suitable derivatives of a generalized Wronskian. The proof of such Jacobi-Trudy formula forces to look at the most general linear ODE, which eventually led us to find, or possibly rediscover[5], the universal basis of solutions alluded above. The universal solution of the Cauchy problem for a (in general non homogeneous, like in Section 3.4) linear ODE with constant coefficients (has a number of consequences, besides those already mentioned.

For example, it shows that many properties of the matrix exponential are purely formal and hold for square matrices with entries in any commutative ring. If, in addition, the latter is an integral domain one can easily prove that the determinant of the exponential of a square matrix is equal to the exponential of its trace. Using this property, we show in Example 5.4.5 an amusing generalization of the celebrated fundamental trigonometric identity $\cos^2 t + \sin^2 t = 1$.

In a second instance one (re)discovers in a natural way a formal Laplace transform defined on $A[[t]]$, which amounts to multiplying the coefficients of t^n of a formal power series by the factorial $n!$ (see Sections 1.4.2 and 3.3). Combinatorial properties of generalized Wronskians associated to a universal fundamental sys-

[4] Bézout's theorem is best known for the projective plane $\mathbb{P}^2$. It says that if C_1 and C_2 are two projective curves of degree d_1 and d_2 respectively, with no component in common, then they intersect at $d_1 d_2$ points, keeping intersection multiplicity into account.

[5] We do not know any explicit reference for this.

tem, following [15] and [16], as well as their relationships with Schubert calculus and derivations of a Grassmann algebra are also briefly discussed in Chapter 4. One shows that wedging altogether the elements of a universal fundamental system is the same as considering the Wronskian of it. The universal Cauchy formula (3.11), that gives the explicit expression of the unique solution to a linear ODE with given initial data, is a consequence of a purely combinatorial property which exhibits an alternative basis of the ring $A[[t]]$ of formal power series in the indeterminate t (Chapter 2). Such combinatorial property leads in a very natural (and probably unavoidable) way to consider linear ODEs of infinite order. These possess a universal basis of solutions, whose elements are indexed by negative integers. The algebra E_r must be replaced in this case by the algebra of polynomials $E_\infty := \mathbb{Q}[e_1, e_2, \ldots]$ in infinitely many indeterminates. The latter will be interpreted in Chapter 7 as the *Fock space* of the theory of representations of infinite dimensional Lie algebras (Oscillator Algebra, Virasoro algebra), which in turn is isomorphic to each *fermion space* of total charge m: the latter can be identified with the $\mathbb{Q}$-algebra generated by certain infinite wedge products of solutions of the linear ODE of infinite order. The very natural isomorphism one obtains in this way, based on the universal Cauchy formula for infinite–order linear ODEs, is nothing but the so-called *boson-fermion* correspondence, as described for example, in the introductory book [19] – see also [3, 27, 30].