Short Course on

Schubert Calculus on Grassmannians and Related Topics

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Abstract

Schubert Calculus is the formalism governing the product structure of the cohomology ring (or Chow intersection ring) of the complex Grassmannian $G(k,n)$ parameterizing k-planes in $\mathbf{C}^n$.

One of the main goals of the short course is to let the audience becoming quickly familiar with Schubert Calculus. This is possible thanks to the introduction of a certain natural *Schubert derivation* on a Grassmann algebra, which is indeed the prototype of a vertex operator used to study the Boson-Fermion correspondence arising in the representation theory of the Heisenberg algebra.

It turns out that the fermionic representation of the latter (which will be introduced in due course) can be seen as a *Wronskian representation*, associated to an infinite-order linear ODE with constant coefficients. Indeed, the formalism of Schubert Calculus can be entirely recovered by means of the natural operation of differentiating the Wronskians of a linear ODE (of finite order) with constant coefficients. If we rephrase Schubert Calculus for Grassmannians in terms of derivatives of generalized Wronskians associated to linear ODEs with constant coefficients, we obtain the "finite-dimensional" version of the Bose-Fermi correspondence, which is nothing but Poincaré's duality between the homology and cohomology of the Grassmannian.

Tentative plan of the lectures:

1. Overview of the course, contents and aims. Schubert Calculus for Grassmannians as generalization of Bézout's theorem. Derivations on a Grassmann algebra: Pieri's formula is Leibniz's rule, Giambelli's formula is integration by parts.

2. Examples of how Schubert Calculus on a Grassman algebra works: the Plücker degree of the Grassmannian of lines and Catalan numbers, the Plücker degree of a Grassmannian in general. Newton's binomial formulas for Schubert Calculus. Numbers of rational curves in $\mathbf{P}^3$ with prescribed flexes or hyperstalls.

3. Linear ODEs with constant coefficients revisited; generic linear ODEs, universal Cauchy theorem on existence and uniqueness for linear ODEs; formal Laplace transform and properties. Generalized

Wronskians associated to the kernel of the generic differential operator. The Jacobi-Trudy formula for generalized Wronskians and relationships to Schubert Calculus: Schubert cycles are Wronskians associated to the module of solutions of a linear ODEs with coefficients in the cohomology ring of the Grassmannian.

4. Linear ODEs (with constant coefficients) of infinite order. Semi-infinite exterior powers of the solution space of a linear ODE of infinite order. Bosonic and fermionic Fock spaces. Fermionic and bosonic representation of the Heisenberg algebra. Each solution of a linear ODE of infinite order is a solution of the Kadomtsev–Petviashvili (KP) equation. Explaining why one says that the KP equation in Hirota bilinear form expresses the equations of an infinite-dimensional Grassmannian's Plücker embedding.

5. Vertex Operators arise from linear ODEs with constant coefficients (already for equations of first order). Description and computations with vertex operators. The Boson-Fermion correspondence is the infinite-dimensional version of Giambelli's formula (or the Jacobi-Trudy formula in the theory of symmetric functions).

References

The short course is partly based on joint work with Taíse Santiago (UFBA, Salvador de Bahia, Brasil), Inna Scherbak (Tel Aviv, Israel) and Parham Salehyan (UNESP, São José do Rio Preto, Brazil).

The main reference on Schubert Calculus on the Grassmann algebra is:

1. L. Gatto, *Schubert Calculus: an Algebraic Introduction*, Publicações Matemáticas do IMPA, 25º Colóquio Brasileiro de Matemática, Instituto Nacional de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2005.

See also

2. L. Gatto, *Schubert Calculus via Hasse–Schmidt Derivations*, Asian J. Math. **9**, No. 3, 315–322, (2005).

3. L. Gatto, T. Santiago, *Schubert calculus on a Grassmann algebra*, Canad. Math. Bull. **52** (2009), no. 2, 200–212.

Some applications to Enumerative Geometry can be found in

4. J. Cordovez, L. Gatto, T. sSantiago, *Newton binomial formulas in Schubert calculus*, Rev. Mat. Complut. **22** (2009), no. 1, 129–152

5. L. Gatto, P. Salehyan, *Families of special Weierstrass points*, C. R. Math. Acad. Sci. Paris **347** (2009), no. 21–22, 1295–1298

The second part of the is inspired by selected chapters of the beautiful books

6. V. G. Kac, *Vertex Algebras for Beginners*, University Lecture Series, Vol. **10**, AMS, 1996.

7. V. G. Kac, A. K. Raina, *Highest Weight Representations of Infinite Dimensional Lie Algebras*, Advanced Studies in Mathematical Physics, Vol. **2**, World Scientific, 1987;

through the coloured lenses provided by the theory of linear ODEs with constant coefficients, as explained in

8. L. Gatto, I. Scherbak, *Linear ODEs, Wronskian and Schubert Calculus*, Moscow Mat. J., **12**, no. 2, (2012), 275–291.

9. L. Gatto, I. Scherbak, *On generalized Wronskians,* in "Contribution in Algebraic Geometry", Impanga Lecture Notes (P. Pragacz Ed.), EMS Congress Series Report, 257–296, 2012.

An expanded and fully-fledged treatment of the topics in the second part – with an eye to the Boson-Fermion correspondence – can be found in

10. L. Gatto, *Linear ODE: an Algebraic Perspective*, Lecture notes from a short course given at XXII Escola de Algebra, Salvador de Bahia, Publicações Mathematicas do IMPA, Vol. **40**, 2012.

Pre-requisites. The course will be taught at an elementary level. The only important requirement is a (solid) background in linear algebra. Attendees are expected to be familiar with the exterior algebra, the basic theory of linear ODEs with constant coefficients, and the statement of Bézout's theorem.