

Chapter 3

Elements of Analytic Geometry

Recall that every $\mathbf{u} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ determines a *direction*, that is, the set

$$[\mathbf{u}] := \{\lambda \cdot \mathbf{u} \mid \lambda \in \mathbb{R}\},$$

of all multiples of $\mathbf{u}$. It could equivalently be denoted by the notation $\mathbb{R} \cdot \mathbf{u}$ (all real multiples of the vector $\mathbf{u}$).

3.1 Affine lines and planes in $\mathcal{E}^n$

We begin immediately with a definition.

3.1.1 Definition. Let $n \geq 1$, $P_0 \in \mathcal{E}^n$ and $\mathbf{u} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$. The affine line passing through P_0 having direction $[\mathbf{u}]$ ¹ is the set:

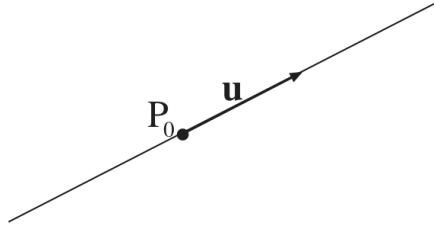
$$P_0 + [\mathbf{u}] := \{P_0 + \mathbf{u}t \mid t \in \mathbb{R}\} \subseteq \mathcal{E}^n. \quad (3.1)$$

In other words, $P \in P_0 + [\mathbf{u}]$ if and only if $\overrightarrow{P_0P} = \mathbf{u}t$ for some $t \in \mathbb{R}$, that is, if and only if the vector $\overrightarrow{P_0P}$ belongs to the direction determined by $\mathbf{u}$. Formally:

$$P \in P_0 + [\mathbf{u}] \iff \overrightarrow{P_0P} \in [\mathbf{u}] \iff \overrightarrow{P_0P} = \mathbf{u}t.$$

Interpreting kinematically the parameter t in (3.1) as *time*, an affine line may be viewed as a model of a *uniform rectilinear motion* that passes through P_0 at the instant $t = 0$ with constant velocity $\mathbf{u}$.

¹One often says, improperly but, according to the author, less precisely, a line parallel to $[\mathbf{u}]$.



3.1.2 Definition. Let $P_1, P_2 \in \mathcal{E}^n$ be distinct. The affine line passing through P_1 and P_2 is the set of points:

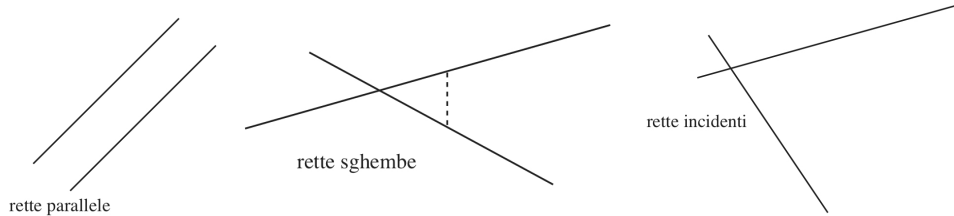
$$P_1 + [\overrightarrow{P_1 P_2}] := \{P_1 + \overrightarrow{P_1 P_2} t \mid t \in \mathbb{R}\}.$$

In other words, $P \in P_1 + [\overrightarrow{P_1 P_2}]$ if and only if $\overrightarrow{P_1 P} \in [\overrightarrow{P_1 P_2}]$, that is, if and only if $\overrightarrow{P_1 P} = \overrightarrow{P_1 P_2} \cdot t$, for some $t \in \mathbb{R}$.

3.2 Parallel, orthogonal, intersecting lines

Let two affine lines $P_1 + [\mathbf{u}]$ and $P_2 + [\mathbf{v}]$ be given. They are said to be

- p) *parallel* if $[\mathbf{u}] = [\mathbf{v}]$;
- i) *intersecting* if $(P_1 + [\mathbf{u}]) \cap (P_2 + [\mathbf{v}]) \neq \emptyset$, that is, if there exists $Q \in \mathcal{E}^n$ such that $\overrightarrow{P_1 Q} \in [\mathbf{u}]$ and $\overrightarrow{P_2 Q} \in [\mathbf{v}]$;
- o) *orthogonal* if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$;
- c) *coincident* if they are parallel and intersecting;
- s) *skew* if they have empty intersection and are not parallel.



3.2.1 Three points $A, B, C \in \mathcal{E}^n$ are said to be *aligned* if and only if $[\overrightarrow{AB}] = [\overrightarrow{AC}]$, that is, if and only if the vector $\overrightarrow{AC}$ is collinear with $\overrightarrow{AB}$. This is equivalent to saying that the point C belongs to the affine line passing through A and B .

3.3 Affine planes

For every pair of vectors $\mathbf{u}, \mathbf{v} \in \mathcal{E}^n$, the set

$$[\mathbf{u}, \mathbf{v}] := \{\lambda \mathbf{u} + \mu \mathbf{v} \mid \lambda, \mu \in \mathbb{R}\}$$

had been introduced, also called a *vector plane*. An *affine plane* in $\mathcal{E}_n$ is obtained by “attaching” a vector plane to a point. More precisely:

3.3.1 Definition. Let $P_0 \in \mathcal{E}^n$ and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ be non-collinear vectors. The affine plane passing through P_0 and generated by (the vector plane determined by) $\mathbf{u}, \mathbf{v}$ is the set of points

$$P_0 + [\mathbf{u}, \mathbf{v}] := \{P_0 + \lambda \mathbf{u} + \mu \mathbf{v} \mid \lambda, \mu \in \mathbb{R}\}. \quad (3.2)$$

In other words, $P \in P_0 + [\mathbf{u}, \mathbf{v}] \iff \overrightarrow{P_0 P} \in [\mathbf{u}, \mathbf{v}] \iff$ there exist $\lambda, \mu \in \mathbb{R}$ such that $\overrightarrow{P_0 P} = \lambda \mathbf{u} + \mu \mathbf{v}$.

3.3.2 Definition. Let $P_1, P_2, P_3 \in \mathcal{E}^n$ be non-aligned. The affine plane generated by P_1, P_2, P_3 is the set of points

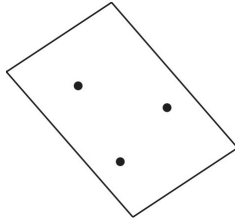
$$P_1 + [\overrightarrow{P_1 P_2}, \overrightarrow{P_1 P_3}] := \{P_1 + \lambda \overrightarrow{P_1 P_2} + \mu \overrightarrow{P_1 P_3} \mid \lambda, \mu \in \mathbb{R}\}.$$

In other words

$$P \in P_1 + [\overrightarrow{P_1 P_2}, \overrightarrow{P_1 P_3}] \iff \overrightarrow{P_1 P} \in [\overrightarrow{P_1 P_2}, \overrightarrow{P_1 P_3}] \iff \exists \lambda, \mu \in \mathbb{R} \mid \overrightarrow{P_1 P} = \lambda \overrightarrow{P_1 P_2} + \mu \overrightarrow{P_1 P_3}.$$

3.3.3 Definition. A subset $\Pi \subseteq \mathcal{E}^n$ is called a plane if and only if for every triple (P_1, P_2, P_3) of non-aligned points one has

$$P \in \Pi \iff \overrightarrow{P_1 P} \in [\overrightarrow{P_1 P_2}, \overrightarrow{P_1 P_3}].$$



-- plane through three points --

3.4 Axes, medians, and altitudes of a triangle

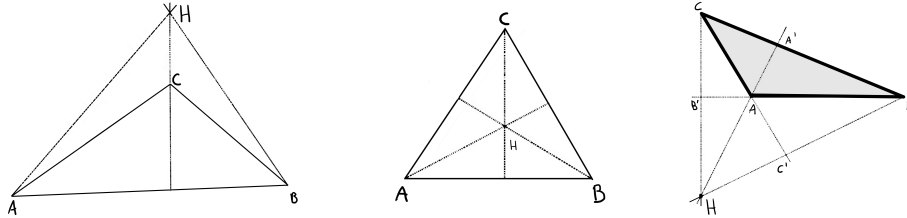
Let $A, B \in \mathcal{E}^n$. A point $M \in \mathcal{E}^n$ is called the *midpoint* of the segment A, B if $M = A + \frac{1}{2}\overrightarrow{AB}$ (equivalently $M = B + \frac{1}{2}\overrightarrow{BA}$). The midpoint exists and is unique. Let M_C, M_B and M_A be the midpoints of the sides $[A, B], [A, C], [B, C]$ of a triangle $[A, B, C]$ ². Then the segments $[C, M_C], [B, M_B], [A, M_A]$ are called the *medians* of the triangle ABC relative, respectively, to its sides $[A, B], [A, C]$ and $[B, C]$. The point G such that $\overrightarrow{AG} + \overrightarrow{BG} + \overrightarrow{CG} = \mathbf{0}$ is called the *barycenter* of the triangle. The *perpendicular bisectors* are the perpendicular bisectors of the segments $[A, B], [B, C], [A, C]$, that is, the lines through M_C, M_B, M_A orthogonal respectively to $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{BC}$. The three perpendicular bisectors have a common intersection, called the *circumcenter*: it is equidistant from A, B, C (Cf. Paragraph 3.10). We now come to other remarkable segments of a triangle, to which we shall devote special attention, the *altitudes*.

3.4.1 The orthocenter and Jacobi's identity. Since every triple of points of $\mathcal{E}^n$ is contained in at least one three-dimensional affine subspace, it is no restriction to consider triangles in $\mathcal{E}^3$ (possibly thought of as a suitable “slice” of $\mathcal{E}^n$). Let then A, B, C be the vertices of an arbitrary triangle $[A, B, C]$ in $\mathcal{E}^3$. The altitude relative to the segment $[A, B]$ is defined to be the segment $[C, C']$ where $C' \in A + \overrightarrow{AB}$ and $\langle \overrightarrow{CC'}, \overrightarrow{AB} \rangle = 0$. Note that C' might not belong to the segment $[A, B]$, although it belongs to the affine line passing through A and B . The altitudes relative to the side $[B, C]$ and to the side $[A, C]$ are defined in the same way.

Note that the cross product

$$(\overrightarrow{AC} \times \overrightarrow{BC}) \times \overrightarrow{AB}$$

is a vector orthogonal to $\overrightarrow{AB}$ and, therefore, the affine line $C + [(\overrightarrow{AC} \times \overrightarrow{BC}) \times \overrightarrow{AB}]$ contains the altitude relative to the side $[A, B]$.



²The author apologizes to female and male readers for the multiple use of square brackets. There should, however, be no risk of confusion. The expression $[A, B, C]$ means triangle with vertices A, B, C as in ??; $[A, B]$ means oriented segment with endpoints A and B (??), while $\overrightarrow{AB}$ is the *direction*(??) of the unique vector of $\mathbb{R}^3$ that translates the point A to the point B .

In the first triangle, the intersection H is not inside the triangle; in the second H is inside; in the third H is outside, as are also the points A' , B' and C' , the *feet* of the altitudes.

Let now

$$H = (C + [(\overrightarrow{AC} \times \overrightarrow{BC}) \times \overrightarrow{AB}]) \cap (B + [(\overrightarrow{AB} \times \overrightarrow{BC}) \times \overrightarrow{AC}]),$$

the intersection point of the lines containing, respectively, the altitude relative to the side $[A, B]$ and the one relative to the side $[A, C]$. Clearly the vector $\overrightarrow{HC}$ is orthogonal (by definition of altitude) to the vector $\overrightarrow{AB}$ (that is $\langle \overrightarrow{HC}, \overrightarrow{AB} \rangle = 0$), just as $\overrightarrow{BH}$ is orthogonal to the vector $\overrightarrow{AC}$. The altitude relative to the side $[B, C]$ is contained in the line $A + [(\overrightarrow{AB} \times \overrightarrow{AC}) \times \overrightarrow{BC}]$. Therefore H belongs to the altitude relative to the side $\overrightarrow{BC}$ if and only if the vector $\overrightarrow{AH}$ is orthogonal to the vector $\overrightarrow{BC}$, in which case the three altitudes would all meet (as indeed they do) at the point H .

The purpose of this paragraph is to provide a proof of the celebrated statement due to the great Russian mathematician Vladimir I. Arnold [?]:

The three altitudes of a triangle have a point in common if and only if Jacobi's identity (3.3) holds.

$$Jac(A, B, C) := (\overrightarrow{AB} \times \overrightarrow{BC}) \times \overrightarrow{AC} + (\overrightarrow{BC} \times \overrightarrow{AC}) \times \overrightarrow{AB} + (\overrightarrow{AC} \times \overrightarrow{AB}) \times \overrightarrow{BC} = 0 \quad (3.3)$$

which is the same one already introduced in Chapter ??, applied to the vectors $\mathbf{u} = \overrightarrow{AB}$, $\mathbf{v} = \overrightarrow{BC}$ and $\mathbf{w} = \overrightarrow{AC}$. Here $Jac(A, B, C)$ will serve as an abbreviation for the left-hand side of (3.3). We shall therefore prove the following

3.4.2 Proposition. *The equality*

$$Jac(A, B, C) = \langle \overrightarrow{AH}, \overrightarrow{BC} \rangle \overrightarrow{CB}$$

holds. Therefore H belongs to the altitude relative to the side $[B, C]$ if and only if Jacobi's identity $Jac(A, B, C) = 0$ holds. Equivalently, the three altitudes of a triangle in three-dimensional Euclidean affine space have a point in common if and only if Jacobi's identity holds.

Proof. Let us begin by applying formula (??) to the vectors $\mathbf{u} = \overrightarrow{AB}$, $\mathbf{v} = \overrightarrow{BC}$ and $\mathbf{w} = \overrightarrow{AC}$:

$$(\overrightarrow{AB} \times \overrightarrow{BC}) \times \overrightarrow{AC} = \langle \overrightarrow{AB}, \overrightarrow{AC} \rangle \overrightarrow{BC} - \langle \overrightarrow{BC}, \overrightarrow{AC} \rangle \overrightarrow{AB}. \quad (3.4)$$

Similarly:

$$(\overrightarrow{BC} \times \overrightarrow{AC}) \times \overrightarrow{AB} = \langle \overrightarrow{BC}, \overrightarrow{AB} \rangle \overrightarrow{AC} - \langle \overrightarrow{AC}, \overrightarrow{AB} \rangle \overrightarrow{BC}. \quad (3.5)$$

The sum of the first two addends of $Jac(A, B, C)$ is, therefore:

$$\begin{aligned} (\overrightarrow{AB} \times \overrightarrow{BC}) \times \overrightarrow{AC} + (\overrightarrow{BC} \times \overrightarrow{AC}) \times \overrightarrow{AB} &= \langle \overrightarrow{AB}, \overrightarrow{AC} \rangle \overrightarrow{BC} - \langle \overrightarrow{BC}, \overrightarrow{AC} \rangle \overrightarrow{AB} \\ &+ \langle \overrightarrow{BC}, \overrightarrow{AB} \rangle \overrightarrow{AC} - \langle \overrightarrow{AC}, \overrightarrow{AB} \rangle \overrightarrow{BC} \\ &= \langle \overrightarrow{BC}, \overrightarrow{AB} \rangle \overrightarrow{AC} - \langle \overrightarrow{BC}, \overrightarrow{AC} \rangle \overrightarrow{AB} \end{aligned} \quad (3.6)$$

where symmetry of the inner product has been used to cancel the addend $\langle \overrightarrow{AB}, \overrightarrow{AC} \rangle \overrightarrow{BC}$ with the addend $-\langle \overrightarrow{AB}, \overrightarrow{AC} \rangle \overrightarrow{BC}$. We now use Chasles' relation (?? af1)) $\overrightarrow{BC} = \overrightarrow{BH} + \overrightarrow{HC}$ together with the fact that $\overrightarrow{CH}$ and $\overrightarrow{BH}$, by hypothesis, are orthogonal, respectively, to $\overrightarrow{AB}$ and $\overrightarrow{AC}$. One obtains

$$(\overrightarrow{AB} \times \overrightarrow{BC}) \times \overrightarrow{AC} + (\overrightarrow{BC} \times \overrightarrow{AC}) \times \overrightarrow{AB} = \langle \overrightarrow{BH}, \overrightarrow{AB} \rangle \overrightarrow{AC} - \langle \overrightarrow{HC}, \overrightarrow{AC} \rangle \overrightarrow{AB} \quad (3.7)$$

We shall make explicit in the same way the third addend of $Jac(A, B, C)$, using Chasles' relation so as to highlight the contributions of the vectors that translate the vertices of the triangle to the point H :

$$\begin{aligned} (\overrightarrow{AC} \times \overrightarrow{AB}) \times \overrightarrow{BC} &= \langle \overrightarrow{AC}, \overrightarrow{BC} \rangle \overrightarrow{AB} - \langle \overrightarrow{AB}, \overrightarrow{BC} \rangle \overrightarrow{AC} \\ &= \langle \overrightarrow{AH} + \overrightarrow{HC}, \overrightarrow{BC} \rangle \overrightarrow{AB} - \langle \overrightarrow{AH} + \overrightarrow{HB}, \overrightarrow{BC} \rangle \overrightarrow{AC} \\ &= \langle \overrightarrow{AH}, \overrightarrow{BC} \rangle \overrightarrow{AB} + \langle \overrightarrow{HC}, \overrightarrow{BC} \rangle \overrightarrow{AB} - \langle \overrightarrow{AH}, \overrightarrow{BC} \rangle \overrightarrow{AC} - \langle \overrightarrow{HB}, \overrightarrow{BC} \rangle \overrightarrow{AC} \end{aligned} \quad (3.8)$$

In conclusion, summing, one has

$$\begin{aligned} Jac(A, B, C) &= \langle \overrightarrow{BH}, \overrightarrow{AB} \rangle \overrightarrow{AC} - \langle \overrightarrow{HC}, \overrightarrow{AC} \rangle \overrightarrow{AB} + \langle \overrightarrow{AH}, \overrightarrow{BC} \rangle \overrightarrow{AB} \\ &+ \langle \overrightarrow{HC}, \overrightarrow{BC} \rangle \overrightarrow{AB} - \langle \overrightarrow{AH}, \overrightarrow{BC} \rangle \overrightarrow{AC} - \langle \overrightarrow{HB}, \overrightarrow{BC} \rangle \overrightarrow{AC} \end{aligned}$$

from which, rearranging the addends

$$= \langle \overrightarrow{BH}, \overrightarrow{AB} \rangle \overrightarrow{AC} - \langle \overrightarrow{HB}, \overrightarrow{BC} \rangle \overrightarrow{AC} - \langle \overrightarrow{HC}, \overrightarrow{AC} \rangle \overrightarrow{AB} + \langle \overrightarrow{HC}, \overrightarrow{BC} \rangle \overrightarrow{AB}$$

$$+ \langle \overrightarrow{AH}, \overrightarrow{BC} \rangle \overrightarrow{AB} - \langle \overrightarrow{AH}, \overrightarrow{BC} \rangle \overrightarrow{AC}. \quad (3.9)$$

The sum (3.9) can now be rewritten using the fact that $\overrightarrow{PQ} = -\overrightarrow{QP}$, for every $P, Q \in \mathcal{E}^3$, the linearity of the inner product, and Chasles' relation ?? af1), obtaining

$$\begin{aligned} & \langle \overrightarrow{BH}, \overrightarrow{AB} + \overrightarrow{BC} \rangle \overrightarrow{AC} + \langle \overrightarrow{HC}, \overrightarrow{CA} + \overrightarrow{BC} \rangle \overrightarrow{AB} + \langle \overrightarrow{AH}, \overrightarrow{BC} \rangle (\overrightarrow{AB} + \overrightarrow{CA}) \\ &= \langle \overrightarrow{BH}, \overrightarrow{AC} \rangle \overrightarrow{AC} + \langle \overrightarrow{AH}, \overrightarrow{BC} \rangle \overrightarrow{CB} + \langle \overrightarrow{CH}, \overrightarrow{AB} \rangle \overrightarrow{AB} \end{aligned} \quad (3.10)$$

Since $\overrightarrow{BH}$ and $\overrightarrow{CH}$ are orthogonal, respectively, to $\overrightarrow{AC}$ and $\overrightarrow{AB}$, one has proved that

$$Jac(A, B, C) = \langle \overrightarrow{AH}, \overrightarrow{BC} \rangle \overrightarrow{CB}.$$

Therefore Jacobi's identity is satisfied if and only if $\overrightarrow{AH}$ is orthogonal to $\overrightarrow{BC}$, that is, if and only if the third altitude of a triangle passes through the point of intersection of the other two, as was to be proved. ■

3.5 Open balls, closed balls, and spheres of affine space

3.5.1 Definition. Let $R \geq 0$ be a real number and let $P_0 \in \mathcal{E}^n$. Consider the three sets:

$$\mathcal{B}_{P_0}(R) : = \{P \in \mathcal{E}^n \mid d(P_0, P) < R\}$$

$$\overline{\mathcal{B}}_{P_0}(R) : = \{P \in \mathcal{E}^n \mid d(P_0, P) \leq R\}$$

$$S_{P_0}^n(R) : = \{P \in \mathcal{E}^{n+1} \mid d(P_0, P) = R\}$$

The set $\mathcal{B}_{P_0}(R)$ is called the open ball of center P_0 and radius R ; the set $\overline{\mathcal{B}}_{P_0}(R)$ is called the closed ball of center P_0 and radius R ; finally, the set $S_{P_0}^n(R)$ is called the n -dimensional Euclidean sphere of center P_0 and radius R .

See Chapter ?? for further details.

A few three-dimensional Euclidean balls...

If $R = 0$, it is easily seen that: $\mathcal{B}_{P_0}(0) = \emptyset$ while

$$\overline{\mathcal{B}}_{P_0}(0) = S_{P_0}^n(0) = \{P_0\}.$$

in particular the empty set is an open ball of $\mathcal{E}^n$, for every $n \geq 1$ (of radius 0), and every point $P_0 \in \mathcal{E}^n$ may be viewed either as a closed ball or as a sphere of center P_0 and radius zero.

3.5.2 Esempio. The open ball of center $P_0 \in \mathcal{E}^1$ is also called an *open interval*. It can be written as:

$$\mathcal{B}_{P_0}(R) = (P_0 - R, P_0 + R)$$

which is the set of all points of $\mathcal{E}^1$ of the form $P_0 + u$ where $-R < u < R$.

The closed ball is the closed interval

$$[P_0 - R, P_0 + R] := \{P_0 + u \mid -R \leq u \leq R\}.$$

The sphere $S_{P_0}^0(R)$ of $\mathcal{E}^1$ is a set made up of only two points:

$$\{P_0 - R, P_0 + R\}.$$

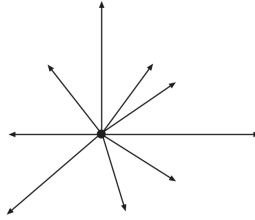
(Observe that in the expression $P_0 \pm R$, the “sum” is between a point and a real number thought of as a vector of $\mathbb{R}^1$).

3.6 Parametric and Cartesian equations of the line and the affine plane

3.6.1 Recall that $\mathbb{R}^n$ has the canonical basis $E_n := (\mathbf{e}_1, \dots, \mathbf{e}_n)$. A standard Cartesian frame is of the form (O, E_n) where $O \in \mathcal{E}^n$ is an arbitrarily chosen point, called from now on the *origin of the frame* $\mathcal{R}$. In this case the Cartesian coordinates of $P \in \mathcal{E}^n$ in the frame $\mathcal{R}$ are $(x_1(P), \dots, x_n(P))$ if and only if³

$$\overrightarrow{OP}(i) = x_i(P),$$

for every $1 \leq i \leq n$. To understand this better, in the standard Cartesian frame, the coordinates of a point are nothing other than the components of the vector $\overrightarrow{OP}$. This observation, in particular, finally allows us to draw a model of a vector space as a set of “arrows” issuing from the origin of a Cartesian coordinate system.



³By $\overrightarrow{OP}(i)$ one shall mean the i -th component of the vector $\overrightarrow{OP}$.

An affine space may be viewed as a model of a vector space whenever an origin is fixed.
 To every point $P \in \mathcal{E}^n$ one associates the vector that moves O to P .

Let now $P, Q \in \mathcal{E}^n$ be two points. Then

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP},$$

whence $\overrightarrow{PQ}(i) = \overrightarrow{OQ}(i) - \overrightarrow{OP}(i) = x_i(Q) - x_i(P)$. More explicitly:

$$\overrightarrow{PQ} = \begin{pmatrix} x_1(Q) - x_1(P) \\ x_2(Q) - x_2(P) \\ \vdots \\ x_n(Q) - x_n(P) \end{pmatrix}.$$

If instead P is a point of $\mathcal{E}^n$ and $\mathbf{u} \in \mathbb{R}^n$, then

$$Q = P + t\mathbf{u} \iff \overrightarrow{PQ} = t\mathbf{u},$$

that is,

$$x_i(Q) - x_i(P) = t\mathbf{u}(i)$$

for all $1 \leq i \leq n$. The equations:

$$\begin{cases} x_1 &= x_1(P) + t\mathbf{u}(1), \\ x_2 &= x_2(P) + t\mathbf{u}(2), \\ \vdots &\vdots \\ x_n &= x_n(P) + t\mathbf{u}(n), \end{cases} \quad (3.11)$$

are called the parametric equations of the line passing through P and having $\mathbf{u}$ as directional vector, where $(x_1, \dots, x_n)$ denotes the coordinates of a point Q varying on the line $r_{P,\mathbf{u}}$. Interpreting the parameter t as time, equations (3.11) may be interpreted as the parametric equations of a uniform rectilinear motion in $\mathcal{E}^n$ with velocity $\mathbf{u}$.

3.6.2 Cartesian equations for the line $r_{P,\mathbf{u}}$ are obtained from (3.11) by eliminating the parameter t , found from any one of the equations as a function of one coordinate, and substituting the expression thus obtained into the other equations. For example, if $\mathbf{u}(1) \neq 0$, one would have, for every $2 \leq i \leq n$:

$$\mathbf{u}(1)x_i - \mathbf{u}(i)x_1 + x_1(P)\mathbf{u}(i) - x_i(P)\mathbf{u}(1) = 0 \quad 2 \leq i \leq n \quad (3.12)$$

which are Cartesian equations for the line $r_{P,\mathbf{u}}$. The reader will verify that the Cartesian equations (3.12) are obtained by imposing that

$$\begin{vmatrix} x_i(Q) - x_i(P) & \mathbf{u}(i) \\ x_j(Q) - x_j(P) & \mathbf{u}(j) \end{vmatrix} = 0, \quad \text{for every } 1 \leq i < j \leq n.$$

3.6.3 Similarly, the *plane* passing through $P_0 \in \mathcal{E}^n$ and *generated by* the vectors $\mathbf{u}, \mathbf{v}$ has parametric equations

$$\begin{cases} x_1 = x_1(P) + \lambda \mathbf{u}(1) + \mu \mathbf{v}(1), \\ x_2 = x_2(P) + \lambda \mathbf{u}(2) + \mu \mathbf{v}(2), \\ \vdots \\ x_n = x_n(P) + \lambda \mathbf{u}(n) + \mu \mathbf{v}(n), \end{cases} \quad (3.13)$$

where $(x_1, \dots, x_n)$ indicate the coordinates of a point Q varying on $\Pi_{P,\mathbf{u},\mathbf{v}}$. Since by hypothesis $[\mathbf{u}] \neq [\mathbf{v}]$, there exists at least one determinant

$$\begin{vmatrix} \mathbf{u}(i) & \mathbf{v}(i) \\ \mathbf{u}(j) & \mathbf{v}(j) \end{vmatrix} \neq 0.$$

If, for example, this were the one for which $i = 1$ and $j = 2$, the system:

$$\begin{cases} \lambda \mathbf{u}(1) + \mu \mathbf{v}(1) = x_1 - x_1(P) \\ \lambda \mathbf{u}(2) + \mu \mathbf{v}(2) = x_2 - x_2(P) \end{cases}$$

has the unique solution (λ, μ) given by:

$$\begin{cases} \lambda = \frac{(x_1 - x_1(P))\mathbf{v}(2) - (x_2 - x_2(P))\mathbf{v}(1)}{\mathbf{u}(1)\mathbf{v}(2) - \mathbf{u}(2)\mathbf{v}(1)}, \\ \mu = \frac{\mathbf{u}(1)(x_2 - x_2(P)) - \mathbf{u}(2)(x_1 - x_1(P))}{\mathbf{u}(1)\mathbf{v}(2) - \mathbf{u}(2)\mathbf{v}(1)}. \end{cases} \quad (3.14)$$

Substituting the right-hand sides of (3.14) into the last $n - 2$ equations of the system (3.13), one thus obtains Cartesian equations for the plane $\Pi_{A,\mathbf{u},\mathbf{v}}$. In other words, a plane in $\mathcal{E}^n$ may be represented by $n - 2$ equations expressing the linear relations among the coordinates of a variable point on the plane. In particular, a plane in $\mathcal{E}^3$ may be represented by one Cartesian equation. Similarly one treats parametric and Cartesian equations of the plane $\Pi_{A,B,C}$ passing through three (non-aligned) points A, B, C , using the fact that

$$\Pi_{A,B,C} = \Pi_{A,\overrightarrow{AB},\overrightarrow{AC}}.$$

3.6.4 Observation. The Cartesian equations of the affine plane passing through three points are precisely the conditions of coplanarity. If A, B, C are not aligned, there exist $1 \leq i < j \leq n$ such that

$$\begin{vmatrix} \overrightarrow{AB}(i) & \overrightarrow{AC}(i) \\ \overrightarrow{AB}(j) & \overrightarrow{AC}(j) \end{vmatrix} \neq 0.$$

and one could prove that

$$\begin{vmatrix} \overrightarrow{AB}(i) & \overrightarrow{AC}(i) & \overrightarrow{AP}(i) \\ \overrightarrow{AB}(j) & \overrightarrow{AC}(j) & \overrightarrow{AP}(j) \\ \overrightarrow{AB}(k) & \overrightarrow{AC}(k) & \overrightarrow{AP}(k) \end{vmatrix} = 0, \quad 1 \leq k \leq n, k \neq i, j$$

are intrinsic equations of the plane in $\mathcal{E}^3$ generated by non-aligned A, B, C .

3.7 Real affine spaces of dimensions 1, 2, 3

In accordance with tradition, the coordinate associated with a standard affine frame of the Euclidean affine line $\mathcal{E}^1$ will be denoted by x ; the affine coordinates of a standard affine frame of the Euclidean plane $\mathcal{E}^2$ will be denoted by (x, y) and those of a standard affine frame of the three-dimensional Euclidean affine space $\mathcal{E}^3$ by (x, y, z) . In all three cases the origin will be understood to be fixed.

3.7.1 The real affine line. The real affine line will be denoted by $\mathcal{E}^1$. The elements of $\mathcal{E}^1$, thought of as a set, are points. The distance between two points A, B is:

$$d(A, B) = |\overrightarrow{AB}|.$$

A Cartesian frame of $\mathcal{E}^1$ is a pair $(O, \mathbf{b})$, where $\mathbf{b} \in \mathbb{R}$ (it is a real number!). If $P \in \mathcal{E}^1$ one says that the coordinate of P is x if and only if $\overrightarrow{OP} = x\mathbf{b}$. To indicate the coordinate of the point P without specifying it, one also writes $x(P)$, thinking of the coordinate as a function

$$x : \mathcal{E}^1 \longrightarrow \mathbb{R}$$

which associates with every P its coordinate in the frame (O, x) . If $\mathbf{b} = 1$, the system is (ortho)normal. If $x(P_1) = x_1$ and $x(P_2) = x_2$, then $d(P_1, P_2) = |\overrightarrow{P_1P_2}| = |x_2 - x_1|$. Although the distance seems to depend on the chosen frame, as has been shown previously, it does not depend on it. If $A, B \in \mathcal{E}^1$, the closed interval $[A, B]$ (also called a segment) is the set of points $P \in \mathcal{E}^1$ such that

$$\overrightarrow{AP} = t\overrightarrow{AB} \tag{3.15}$$

with $0 \leq t \leq 1$. The open interval (A, B) is the one for which $0 < t < 1$. Finally, one has the half-closed intervals $[A, B)$ and $(A, B]$, the locus of all points P satisfying (3.15).

3.7.2 The real Euclidean affine plane $\mathcal{E}^2$. Let (O, x, y) be a Cartesian frame. In other words, (x, y) is a function:

$$(x, y) : \mathcal{E}^2 \longrightarrow \mathbb{R}^2$$

which associates with every $P \in \mathcal{E}^2$ its coordinates $(x, y)(P) = (x(P), y(P))$. We say that $(x(P), y(P)) = (x_P, y_P)$ if and only if $\overrightarrow{OP} = x_P \mathbf{e}_1 + y_P \mathbf{e}_2$, where, as usual,

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The affine line passing through $P_0 = (x_0, y_0)$ and parallel to the vector $\mathbf{u}$, in accordance with (3.1), is given by:

$$\overrightarrow{P_0 P} = t\mathbf{u}$$

If (x, y) denote the coordinates of the variable point P on the line, one has

$$\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = t \begin{pmatrix} \mathbf{u}(1) \\ \mathbf{u}(2) \end{pmatrix} \quad (3.16)$$

whence

$$\begin{cases} x - x_0 = t\mathbf{u}(1) \\ y - y_0 = t\mathbf{u}(2) \end{cases} \iff \begin{cases} x = x_0 + t\mathbf{u}(1) \\ y = y_0 + t\mathbf{u}(2) \end{cases}$$

which is the explicit form of the parametric equations of the line passing through the point P_0 and parallel to the direction determined by $\mathbf{u}$. Equation (3.16) expresses the collinearity of two vectors, which may therefore be expressed by means of the determinant condition:

$$0 = \begin{vmatrix} x - x_0 & \mathbf{u}(1) \\ y - y_0 & \mathbf{u}(2) \end{vmatrix} = \mathbf{u}(2)(x - x_0) - \mathbf{u}(1)(y - y_0)$$

Therefore $\mathbf{u}(2)(x - x_0) - \mathbf{u}(1)(y - y_0) = 0$ is a Cartesian equation of the line through P_0 parallel to $[\mathbf{u}]$. In common textbooks one writes the Cartesian equation of the line in the form

$$a(x - x_0) + b(y - y_0) = 0, \quad (3.17)$$

or, again:

$$ax + by - ax_0 - by_0 = 0.$$

which is an equation⁴ of the line passing through $P_0(x_0, y_0)$ parallel to the direction:

$$\begin{bmatrix} -b \\ a \end{bmatrix}.$$

One also writes:

$$ax + by + c = 0, \quad (3.18)$$

absorbing into the coefficient c the information $-ax_0 - by_0$, which is constant as (x_0, y_0) varies on the same line. Equation (3.18) is indeed a Cartesian equation of a line, as is easily seen by writing it in parametric form. Indeed at least one of the coefficients a and b is nonzero (if both were zero, necessarily $c = 0$ and (3.18) would be the identity $0 = 0$). If $b \neq 0$, we would write:

$$\begin{cases} x = t, \\ y = -\frac{a}{b}t - \frac{c}{b}, \end{cases}$$

that is,

$$\begin{pmatrix} x \\ y + \frac{c}{b} \end{pmatrix} = t \begin{pmatrix} 1 \\ -\frac{a}{b} \end{pmatrix}, \quad (3.19)$$

and the line therefore certainly passes through $P_0 = (0, -c/b)$, obtained by setting $t = 0$ in (3.19). The line (3.17) is orthogonal to the direction $\begin{bmatrix} a \\ b \end{bmatrix}$. Indeed, if (x, y) are the coordinates of any point on the line, the vector $\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}$ is parallel to the direction of the line (3.17). Now:

$$0 = a(x - x_0) + b(y - y_0) = (a, b) \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = \left\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} \right\rangle,$$

proving that $(a, b)^T$ is orthogonal to $(x - x_0, y - y_0)^T$. To summarize. In the equation $ax + by + c = 0$ of a line in the plane, the vector $(a, b)^T$ is orthogonal to the line, while the vector $(-b, a)^T$ is parallel to it. Verify once again, and for the sake of emphasis, that

$$\left\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} -b \\ a \end{pmatrix} \right\rangle = (a, b) \cdot \begin{pmatrix} -b \\ a \end{pmatrix} = -ab + ab = 0.$$

⁴Why an equation and not the equation?

3.7.3 Pencils of lines. Thinking of letting $\mathbf{u} := \begin{pmatrix} a \\ b \end{pmatrix}$ vary in the set of all nonzero vectors of $\mathbb{R}^2$, or better, letting $[\mathbf{u}] \in \mathbb{P}^1$ vary, equation (3.17) is in fact the equation of what is called a *pencil of lines*. It is, indeed, the equation of the pencil of lines passing through the point $P_0(x_0, y_0)$. A line belongs to the pencil if the point (x_0, y_0) belongs to it.

3.7.4 Esempio. Let us find the equation of the line passing through the point $(1, 2)$ orthogonal to the line r of Cartesian equation $3x + 2y - 1 = 0$. Observe that $(1, 2) \notin r$. The required line, if it exists, belongs to the pencil of lines passing through the point $(1, 2)$:

$$a(x - 1) + b(y - 2) = 0.$$

It is therefore necessary that $\langle \begin{pmatrix} -b \\ a \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \end{pmatrix} \rangle = 0$, that is $3a + 2b = 0$. It is enough to choose $a = -2$ and $b = 3$. Therefore the required line is

$$-2(x - 1) + 3(y - 2) = 0 \quad \text{that is} \quad 2x - 3y + 4 = 0.$$

3.8 Planes and lines in three-dimensional affine space

3.8.1 Definition. *The Euclidean affine line passing through the point P_0 and parallel to the direction $\mathbf{u}$ is the set:*

$$r_{P_0, \mathbf{u}} := P_0 + \mathbf{u} \cdot t = \{P \in \mathcal{E}^3 \mid \overrightarrow{P_0 P} = t\mathbf{u}, t \in \mathbb{R}\}$$

From equation (3.8.1) one immediately obtains the *parametric equations of the line*. Indeed, if (x, y, z) are the coordinates of a generic point P on the line, equation (3.8.1) may be rewritten as:

$$\begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix} = t \begin{pmatrix} l \\ m \\ n \end{pmatrix} = \begin{pmatrix} lt \\ mt \\ nt \end{pmatrix},$$

where, as is traditional, one sets $l := \mathbf{u}(1)$, $m := \mathbf{u}(2)$ and $n := \mathbf{u}(3)$. Recalling that two vectors are equal if and only if all rows are equal, the equality $\overrightarrow{P_0 P}(i) = \mathbf{u}(i)$, for every $1 \leq i \leq 3$, gives the system

$$\begin{cases} x - x_0 &= lt, \\ y - y_0 &= mt, \\ z - z_0 &= nt, \end{cases}$$

whence again

$$\begin{cases} x = x_0 + lt, \\ y = y_0 + mt, \\ z = z_0 + nt, \end{cases} \quad (3.20)$$

called the parametric equations of the line passing through the point $P_0(x_0, y_0, z_0)$ and parallel to the direction $[\mathbf{u}]$ of the vector $\mathbf{u}$ (or also: generated by the vector $\mathbf{u}$) with components $(l, m, n)^T$. The vector $\mathbf{u}$ is called a *directional vector*, and the components of $\mathbf{u}/|\mathbf{u}|$ are often called the *direction cosines* of the direction determined by $\mathbf{u}$:

$$\frac{\mathbf{u}}{|\mathbf{u}|} = \begin{pmatrix} \frac{l}{\sqrt{l^2 + m^2 + n^2}} \\ \frac{m}{\sqrt{l^2 + m^2 + n^2}} \\ \frac{n}{\sqrt{l^2 + m^2 + n^2}} \end{pmatrix}.$$

To obtain Cartesian equations it is enough to eliminate the parameter t from one of the equations and substitute it into the second and third. For example, if $l \neq 0$, multiplying the other two equations by l , one has:

$$\begin{cases} l(y - y_0) = mlt, \\ l(z - z_0) = nlt, \end{cases}$$

whence, substituting,

$$\begin{cases} l(y - y_0) = m(x - x_0) \\ l(z - z_0) = n(x - x_0) \end{cases}$$

or, again:

$$\begin{cases} ly - mx - ly_0 + mx_0 = 0 \\ lz - nx - lz_0 + nx_0 = 0, \end{cases} \quad (3.21)$$

which is a pair of Cartesian equations for the line given by (3.20). Each of them is the Cartesian equation of a plane. From (3.13), the parametric equations of a plane in $\mathcal{E}^3$ passing through $P_0(x_0, y_0, z_0)$ and generated by $\mathbf{u}$ and $\mathbf{v}$ are of the form:

$$\begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix} = \lambda \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \mu \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

where one has set $\mathbf{u}(i) = u_i$ and $\mathbf{v}(i) = v_i$, for $1 \leq i \leq 3$,

$$\begin{cases} x = x_0 + \lambda u_1 + \mu v_1 \\ y = y_0 + \lambda u_2 + \mu v_2 \\ z = z_0 + \lambda u_3 + \mu v_3 \end{cases} \quad (3.22)$$

while Cartesian equations may be obtained, besides eliminating the two parameters found from a pair of equations and substituting them into the third, through the condition of [coplanarity](#):

$$\Pi_{P_0, \mathbf{u}, \mathbf{v}} := \{P \in \mathcal{E}^3 \mid \overrightarrow{P_0 P} \wedge \mathbf{u} \wedge \mathbf{v} = 0\}, \quad (3.23)$$

an equation that may be written as:

$$\langle \overrightarrow{P_0 P}, \mathbf{u} \times \mathbf{v} \rangle = 0,$$

which expresses the fact that $\Pi_{P_0, \mathbf{u}, \mathbf{v}}$ is also the set of points $P \in \mathcal{E}^3$ such that $\overrightarrow{P_0 P}$ is a vector orthogonal to $\mathbf{u} \times \mathbf{v}$.

3.8.2 Cartesian equations of the plane are obtained by expanding the determinant (3.23):

$$\begin{aligned} 0 = \begin{vmatrix} u_1 & v_1 & x - x_0 \\ u_2 & v_2 & y - y_0 \\ u_3 & v_3 & z - z_0 \end{vmatrix} &= \begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix} (x - x_0) + \begin{vmatrix} v_1 & u_1 \\ v_3 & u_3 \end{vmatrix} (y - y_0) + \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} (z - z_0) = \\ &= a(x - x_0) + b(y - y_0) + c(z - z_0), \end{aligned} \quad (3.24)$$

where (a, b, c) are precisely the components of $\mathbf{u} \times \mathbf{v}$, with respect to the basis $\mathbf{i}, \mathbf{j}, \mathbf{k}$ of $\mathbb{R}^3$:

$$a := \begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix}, \quad b := \begin{vmatrix} v_1 & u_1 \\ v_3 & u_3 \end{vmatrix}, \quad c := \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix}. \quad (3.25)$$

The expression (3.24) is generally written in the form

$$ax + by + cz + d = 0 \quad (3.26)$$

which is the form in which all planes passing through a point and parallel to two vectors may be written. having set:

$$d = -ax_0 - by_0 - cz_0. \quad (3.27)$$

Therefore $P(x, y, z) \in \Pi_{P_0, \mathbf{u}, \mathbf{v}}$ if and only if the coordinates of the point P satisfy equation (3.26), with a, b, c , given by (3.25) and d given by (3.27). Conversely,

$$ax + by + cz + d = 0,$$

with

$$\mathbf{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \neq \mathbf{0},$$

is an equation of a plane. Indeed, if for example $c \neq 0$, one has:

$$\begin{cases} x = \lambda, \\ y = \mu, \\ z = -\frac{a}{c}\lambda - \frac{b}{c}\mu - \frac{d}{c}, \end{cases}$$

which are parametric equations of the plane passing through $P_0 = (0, 0, -(d/c))$ and parallel to the vectors:

$$\mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ -\frac{a}{c} \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ -\frac{b}{c} \end{pmatrix}.$$

Moreover $\mathbf{u}$ is a vector orthogonal to the plane of equation (3.26). If Π has equation given by (3.26), one denotes by $\mathbf{n}_\Pi$ the

$$\mathbf{n}_\Pi := \begin{pmatrix} \frac{a}{\sqrt{a^2 + b^2 + c^2}} \\ \frac{b}{\sqrt{a^2 + b^2 + c^2}} \\ \frac{c}{\sqrt{a^2 + b^2 + c^2}} \end{pmatrix}$$

the unit vector orthogonal to the plane, often called the *normal unit vector* to the plane Π .

3.8.3 The previous discussion says that the Cartesian equations (3.21) of a line are each the equation of a plane. They express a line as the intersection of two planes.

3.8.4 Conversely, let Π and Π' be two planes with equations respectively $ax + by + cz + d = 0$ and $a'x + b'y + c'z + d' = 0$. Consider the intersection:

$$\Pi \cap \Pi' = \left\{ P(x, y, z) \in \mathcal{E}_3 \mid \begin{cases} ax + by + cz + d = 0 \\ a'x + b'y + c'z + d' = 0 \end{cases} \right\} \quad (3.28)$$

One has that either $\Pi \cap \Pi'$ is a plane, or $\Pi \cap \Pi' = \emptyset$, or else $\Pi \cap \Pi'$ is a line. Indeed, if $(a', b', c', d') = \lambda(a, b, c, d)$, $\lambda \neq 0$ (that is, the coefficients of the equations of the two planes are proportional), the two planes coincide and, therefore, the intersection is all of $\Pi = \Pi'$. If $(a', b', c') = \lambda(a, b, c)$, $\lambda \neq 0$, but $d' \neq \lambda d$, the intersection is empty. Indeed, in this case the system is of the form:

$$\begin{cases} ax + by + cz + d = 0 \\ \lambda ax + \lambda by + \lambda cz + d' = 0 \end{cases}$$

Therefore, multiplying the first equation by λ and subtracting the second from the first, one obtains

$$\lambda d - d' = 0$$

an absurdity, since one has supposed $d' \neq \lambda d$. Finally, if (a, b, c) and (a', b', c') are not proportional triples, there exists at least one square submatrix 2×2 of

$$\begin{pmatrix} a & b & c \\ a' & b' & c' \end{pmatrix}$$

having nonzero determinant. Suppose it is the one for which $ab' - a'b \neq 0$. Then the system (3.28) may be rewritten as

$$\begin{cases} ax + by = -cz - d \\ a'x + b'y = -c'z - d' \end{cases}$$

Setting $z = t$, one has, solving the system as a function of t , using Cramer's rule

(Cf. ??):

$$\begin{cases} x = \frac{\begin{vmatrix} -ct - d & b \\ -c't - d' & b' \end{vmatrix}}{\begin{vmatrix} a & b \\ a' & b' \end{vmatrix}} \\ y = \frac{\begin{vmatrix} a & -ct - d \\ a' & -c't - d' \end{vmatrix}}{\begin{vmatrix} a & b \\ a' & b' \end{vmatrix}} \\ z = t \end{cases}$$

that is, again

$$\begin{cases} x = \frac{bc' - b'c}{ab' - a'b}t - \frac{b'd - bd'}{ab' - a'b}, \\ y = \frac{a'c - ac'}{ab' - a'b}t + \frac{a'd - ad'}{ab' - a'b}, \\ z = t. \end{cases} \quad (3.29)$$

In vector form:

$$\begin{pmatrix} x + \frac{b'd - bd'}{ab' - a'b} \\ y - \frac{a'd - ad'}{ab' - a'b} \\ z \end{pmatrix} = t \begin{pmatrix} \frac{bc' - b'c}{ab' - a'b} \\ \frac{a'c - ac'}{ab' - a'b} \\ 1 \end{pmatrix}$$

or, rescaling the parameter t :

$$\begin{pmatrix} x + \frac{b'd - bd'}{ab' - a'b} \\ y - \frac{a'd - ad'}{ab' - a'b} \\ z \end{pmatrix} = t \begin{pmatrix} bc' - b'c \\ a'c - ac' \\ ab' - a'b \end{pmatrix},$$

which shows that (3.29) are precisely the parametric equations of a line passing

through the point with coordinates

$$\left(\frac{b'd - bd'}{ab' - a'b}, \frac{a'd - ad'}{ab' - a'b}, 0 \right)$$

and parallel to the vector

$$(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \times (a'\mathbf{i} + b'\mathbf{j} + c'\mathbf{k}).$$

3.8.5 Relative position of planes. Two planes are said to be parallel if either they are coincident or their normal vectors are parallel. In other words, Π and Π' are parallel if and only if $(a', b', c', d') = \lambda(a, b, c, d)$ or $\lambda(a', b', c') + \mu(a, b, c) = (0, 0, 0)$ for some $\lambda, \mu \neq 0$. Two nonparallel planes are said to be *intersecting*.

pair of intersecting, parallel and orthogonal planes
(the latter also, of course, intersecting)

The condition of nonparallelism is equivalent to requiring that the two vectors (a, b, c) and (a', b', c') be linearly independent. Two planes are *perpendicular*, or *orthogonal*, if and only if their normal vectors are orthogonal:

$$\Pi \perp \Pi' \iff \langle (a, b, c)^T, (a', b', c')^T \rangle = 0$$

Let Π and Π' be two intersecting planes. They intersect along the line

$$\begin{cases} ax + by + cz + d = 0 \\ a'x + b'y + c'z + d' = 0 \end{cases}$$

As the pair (λ, μ) varies, with λ and μ never simultaneously zero, consider the linear combination:

$$\lambda(ax + by + cz + d) + \mu(a'x + b'y + c'z + d') = 0. \quad (3.30)$$

For every choice of $(\lambda, \mu) \neq (0, 0)$, equation (3.30) is the Cartesian equation of a plane. Indeed, expanding the products and collecting x , y and z :

$$(\lambda a + \mu a')x + (\lambda b + \mu b')y + (\lambda c + \mu c')z + \lambda d + \mu d' = 0,$$

which is of the form

$$A(\lambda, \mu)x + B(\lambda, \mu)y + C(\lambda, \mu)z + D(\lambda, \mu) = 0, \quad (3.31)$$

that is, it is the equation of a plane. Clearly, for every choice of $(\lambda : \mu)$, equation (3.31) is the equation of a plane containing the line $r := \Pi \cap \Pi'$. Indeed, if $P_0(x_0, y_0, z_0)$ is a point of r , then it satisfies both the equation of Π and the equation of Π' , and therefore also equation (3.30). Conversely, suppose that $\Pi'' : a''x + b''y + c''z + d'' = 0$ is the equation of a plane containing r . Let $P_0(x_0, y_0, z_0)$ be a point of r . Then the equations of Π , Π' and Π'' may be written as:

$$\begin{cases} a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \\ a'(x - x_0) + b'(y - y_0) + c'(z - z_0) = 0 \\ a''(x - x_0) + b''(y - y_0) + c''(z - z_0) = 0 \end{cases}$$

Since Π'' contains r , its normal vector must be perpendicular to r . Since a directional vector of r is precisely $\mathbf{n}_\Pi \times \mathbf{n}_{\Pi'}$, one has:

$$\langle (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \times (a'\mathbf{i} + b'\mathbf{j} + c'\mathbf{k}), a''\mathbf{i} + b''\mathbf{j} + c''\mathbf{k} \rangle = 0,$$

that is, by the definition of the vector product:

$$\begin{vmatrix} a & a' & a'' \\ b & b' & b'' \\ c & c' & c'' \end{vmatrix} = 0$$

Since the first two columns are linearly independent (that is, the triples are not proportional), by ?? there exist $\lambda, \mu \in \mathbb{R}$ not both zero such that

$$\begin{pmatrix} a'' \\ b'' \\ c'' \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \mu \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix}$$

that is, the plane Π'' may be written as:

$$0 = a''x + b''y + c''z + d'' = (\lambda a + \mu a')(x - x_0) + (\lambda b + \mu b')(y - y_0) + (\lambda c + \mu c')(z - z_0)$$

and is therefore of the form (3.30).

3.8.6 Definition. Equation (3.30) is said to be an equation of the pencil of planes having the line $r := \Pi \cap \Pi'$ as axis.

– Some planes of a pencil of planes –

A pencil of planes admits infinitely many equations: as many as there are choices of a pair of intersecting planes belonging to the pencil.

3.8.7 Given the parametric equations of a line r in the form (3.20), an equation of the pencil of planes passing through r is given by

$$\lambda(ly - mx - ly_0 + mx_0) + \mu(lz - ny - lz_0 + ny_0) = 0$$

Indeed, equations (3.21) are Cartesian equations of two planes containing r .

3.8.8 Relative position of lines. Let r and r' be two lines in the space $\mathcal{E}^3$. They are said to be *coincident* if $r = r'$, they are said to be *parallel* if the directional vectors are proportional (equivalently, if they have the same direction cosines), they are said to be *intersecting* if $r \neq r'$ and $r \cap r' \neq \emptyset$, they are said to be *coplanar* if there exists a plane containing both of them, they are said to be *skew* if they are not coplanar. Two coincident lines are regarded as parallel. One also says that a line is parallel to itself.

3.8.9 Exercise. Prove that the relation of parallelism is an equivalence relation, that is, that it is reflexive, symmetric and transitive.

3.8.10 Esempio. Consider the lines

$$r := \begin{cases} x &= 1 + 2t, \\ y &= 1 - t, \\ z &= 2 + 2t, \end{cases} \quad \text{and} \quad s := \begin{cases} x &= 1 \\ y &= 1 - t \\ z &= 1 + t. \end{cases}$$

They are not coplanar. Indeed, if $ax + by + cz + d = 0$ were the equation of a plane containing both of them, the equations

$$a(1 + 2t) + b(1 - t) + c(2 + 2t) + d = 0,$$

and

$$a + b(1 - t) + c(1 + t) = 0,$$

obtained by substituting in place of x, y, z the parametric equations of the first and second line, should both be satisfied for every $t \in \mathbb{R}$. For this to be possible, it is necessary that the system

$$\begin{cases} a + b + 2c + d = 0 \\ 2a - b + 2c = 0 \\ a + b + c = 0 \\ -b + c = 0 \end{cases}$$

have a nonzero solution. However, the fourth equation gives $b = c$ and, therefore, the second gives $c = -2a$. The third equation then gives $a - 2a + 2a = 0$, whence $a = 0$. Then $b = c = 0$ and, therefore, using the first equation, one has $d = 0$. Hence the two lines are not coplanar.

3.9 Distances.

3.9.1 Let $P_1, P_2 \in \mathcal{E}^3$. The distance between two points P_1, P_2 has already been defined as

$$d(P_1, P_2) = |\overrightarrow{P_1 P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

where $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$.

3.9.2 Definition (point-plane distance). Let Π be a plane and let P_0 be a point. The distance between P_0 and Π is defined by:

$$d(P_0, \Pi) := \inf\{d(P_0, P) \mid P \in \Pi\}.$$

First of all, let $Q \in \Pi$ be such that $\overrightarrow{P_0 Q}$ is parallel to $\mathbf{n}_\Pi$. Such a point exists since it is the intersection of the plane Π with the line passing through P_0 and parallel to $\mathbf{n}_\Pi$. Then $\langle \overrightarrow{P_0 Q}, \mathbf{n}_\Pi \rangle = |\overrightarrow{P_0 Q}|$. Moreover

$$|\overrightarrow{P_0 Q}| = |\langle \overrightarrow{P_0 P} + \overrightarrow{P Q}, \mathbf{n}_\Pi \rangle| = |\langle \overrightarrow{P_0 P}, \mathbf{n}_\Pi \rangle + \langle \overrightarrow{P Q}, \mathbf{n}_\Pi \rangle| = |\langle \overrightarrow{P_0 P}, \mathbf{n}_\Pi \rangle|$$

Indeed, since $\overrightarrow{P Q}$ is a vector of the plane, it is orthogonal to $\mathbf{n}_\Pi$. Therefore $|\langle \overrightarrow{P_0 P}, \mathbf{n}_\Pi \rangle| = d(P_0, Q) \leq d(P_0, P)$ for every $P \in \Pi$ and, consequently:

$$d(P_0, \Pi) = |\langle \overrightarrow{P_0 P}, \mathbf{n}_\Pi \rangle|,$$

for every $P \in \Pi$. Let P then have coordinates (x, y, z) . One has

$$\begin{aligned} d(P_0, \Pi) &= \left| \left\langle \overrightarrow{P_0 P}, \mathbf{n}_\Pi \right\rangle \right| = \left| \left\langle \begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix}, \begin{pmatrix} \frac{a}{\sqrt{a^2 + b^2 + c^2}} \\ \frac{b}{\sqrt{a^2 + b^2 + c^2}} \\ \frac{c}{\sqrt{a^2 + b^2 + c^2}} \end{pmatrix} \right\rangle \right| = \\ &= \frac{|a(x - x_0) + b(y - y_0) + c(z - z_0)|}{\sqrt{a^2 + b^2 + c^2}} \end{aligned}$$

Since (x, y, z) are the coordinates of a generic point of Π , one has

$$ax + by + cz = -d$$

whence, finally,

$$d(P_0, \Pi) = \frac{|-ax_0 - by_0 - cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}} \quad (3.32)$$

3.9.3 Exercise. Prove that a point P_0 belongs to a plane Π if and only if $d(P_0, \Pi) = 0$.

3.9.4 Definition (point-line distance). Let $P_0 \in \mathcal{E}^3$ and let r be a line. The distance between P_0 and r is

$$d(P_0, r) = \inf\{d(P_0, P) \mid P \in r\}$$

This lower bound is in fact a minimum, as shown by the following argument. Let Q be the unique point such that $\overrightarrow{P_0Q}$ is orthogonal to the line. It is obtained by intersecting r with the plane through P_0 orthogonal to r . Then, if $P \in r$, one has:

$$|\overrightarrow{P_0Q}|^2 = |\overrightarrow{P_0P}|^2 - |\overrightarrow{PQ}|^2 \leq |\overrightarrow{P_0P}|^2$$

by the Pythagorean theorem. Therefore $d(P_0, r) = d(P_0, Q)$.

One must now determine a unit vector $\mathbf{b}$ belonging to the direction determined by the vector $\overrightarrow{P_0Q}$. Let $\mathbf{0} \neq \mathbf{v}$ be the directional vector of r . Then

$$(\mathbf{v} \times \overrightarrow{P_0P}) \times \mathbf{v}$$

is a vector parallel to $\overrightarrow{P_0Q}$. Indeed

$$\langle \mathbf{v} \times \overrightarrow{P_0P}, \overrightarrow{P_0Q} \rangle = \langle \mathbf{v} \times (\overrightarrow{P_0Q} + \overrightarrow{QP}), \overrightarrow{P_0Q} \rangle = \langle \mathbf{v} \times \overrightarrow{QP}, \overrightarrow{P_0Q} \rangle = 0,$$

where the last equality comes from the fact that $\mathbf{v} \times \overrightarrow{QP} = 0$ since $\mathbf{v} // \overrightarrow{QP}$. Therefore P_0Q is orthogonal both to $\mathbf{v} \times \overrightarrow{P_0P}$ and to $\mathbf{v}$ (by construction) and, consequently, is parallel to $\mathbf{v} \times (\mathbf{v} \times \overrightarrow{P_0P})$, which has the same properties. Then the unit vector

$$\mathbf{b} = \frac{\mathbf{v} \times (\mathbf{v} \times \overrightarrow{P_0P})}{|\mathbf{v} \times (\mathbf{v} \times \overrightarrow{P_0P})|} \quad (3.33)$$

may be chosen as $\mathbf{b}$. Since the distance sought is equal to the modulus of the orthogonal projection of $\overrightarrow{P_0P}$ onto the direction $[\mathbf{b}]$, that is $|\langle \overrightarrow{P_0P}, \mathbf{b} \rangle|$, one has:

$$\begin{aligned}
d(P_0, r) &= |\langle \overrightarrow{P_0P}, \mathbf{b} \rangle| = && \text{(expression for the point-line distance)} \\
&= \frac{|\langle \overrightarrow{P_0P}, \mathbf{v} \times (\mathbf{v} \times \overrightarrow{P_0P}) \rangle|}{|\mathbf{v} \times (\mathbf{v} \times \overrightarrow{P_0P})|} = && \text{(expression (3.33) for } \mathbf{b} \text{)} \\
&= \frac{|\langle \overrightarrow{P_0P} \times \mathbf{v}, \mathbf{v} \times \overrightarrow{P_0P} \rangle|}{|\mathbf{v}| \cdot |\overrightarrow{P_0P} \times \mathbf{v}|} = && \text{(property (??) of the scalar triple product)} \\
&= \frac{|\overrightarrow{P_0P} \times \mathbf{v}|^2}{|\mathbf{v}| \cdot |\overrightarrow{P_0P} \times \mathbf{v}|} = && \text{(definition of the norm of a vector)} \\
&= \frac{|\overrightarrow{P_0P} \times \mathbf{v}|}{|\mathbf{v}|}. && \text{(simplification of numerator and denominator)}
\end{aligned}$$

whence, finally, calling $\vec{e}_{\mathbf{v}}$ the unit vector $\mathbf{v}/|\mathbf{v}|$, and using the properties of the norm and the linearity of the vector product:

$$d(P_0, r) = \frac{|\overrightarrow{P_0P} \times \mathbf{v}|}{|\mathbf{v}|} = \left| \overrightarrow{P_0P} \times \frac{\mathbf{v}}{|\mathbf{v}|} \right| = |\overrightarrow{P_0P} \times \vec{e}_{\mathbf{v}}|. \quad (3.34)$$

3.9.5 Esempio. Let the line

$$r := \begin{cases} x &= t + 1, \\ y &= t + 1, \\ z &= 2t - 3. \end{cases}$$

and the point $P_0(2, 1, 1)$ be given. To compute $d(P_0, r)$ one considers any point of the line, for example the point $P(1, 1, -3)$ obtained by substituting $t = 0$. Therefore

$$\overrightarrow{P_0P} = \begin{pmatrix} -1 \\ 0 \\ -4 \end{pmatrix} \quad \text{and} \quad \vec{e}_{\mathbf{v}} = \begin{pmatrix} \frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{3} \end{pmatrix},$$

whence

$$d(P_0, r) = |\overrightarrow{P_0P} \times \vec{e}_v| = (\text{performing the calculations}) = 2\sqrt{33}/3.$$

3.9.6 Definition (line-line distance). Let r_1 and r_2 be two lines of $\mathcal{E}^3$ with directional vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. The distance between r_1 and r_2 is:

$$d(r_1, r_2) = \inf\{d(P_1, P_2) \mid P_1 \in r_1 \quad \wedge \quad P_2 \in r_2\}$$

If $\mathbf{b}$ is a unit vector orthogonal to $\mathbf{v}_1$ and $\mathbf{v}_2$, one has:

$$d(r_1, r_2) = |\langle \overrightarrow{P_1P_2}, \mathbf{b} \rangle|.$$

Since $\mathbf{b}$ may be computed as

$$\mathbf{b} = \frac{\mathbf{v}_1 \times \mathbf{v}_2}{|\mathbf{v}_1 \times \mathbf{v}_2|},$$

substituting, one has:

$$d(r_1, r_2) = |\langle \overrightarrow{P_1P_2}, \mathbf{b} \rangle| = \frac{|\langle \overrightarrow{P_1P_2}, \mathbf{v}_1 \times \mathbf{v}_2 \rangle|}{|\mathbf{v}_1 \times \mathbf{v}_2|} = \frac{|\det(\mathbf{v}_1, \mathbf{v}_2, \overrightarrow{P_1P_2})|}{|\mathbf{v}_1 \times \mathbf{v}_2|},$$

having used the definition of vector product (or scalar triple product, see ?? and ??).

3.9.7 Areas and Volumes. Four non-aligned points in $\mathcal{E}^3$ define a *tetrahedron* $\mathcal{P}$:

$$\mathcal{P}(P_1, P_2, P_3, P_4) := \{P \mid \overrightarrow{P_1P} = \lambda \overrightarrow{P_1P_2} + \mu \overrightarrow{P_1P_3} + \nu \overrightarrow{P_1P_4}\}$$

as all triples (λ, μ, ν) of nonnegative real numbers less than or equal to one vary ($0 \leq \lambda, \mu, \nu \leq 1$). The volume of $\mathcal{P}(P_1, P_2, P_3, P_4)$ is $\frac{1}{6}$ of the volume $\mathcal{V}_{\mathcal{P}}$ (Cf. ??) of the parallelepiped of the three vectors $\overrightarrow{P_1P_2}, \overrightarrow{P_1P_3}, \overrightarrow{P_1P_4}$:

$$\text{Vol}(P_1, P_2, P_3, P_4) = |\overrightarrow{P_1P_2} \wedge \overrightarrow{P_1P_3} \wedge \overrightarrow{P_1P_4}|.$$

Similarly, the area $A(P_1, P_2, P_3)$ of a parallelogram with vertices P_1, P_2, P_3, P_4 where

$$P_4 = P_1 + \overrightarrow{P_1P_2} + \overrightarrow{P_1P_3}$$

is (see ??):

$$A(P_1, P_2, P_3) = |\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}|. \quad (3.35)$$

Solved Exercise. Let $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$ be three points of the affine plane with coordinates given in a fixed standard orthonormal system. Find a formula for the area of the triangle $[P_1, P_2, P_3]$.

Solution. First of all, let us *embed* $\mathcal{E}^2$ into $\mathcal{E}^3$ by thinking of it as the locus of points $P(x, y, z)$ of $\mathcal{E}^3$ such that $z = 0$, so that we think of the given points in $\mathcal{E}^3$ with coordinates $P_i(x_i, y_i, 0)$ for $1 \leq i \leq 3$. Then one applies the formula for the area of a parallelogram in $\mathcal{E}^3$, which is twice the area of the triangle sought. One has

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{vmatrix} \mathbf{i} & x_2 - x_1 & x_3 - x_1 \\ \mathbf{j} & y_2 - y_1 & y_3 - y_1 \\ \mathbf{k} & 0 & 0 \end{vmatrix} = \begin{vmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{vmatrix}$$

whence, finally,

$$A([P_1, P_2, P_3]) = \frac{1}{2} \left\| \begin{vmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{vmatrix} \right\| = \frac{1}{2} |\overrightarrow{P_1P_2} \wedge \overrightarrow{P_1P_3}|.$$

3.9.8 Esempio. If $P_1(1, 2)$, $P_2(1, -5)$ and $P_3(3, 1)$, the area of the triangle with vertices P_1, P_2, P_3 is:

$$\frac{1}{2} |\overrightarrow{P_1P_2} \wedge \overrightarrow{P_1P_3}| = \frac{1}{2} \left| \begin{pmatrix} 0 \\ -7 \end{pmatrix} \wedge \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right| = \frac{14}{2} = 7.$$

3.10 Spheres and Circles

3.10.1 Let $P_0 \in \mathcal{E}_n$ and let $r \geq 0$. The sphere of center P_0 and radius r is, by definition:

$$S_{P_0}^{n-1}(r) = \{P \mid d(P_0, P) = r\}.$$

It is very easy to write Cartesian equations of a sphere $S_{P_0}^{n-1}(R)$. Indeed, if $P_0(\alpha_1, \dots, \alpha_n)$, one has:

$$|\overrightarrow{P_0P}| = \left| \begin{pmatrix} x_1 - \alpha_1 \\ \vdots \\ x_n - \alpha_n \end{pmatrix} \right| = R$$

that is:

$$(x_1 - \alpha_1)^2 + \dots + (x_n - \alpha_n)^2 = R^2.$$

If $n = 2$ the sphere $S_{P_0}^1(r)$ is called the *circle of center P_0 and radius r* . If $n = 3$ the sphere $S_{P_0}^2(r)$ is called the *sphere of center P_0 and radius r* .

3.10.2 Let $P_0 = (\alpha, \beta, \gamma) \in \mathcal{E}^3$. The Cartesian equation of the sphere of radius r and center P_0 is given by:

$$(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 = R^2$$

Expanding the squares on the left-hand side and subtracting R^2 , one has:

$$x^2 + y^2 + z^2 - 2\alpha x - 2\beta y - 2\gamma z + \alpha^2 + \beta^2 + \gamma^2 - R^2 = 0$$

that is, also, as is traditionally written:

$$x^2 + y^2 + z^2 + Ax + By + Cz + D = 0$$

where one has set

$$A = -2\alpha, \quad B = -2\beta, \quad C = -2\gamma, \quad D = \alpha^2 + \beta^2 + \gamma^2 - R^2.$$

These equations may be written more usefully:

$$\alpha = -A/2; \quad \beta = -B/2; \quad \gamma = -C/2; \quad R^2 = \alpha^2 + \beta^2 + \gamma^2 - D, \quad (3.36)$$

where the elements of the sphere (that is, coordinates of the center and radius) are highlighted from the coefficients A, B, C, D of one of its general equations.

3.10.3 Esempio. Consider the sphere of equation:

$$x^2 + y^2 + z^2 + 4x + 6y - 7z + 10 = 0$$

The coordinates of the center are $C(-2, -3, 7/2)$ while the radius is given by $R^2 = 4 + 9 + 49/4 - 10 = 61/4$, whence $r = \sqrt{61}/2$.

3.10.4 Definition. Let Π be a plane of $\mathcal{E}^3$. A circle C of Π is the locus of points of Π equidistant from a point $P_0 \in \Pi$. The distance between any point of C and the center P_0 is called the radius.

3.10.5 Example (Embedding a circle into three-dimensional space). In the Euclidean affine plane $\mathcal{E}^2$ with frame $(O; x, y)$, the circle centered at the origin and of radius 1 has equation:

$$x^2 + y^2 = 1.$$

Such a circle may be viewed in three-dimensional space through the *embedding* $\mathcal{E}^2 \rightarrow \mathcal{E}^3$ defined by $(x, y) \mapsto (x, y, 0)$. In this case the same circle, viewed in the space $\mathcal{E}^3$, may be represented by two Cartesian equations:

$$\begin{cases} x^2 + y^2 = 1 \\ z = 0 \end{cases}$$

that is, as the intersection of a *cylinder* in $\mathcal{E}^3$ cut by the slice $z = 0$.

Indeed, the locus of points of the form $(x, y, 0)$ such that $x^2 + y^2 = 1$ describes precisely a circle in the plane $z = 0$ of $\mathcal{E}^3$. Cartesian equations of a circle of radius r are easily obtained. Indeed, if (α, β, γ) are the coordinates of the center of the circle, belonging to the plane Π with Cartesian equations $ax + by + cz + d = 0$ (that is, $a\alpha + b\beta + c\gamma + d = 0$), the circle is the intersection of this plane with the sphere of center (α, β, γ) and radius r :

$$\mathcal{C} : \begin{cases} (x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 &= R^2 \\ ax + by + cz + d &= 0 \end{cases}$$

Indeed, if $P \in \mathcal{C}$ then $P \in \Pi$. Moreover, since it belongs to the sphere of radius r centered at (α, β, γ) , it must be at distance r from the center.

3.10.6 Since a circle may be viewed as the intersection of a sphere and a plane, it is natural to ask whether it is true that the intersection of a sphere and a plane is a circle. Let S be the sphere and Π the plane. Consider the line s passing through the center of the sphere and orthogonal to the plane. If $P_1, P_2 \in S \cap \Pi$, then $d(P_1, s) = d(P_2, s)$. Indeed, letting $P_0 := s \cap \Pi$ and C be the center of the sphere, one has:

$$\overline{P_0 P}^2 = \overline{CP}^2 - \overline{CP_0}^2,$$

for every $P \in \Pi \cap S$. Moreover $|\overrightarrow{CP}| = R$, that is, one obtains precisely a circle of center P_0 and radius $r^2 = R^2 - d(C, \Pi)^2$ (where $d(C, \Pi)$ is the distance of the plane from the center of the sphere).

3.10.7 Solved Exercise⁵ Find equations for the circle passing through the following points $P_1(-1, 0, 1)$, $P_2(1, -1, 0)$ and $P_3(1, 0, -1)$.

Solution. This circle is the intersection of the plane $\Pi : x + y + z = 0$ containing the three given points and of any sphere of the pencil passing through the three given points. If in the equation of the generic sphere $x^2 + y^2 + z^2 + Ax + By + Cz + D = 0$ one imposes passage through the three given points, one easily obtains $A = B = C$ and $D = -2$. One may then choose $A = 0$, whence

$$\begin{cases} x^2 + y^2 + z^2 - 6 = 0 \\ x + y + z = 0 \end{cases}$$

which are Cartesian equations of the required circle.

⁵Politecnico di Torino, First Faculty of Engineering, Written Examination in Geometry of 15 June 2009. The exercise was worth 4 points out of 30.

3.10.8 Pencils of spheres. Let S_1 and S_2 be two spheres with Cartesian equations respectively:

$$x^2 + y^2 + z^2 + A_1x + B_1y + C_1z + D_1 = 0 \quad \text{and} \quad x^2 + y^2 + z^2 + A_2x + B_2y + C_2z + D_2 = 0$$

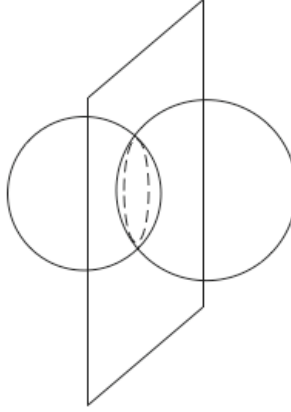
The family of spheres parametrized by the directions of the plane $(\lambda : \mu) \in \mathbb{P}^1$:

$$\lambda(x^2 + y^2 + z^2 + A_1x + B_1y + C_1z + D_1) + \mu(x^2 + y^2 + z^2 + A_2x + B_2y + C_2z + D_2) = 0. \quad (3.37)$$

is called a *pencil of spheres*, and will often be written briefly, and by abuse of notation, as

$$\lambda S_1 + \mu S_2 = 0$$

thus confusing the equations of the spheres with the spheres themselves.



Expanding equation (3.37), one has:

$$\begin{aligned} & (\lambda + \mu)x^2 + (\lambda + \mu)y^2 + (\lambda + \mu)z^2 + \\ & + (\lambda A_1 + \mu A_2)x + (\lambda B_1 + \mu B_2)y + (\lambda C_1 + \mu C_2)z + (\lambda D_1 + \mu D_2) = 0 \end{aligned} \quad (3.38)$$

If $\lambda + \mu \neq 0$, then (3.38) is indeed a sphere: in fact it is of the form

$$x^2 + y^2 + z^2 + A(\lambda, \mu)x + B(\lambda, \mu)y + C(\lambda, \mu)z + D(\lambda, \mu) = 0$$

where one has set:

$$\begin{aligned} A(\lambda, \mu) &= \frac{\lambda A_1 + \mu A_2}{\lambda + \mu} & B(\lambda, \mu) &= \frac{\lambda B_1 + \mu B_2}{\lambda + \mu} \\ C(\lambda, \mu) &= \frac{\lambda C_1 + \mu C_2}{\lambda + \mu} & D(\lambda, \mu) &= \frac{\lambda D_1 + \mu D_2}{\lambda + \mu} \end{aligned}$$

However, if $\lambda = -\mu$, equation (3.37) or, equivalently, (3.38), may be rewritten as:

$$(A_1 - A_2)x + (B_1 - B_2)y + (C_1 - C_2)z + (D_1 - D_2) = 0$$

which is the equation of a plane. It is convenient to regard this plane as a sphere of infinite radius, so as to be able to say that for every $(\lambda : \mu)$, $\lambda S_1 + \mu S_2$ is a sphere. Such a plane is called the *radical plane* of the pencil. It is evident that S_1 and S_2 intersect along one and the same circle, given by the intersection of S_1 or S_2 with the radical plane. Moreover

3.10.9 Proposition. *All the spheres of the pencil contain the same circle $S_1 \cap S_2$.*

Proof. Indeed, if $P_0(x_0, y_0, z_0) \in S_1 \cap S_2$, then for every $(\lambda : \mu) \in \mathbb{P}^1$, one has

$$\lambda S_1(x_0, y_0, z_0) + \mu S_2(x_0, y_0, z_0) = 0$$

since the coordinates (x_0, y_0, z_0) satisfy both the equation of S_1 and that of S_2 . ■

3.10.10 Tangent plane at a point. As has been seen, a plane may intersect a sphere in a real circle. This happens if the distance of the plane from the center of the sphere is less than the radius of the sphere itself. A sphere and a plane intersect along an imaginary circle if the distance of the plane from the sphere is greater than the radius R . A plane whose distance from the center of the sphere is exactly R is called *tangent*, at the unique real point of contact.

Solved Exercise. Let the sphere $\mathcal{S} : x^2 + y^2 + z^2 - 2x - 4y + 6z - 22 = 0$ be given, and let us find the equation of the tangent plane at the point P_0 with coordinates $(1, 2, 3)$.

Solution. First of all, observe that $P_0(1, 2, 3) \in \mathcal{S}$. The radius of the sphere is given by (3.36)

$$R^2 = 1 + 4 + 9 + 22 = 36$$

that is $R = 6$, while the center C has coordinates $(1, 2, -3)$. The generic plane passing through the point $(1, 2, 3)$ has equation

$$a(x - 1) + b(y - 2) + c(z - 3) = 0$$

The tangency condition $d(C, \Pi) = 6$ is written as:

$$\frac{|6c|}{\sqrt{a^2 + b^2 + c^2}} = 6$$

However, this information is not necessary. Indeed: $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = [\overrightarrow{CP}]$, that is $\begin{bmatrix} a \\ b \\ c \end{bmatrix} =$

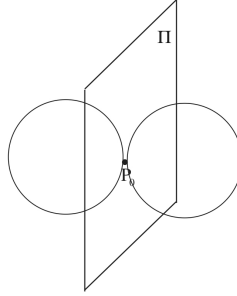
$\begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix}$. Therefore the equation of the required plane is $c(z - 3) = 0$, where c is any nonzero real number.

3.10.11 Solved Exercise. Write the equation of the sphere tangent to the plane $\Pi : x + y + z - 1 = 0$ at the point $P_0(2, 1, -2)$ and having radius equal to 3.

Solution. The solution proposed here is not necessarily the shortest one, but it is interesting because it illustrates the method of pencils for solving certain classes of problems. The required sphere evidently belongs to the pencil of spheres tangent to the plane Π at the point P_0 . Therefore the plane Π is itself the radical plane of the pencil, which, in turn, may be regarded in every respect as a sphere of the same pencil. The equation

$$(x - 2)^2 + (y - 1)^2 + (z + 2)^2 + 2\lambda(x + y + z - 1) = 0 \quad (3.39)$$

is the equation of the required pencil, generated by the radical plane and by the sphere centered at the point P_0 and of radius 0!.



At this point it is enough to find λ so as to select the spheres of radius 3. Equation (3.39) may be rewritten as:

$$x^2 + y^2 + z^2 + (2\lambda - 4)x + (2\lambda - 2)y + (4 + 2\lambda)z + 9 - 2\lambda = 0$$

The coordinates of the center $C(\lambda)$ of all the spheres of the pencil are:

$$C(\lambda) = (2 - \lambda, 1 - \lambda, -2 - \lambda)$$

and therefore the squared radius is given by, using (3.36):

$$9 = R(\lambda)^2 = (2 - \lambda)^2 + (1 - \lambda)^2 + (2 + \lambda)^2 - 9 + 2\lambda$$

whence:

$$3\lambda^2 = 9,$$

that is $\lambda = \pm\sqrt{3}$. The equations of the required spheres are then:

$$x^2 + y^2 + z^2 + (\pm 2\sqrt{3} - 4)x + (\pm 2\sqrt{3} - 2)y + (4 \pm 2\sqrt{3})z + 9 \mp 2\sqrt{3} = 0.$$

3.10.12 An example in four-dimensional Euclidean affine space. The purpose of this paragraph is *recreational* and aims to let the student, or the reader in general, touch with hand the phenomenology often described in popular expositions, when it is imagistically asserted that a *surgeon in four dimensions can operate on a patient, for example to perform an appendectomy, without having to cut open the abdomen*. To account for this, one considers the unit sphere centered at the origin of a frame of $\mathcal{E}^3$:

$$x_1^2 + x_2^2 + x_3^2 = 1, \quad (3.40)$$

and embeds it in $\mathcal{E}^4$, analogously to how, in Example 3.10.5, the circle of the plane $\mathcal{E}^2$ had been embedded into the space $\mathcal{E}^3$. The Cartesian equations of the sphere (3.40) embedded in $\mathcal{E}^4$ with coordinates (x_1, x_2, x_3, x_4) are:

$$\begin{cases} x_1^2 + x_2^2 + x_3^2 &= 1, \\ x_4 &= 0. \end{cases}$$

Note that the locus of points $(x_1, x_2, x_3, 0)$ of $\mathcal{E}^4$ such that $x_1^2 + x_2^2 + x_3^2 = 1$ satisfies precisely the definition of a sphere of radius 1 and center the origin $(0, 0, 0, 0)$, in the sense of 3.10.1. Now consider the line of $\mathcal{E}^4$ defined by the parametric equations

$$r := \begin{cases} x_1 &= t \\ x_2 &= t \\ x_3 &= t \\ x_4 &= t \end{cases}$$

The line r evidently passes through the origin, which is also the center of the sphere. Possible intersections between the line and the sphere are obtained for the values of t that satisfy the system:

$$\begin{cases} \begin{cases} x_1^2 + x_2^2 + x_3^2 &= 1, \\ x_4 &= 0 \end{cases} \\ \begin{cases} x_1 &= t, \\ x_2 &= t, \\ x_3 &= t, \\ x_4 &= t. \end{cases} \end{cases} \quad (3.41)$$

Substituting t in place of x_1, x_2, x_3 in the first of (3.41), one obtains

$$3t^2 = 1$$

whence $t = \pm\sqrt{3}/3$. Then x_4 should simultaneously equal 0 and $\pm\sqrt{3}/3$. Impossible! One has therefore found a line passing through the center of a sphere without “piercing” (we say: intersecting) its surface. If medical science could learn to embed a patient in a four-dimensional space, surgery would benefit dramatically, not to mention the patients!!!

3.11 Exercises

3.11.1 Find a family of lines passing through the center of the sphere embedded in $\mathcal{E}^4$ of example 3.10.12, none of which intersects the surface of the sphere itself. More serious and down-to-earth (three-dimensional!) exercises follow.

Soluzione: [?, 12.4.1]



— surgery —

3.11.2 Given the plane Π with Cartesian equation $3x - 5y + 2z - 3 = 0$ and the point $P(1, 2, 3)$, find the coordinates of the point symmetric with respect to Π . (Hint: the symmetric point is a point Q such that $\vec{PQ} \perp \Pi$ and such that the midpoint between P and Q lies on the plane.) *Soluzione:* [?, 3.5.3]

3.11.3 Let the point $P_0(2, 1, 4)$ and the line r with parametric equations

$$(x, y, z) = (2 + 2t, 1 - t, 3 - 4t).$$

be given.

1. Find the equations of the plane containing the line r and the point P_0 ;
2. Find the Cartesian equations of the plane Π' containing the line r and orthogonal to Π ;

3. Find the coordinates of the point P_1 symmetric to P_0 with respect to the line r ; (hint: it is the same as the symmetric point of P_0 with respect to the plane Π').

Soluzione: [?, 3.5.4]

3.11.4 Determine the radius and center of the circle that is the intersection of the sphere passing through the points $P_1(2, 1, 2)$, $P_2(3, 1, 1)$ and $P_3(2, 2, 1)$ and the plane passing through the same points.

Soluzione: [?, 12.3.1]

3.11.5 Let the sphere with equations $S : x^2 + y^2 + z^2 - 3x + 5y - 7z - 4 = 0$ and the plane $\pi : 2x - y + z + 7 = 0$ be given. Determine the intersection $S \cap \pi$.

Soluzione: [?, 12.3.2]

3.11.6 Write the equations of the planes through the line $r : x - 1 = y - 2z = 0$ that intersect the sphere $\mathcal{S} := x^2 + (y - 1)^2 + (z - 1)^2 = 9$ in circles of radius $2\sqrt{2}$.

Soluzione: [?, 12.3.3]

3.11.7

1. Write the equation of the sphere $\mathcal{S}$ that contains the circle $\mathcal{C}$ with equations:

$$\begin{cases} x^2 + y^2 - 2x + y = 0 \\ z = 0 \end{cases}$$

and passes through the point with coordinates $(1, 1, 1)$.

2. Determine which among the following planes are secant, tangent, or exterior to the sphere;

(a) $x + y + z - 1 = 0$;

(b) $x - 2y + z = 0$;

(c) $x + y + z - 3 = 0$;

(d) $2x - 3y - 4z + 7 = 0$;

iii) $2x + y + 8z - 1 = 0$

3. Find the equation of the tangent plane to the sphere $\mathcal{S}$ at the point $(1, 1, 1)$.

4. Find the equations of the tangent line to the circle at the point $(0, 0, 0)$,

Soluzione: [?, 12.3.5]

3.11.8 Let the sphere $\mathcal{S}$ with Cartesian equation $4x^2 + 4y^2 + 4z^2 - 2x - y + 3z - 5 = 0$ be given.

1. Determine the coordinates of the center and the radius.
2. Let the sphere $\mathcal{S}$ with Cartesian equation $4x^2 + 4y^2 + 4z^2 - 2x - y + 3z - 5 = 0$ and the plane $\Pi := x + 2y - z - 3 = 0$ be given. Find Cartesian equations for the circle $\mathcal{S} \cap \Pi$.

Soluzione: [?, 12.3.6]

3.11.9

- i) Determine the equation of the circle $\mathcal{C}$ passing through the points $A(4, 1, 1)$, $B(2, 2, 2)$ and $C(1, 2, 3)$;
- ii) Determine the radius and the coordinates of the center of $\mathcal{C}$;
- iii) Consider the point $V(1, 1, 1)$. Write parametric equations and, if possible, Cartesian equations of the surface that is the locus of the lines passing through $V(1, 1, 1)$ and *resting*⁶ on $\mathcal{C}$.

Soluzione: [?, 12.3.7]

3.11.10 Write the equations of the lines in the plane $\pi : x - y + z = 0$, orthogonal to the line $x + z = 1 - y$, and at distance $d = \sqrt{2}/2$ from the point $P(1, 0, 0)$

Soluzione: [?, 11.8.5]

3.11.11 Find the distances between the following pairs of objects (point-line, point-plane, line-line)

1. $r_1 := x - y = z - 2x + 1 = 0$ $r_2 := 2x - y - z = z - 2x - 1 = 0$;
2. $r_1 := 2x - y - z = x = 0$ $r_2 := 2x - y - z = 3z - 2x - 1 = 0$;
3. $r_1 := x = z = 0$ $r_2 := 2x - y - z = z - 2x - 1 = 0$;
4. $r_1 := x - y = z - 2x + 1 = 0$ $\pi_1 := 2x - y - z = z - 2x - 1 = 0$.

Soluzione: [?, 11.8.6]

⁶As Italian mathematicians of the first half of the twentieth century used to say.

3.11.12 In the three-dimensional Euclidean affine space $\mathcal{E}^3$, let the plane π and the sphere S with equations, respectively,

$$\pi : x + y - z - 1 = 0, \quad S : x^2 + y^2 + z^2 - 4x - 2y + 4z = 0$$

- i) Determine the center and radius of S .
- ii) Write Cartesian equations for the circle $S \cap \pi$.
- iii) Determine the radius and center of the circle $S \cap \pi$.
- iv) Setting $\pi_h : x + y - z + h = 0$, determine for which value or values of h the plane π_h intersects the sphere S along a great circle.

Soluzione: [?, 12.3.9]

3.11.13 Consider the circle $S^1 := \{(x, y) \in \mathcal{E}^2 \mid x^2 + y^2 = 1\}$ and the sphere

$$S^2 := \{(x, y, z) \in \mathcal{E}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

Let the same symbol N denote the north pole $(0, 1)$ of the circle S^1 and the north pole $(0, 0, 1)$ of the sphere S^2 . Prove that the *punctured spheres* $S^1 \setminus \{N\}$ and $S^2 \setminus \{N\}$ may be endowed with a vector-space structure isomorphic respectively to $\mathbb{R}$ and $\mathbb{R}^2$ (*Hint: reconsider exercise ??*).

Soluzione: [?, 6.5.6]