

Chapter 2

The three-dimensional Euclidean vector space

2.1 Let us recall that...

2.1.1 ... the Euclidean vector space $\mathbb{R}^n$ is called the *Euclidean vector plane* if $n = 2$ and the *three-dimensional Euclidean vector space* if $n = 3$. For reasons of tradition, the vectors of the canonical basis E_2 of $\mathbb{R}^2$ will be denoted by $(\mathbf{i}, \mathbf{j})$ instead of $(\mathbf{e}_1, \mathbf{e}_2)$:

$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Likewise, the vectors of the canonical basis E_3 of $\mathbb{R}^3$ will be denoted, instead of by $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, by $(\mathbf{i}, \mathbf{j}, \mathbf{k})$:

$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (2.1)$$

The vector plane $\mathbb{R}^2$ can be identified with the *vector subspace* of $\mathbb{R}^3$ (Cf. paragraph ??) generated by $\mathbf{i}, \mathbf{j}$:

$$\mathbb{R}^2 = [\mathbf{i}, \mathbf{j}] = \{\mathbf{u} \in \mathbb{R}^3 \mid \mathbf{u}(3) = 0\}.$$

With respect to the scalar product $\langle \rangle$ defined respectively on $\mathbb{R}^2$ and $\mathbb{R}^3$, the canonical bases E_2 and E_3 are *orthonormal*, as already seen in general for the basis E_n of $\mathbb{R}^n$, Cf. Example ??. In particular (Cf. Section ??):

$$\mathbf{i}^\perp := [\mathbf{j}, \mathbf{k}], \quad \mathbf{j}^\perp := [\mathbf{k}, \mathbf{i}], \quad \mathbf{k}^\perp := [\mathbf{i}, \mathbf{j}].$$

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Every vector $\mathbf{u} \in \mathbb{R}^3$ can therefore be written uniquely as:

$$\mathbf{u} = \langle \mathbf{u}, \mathbf{i} \rangle \mathbf{i} + \langle \mathbf{u}, \mathbf{j} \rangle \mathbf{j} + \langle \mathbf{u}, \mathbf{k} \rangle \mathbf{k} = u(1)\mathbf{i} + u(2)\mathbf{j} + u(3)\mathbf{k},$$

with $\mathbf{i}, \mathbf{j}, \mathbf{k}$ given by (2.1).

In general, if $\mathcal{B} := (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ is an orthonormal basis, a vector $\mathbf{u} \in \mathbb{R}^3$ can be decomposed uniquely along the directions $[\mathbf{b}_1]$, $[\mathbf{b}_2]$ and $[\mathbf{b}_3]$ according to the equality:

$$\mathbf{u} = \langle \mathbf{u}, \mathbf{b}_1 \rangle \mathbf{b}_1 + \langle \mathbf{u}, \mathbf{b}_2 \rangle \mathbf{b}_2 + \langle \mathbf{u}, \mathbf{b}_3 \rangle \mathbf{b}_3.$$

2.1.2 In the first part of this chapter one wishes to deal with the following problem. Given two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$, in which cases, and how, can a third vector $\mathbf{w} \in \mathbb{R}^2$ be decomposed as a linear combination of $\mathbf{u}$ and $\mathbf{v}$? Similarly, given three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$, in which cases, and how, can a fourth vector be decomposed along the directions determined by them? In Chapter ?? this question has already been answered in general, but only in the case in which the directions were orthogonal (and hence using the scalar product).

The general answer to these questions is important, already in the case of $\mathbb{R}^2$ and $\mathbb{R}^3$, because they provide the occasion for the introduction of a fundamental tool of linear algebra, called the *determinant*, which will be taken up again in a more general and deeper way in Chapter ??. One begins with the simplest case, the determinant of a pair of vectors, on which the notion of *area of a parallelogram* is based.

2.2 Determinants of pairs of vectors

2.2.1 In what follows

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

will denote three arbitrary vectors of $\mathbb{R}^2$. The *determinant of an ordered pair*

$$(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^2 \times \mathbb{R}^2$$

is the scalar $\mathbf{u} \wedge \mathbf{v}$ defined by the equality¹:

$$\mathbf{u} \wedge \mathbf{v} =: \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} =: u_1 v_2 - u_2 v_1. \quad (2.2)$$

¹Naturally the same definition also applies if $\mathbb{K} = \mathbb{C}$. The general definition of determinant will be given in Chapter ??.

2.2.2 Linearity and antisymmetry. The determinant is *linear with respect to the first argument*, that is:

$$(\lambda \mathbf{u} + \mu \mathbf{v}) \wedge \mathbf{w} = \lambda \cdot \mathbf{u} \wedge \mathbf{w} + \mu \cdot \mathbf{v} \wedge \mathbf{w}. \quad (2.3)$$

for every choice of $\lambda, \mu \in \mathbb{R}$ and every choice of $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$. The verification is done... *by hand*:



– In the exam the use of any electronic device is not allowed –

$$\begin{aligned} (\lambda \mathbf{u} + \mu \mathbf{v}) \wedge \mathbf{w} &= \begin{pmatrix} \lambda u_1 + \mu v_1 \\ \lambda u_2 + \mu v_2 \end{pmatrix} \wedge \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \\ &= (\lambda u_1 + \mu v_1)w_2 - (\lambda u_2 + \mu v_2)w_1 = \\ &= \lambda(u_1 w_2 - u_2 w_1) + \mu(v_1 w_2 - v_2 w_1) = \\ &= \lambda \cdot \mathbf{u} \wedge \mathbf{w} + \mu \cdot \mathbf{v} \wedge \mathbf{w}, \end{aligned}$$

as desired. Moreover the determinant is *antisymmetric*, that is

$$\mathbf{v} \wedge \mathbf{u} = -\mathbf{u} \wedge \mathbf{v}, \quad (2.4)$$

for every $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$. Indeed

$$\mathbf{v} \wedge \mathbf{u} = v_1 u_2 - u_1 v_2 = -(u_1 v_2 - u_2 v_1) = -\mathbf{u} \wedge \mathbf{v}.$$

Equations (2.3) and (2.4) imply *linearity with respect to the second argument*:

$$\mathbf{u} \wedge (\lambda \mathbf{v} + \mu \mathbf{w}) = \lambda \mathbf{u} \wedge \mathbf{v} + \mu \mathbf{u} \wedge \mathbf{w}. \quad (2.5)$$

Indeed, for every choice of $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ and $\lambda, \mu \in \mathbb{R}$ one has:

$$\begin{aligned} \mathbf{u} \wedge (\lambda \mathbf{v} + \mu \mathbf{w}) &= -(\lambda \mathbf{v} + \mu \mathbf{w}) \wedge \mathbf{u} && \text{(by (2.4))} \\ &= -\lambda \cdot \mathbf{v} \wedge \mathbf{u} - \mu \cdot \mathbf{w} \wedge \mathbf{u} && \text{(by (2.3))} \\ &= \lambda \cdot \mathbf{u} \wedge \mathbf{v} + \mu \cdot \mathbf{u} \wedge \mathbf{w}, && \text{(by (2.4))} \end{aligned}$$

as desired. The determinant is therefore a *bilinear function*, in the sense that it is linear both with respect to the first and to the second argument. Moreover, as is easily checked,

$$\mathbf{i} \wedge \mathbf{j} = 1.$$

It is not difficult to prove, as will be shown later and more generally, that the determinant function

$$\begin{cases} \det : \mathbb{R}^2 \wedge \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (\mathbf{u}, \mathbf{v}) & \longmapsto & \mathbf{u} \wedge \mathbf{v}, \end{cases}$$

is the unique *form*² bilinear antisymmetric such that $\mathbf{i} \wedge \mathbf{j} = 1$.

2.2.3 Observations.

- i) The equality $\mathbf{u} \wedge \mathbf{u} = 0$ holds for every $\mathbf{u} \in \mathbb{R}^2$. Indeed, since antisymmetry (2.4) holds for every pair of vectors $(\mathbf{u}, \mathbf{v})$, it holds in particular for the pair $(\mathbf{u}, \mathbf{u})$. Hence:

$$\mathbf{u} \wedge \mathbf{u} = -\mathbf{u} \wedge \mathbf{u},$$

that is $2\mathbf{u} \wedge \mathbf{u} = 0$. Since³ $2 \neq 0$, it follows that $\mathbf{u} \wedge \mathbf{u} = 0$.

- ii) the determinant of the pair $(\mathbf{u}, \mathbf{v})$ is zero if at least one of the two vectors is zero, as a consequence of bilinearity. Indeed:

$$\mathbf{0} \wedge \mathbf{v} = (0 \cdot \mathbf{u}) \wedge \mathbf{v} = 0 \cdot \mathbf{u} \wedge \mathbf{v} = 0$$

where $\mathbf{u}$ is an arbitrary vector.

2.2.4 Proposition. *Two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ are collinear (see paragraph ??) if and only if*

$$\mathbf{u} \wedge \mathbf{v} = 0.$$

Proof. If $\mathbf{u} \in [\mathbf{v}]$ then $\mathbf{u} = \lambda \mathbf{v}$ whence

$$\mathbf{u} \wedge \mathbf{v} = \lambda \mathbf{v} \wedge \mathbf{v} = \lambda(\mathbf{v} \wedge \mathbf{v}) = 0,$$

by Observation 2.2.3 i). Conversely, suppose that $\mathbf{u} \wedge \mathbf{v} = 0$. If $\mathbf{u} = \mathbf{0}$ there is nothing to prove by 2.2.3 ii). If $\mathbf{u} \neq 0$, it has at least one nonzero component. Suppose, for instance, that $u_1 \neq 0$. If λ is the unique scalar such that $v_1 = \lambda u_1$, then:

²A *bilinear form* is a homogeneous polynomial expression of degree two in the components of two vectors of $\mathbb{K}^n$, where $\mathbb{K}$ is a field.

³Unfortunately the condition $2 \neq 0$ is no joke. There exist fields in which $1 + 1 = 0$, the so-called *fields of characteristic 2*. This is not the case, however, for $\mathbb{R}$ or $\mathbb{C}$ or for any field containing $\mathbb{Q}$ as a subfield.

$$\begin{aligned}
0 = \mathbf{u} \wedge \mathbf{v} &= \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} = && \text{(notation (2.2))} \\
&= u_1 v_2 - u_2 v_1 = && \text{(definition (2.2) of determinant)} \\
&= u_1 v_2 - u_2 (\lambda u_1) = && \text{(hypothesis that } v_1 = \lambda u_1) \\
&= u_1 (v_2 - \lambda u_2) && (u_1 \text{ factored out).}
\end{aligned}$$

hence $v_2 = \lambda u_2$, since $u_1 \neq 0$. Therefore $\mathbf{v} = \lambda \mathbf{u}$ and the two vectors are collinear. ■

2.3 Linear systems of two equations in two unknowns.

2.3.1 Let

$$\begin{cases} a_1 x + b_1 y = c_1, \\ a_2 x + b_2 y = c_2, \end{cases} \quad (2.6)$$

be a system of two first-degree equations in the unknowns x and y . Equality (2.6) can be written equivalently in the form

$$\begin{pmatrix} a_1 x + b_1 y \\ a_2 x + b_2 y \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

that is, again,

$$x\mathbf{a} + y\mathbf{b} = \mathbf{c}, \quad (2.7)$$

where

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \quad \text{and} \quad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Equality (2.7) obviously implies:

$$(x\mathbf{a} + y\mathbf{b}) \wedge \mathbf{b} = \mathbf{c} \wedge \mathbf{b}$$

whence,

$$x\mathbf{a} \wedge \mathbf{b} + y\mathbf{b} \wedge \mathbf{b} = \mathbf{c} \wedge \mathbf{b},$$

having used linearity (2.3) of the determinant with respect to the first argument. Since $\mathbf{b} \wedge \mathbf{b} = 0$, one finally gets:

$$x\mathbf{a} \wedge \mathbf{b} = \mathbf{c} \wedge \mathbf{b}.$$

Similarly one obtains

$$\mathbf{a} \wedge (x\mathbf{a} + y\mathbf{b}) = \mathbf{a} \wedge \mathbf{c},$$

whence

$$y \cdot \mathbf{a} \wedge \mathbf{b} = \mathbf{a} \wedge \mathbf{c}.$$

If the determinant of the coefficients of the equation is different from zero, the system has a unique solution given by:

$$(x, y) = \left(\frac{\mathbf{c} \wedge \mathbf{b}}{\mathbf{a} \wedge \mathbf{b}}, \frac{\mathbf{a} \wedge \mathbf{c}}{\mathbf{a} \wedge \mathbf{b}} \right). \quad (2.8)$$

Equation (2.8) expresses the classical *Cramer's rule*, which solves systems of two linear equations in two unknowns, also called *Cramer systems* 2×2 .

2.3.2 Exercise.

- 1) Show that if $\mathbf{a} \wedge \mathbf{b} = 0$ and at least one among $\mathbf{c} \wedge \mathbf{b}$ and $\mathbf{a} \wedge \mathbf{c}$ is different from zero, then the system (2.6) admits no solutions;
- 2) Give an example of $\mathbf{a}, \mathbf{b}, \mathbf{c}$ such that $\mathbf{a} \wedge \mathbf{b} = \mathbf{a} \wedge \mathbf{c} = \mathbf{c} \wedge \mathbf{b} = 0$ for which the system (2.6) has infinitely many solutions;
- 3) Give an example of $\mathbf{a}, \mathbf{b}, \mathbf{c}$ such that $\mathbf{a} \wedge \mathbf{b} = \mathbf{a} \wedge \mathbf{c} = \mathbf{c} \wedge \mathbf{b} = 0$ for which the system (2.6) has no solutions.

2.3.3 Decomposition of vectors along two given directions. Let $\mathbf{u}$ and $\mathbf{v}$ be non-collinear vectors. Then every $\mathbf{w} \in \mathbb{R}^2$ can be decomposed as the sum of two vectors $\mathbf{w}_1 + \mathbf{w}_2$ such that $\mathbf{w}_1 \in [\mathbf{u}]$ and $\mathbf{w}_2 \in [\mathbf{v}]$. Indeed there exist $\lambda, \mu \in \mathbb{R}$ such that

$$\mathbf{w} = \lambda\mathbf{u} + \mu\mathbf{v},$$

because, rewriting (2.8), it suffices to take

$$\lambda = \frac{\mathbf{w} \wedge \mathbf{v}}{\mathbf{u} \wedge \mathbf{v}} \quad \text{and} \quad \mu = \frac{\mathbf{u} \wedge \mathbf{w}}{\mathbf{u} \wedge \mathbf{v}},$$

so that:

$$\mathbf{w} = \frac{\mathbf{w} \wedge \mathbf{v}}{\mathbf{u} \wedge \mathbf{v}} \cdot \mathbf{u} + \frac{\mathbf{u} \wedge \mathbf{w}}{\mathbf{u} \wedge \mathbf{v}} \cdot \mathbf{v}.$$

In conclusion, if two vectors $\mathbf{u}$ and $\mathbf{v}$ are not collinear, then $\mathbb{R}^2 = [\mathbf{u}, \mathbf{v}]$, that is, $\mathbf{u}, \mathbf{v}$ are generators of $\mathbb{R}^2$. It will be seen that the vector plane $\mathbb{R}^2$ can be embedded into the vector space $\mathbb{R}^3$ of columns of three real numbers.

2.4 Collinearity and coplanarity in $\mathbb{R}^3$

It will be useful to consider, in what follows, for each $1 \leq i \leq 3$, the function

$$q_i : \mathbb{R}^3 \rightarrow \mathbb{R}^2$$

that “forgets” the i -th component:

$$q_1(\mathbf{u}) = \begin{pmatrix} u_2 \\ u_3 \end{pmatrix}, \quad q_2(\mathbf{u}) = \begin{pmatrix} u_1 \\ u_3 \end{pmatrix}, \quad q_3(\mathbf{u}) = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad (2.9)$$

where $\mathbf{u}$ is as in (2.11). The reader will verify as an exercise that q_i is a *linear map*, for every $i \in \{1, 2, 3\}$, that is, it “respects” linear combinations in the following sense:

$$q_i(\lambda \mathbf{u} + \mu \mathbf{v}) = \lambda \cdot q_i(\mathbf{u}) + \mu \cdot q_i(\mathbf{v}). \quad (2.10)$$

2.4.1 Thus, as seen in general in $\mathbb{R}^n$, a nonzero column

$$\mathbf{u} := \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \in \mathbb{R}^3 \quad (2.11)$$

generates a *vector line* of $\mathbb{R}^3$, denoted by $[\mathbf{u}]$, which is the set $\{\lambda \mathbf{u} \mid \lambda \in \mathbb{R}\}$, while $[\mathbf{0}] = \{\mathbf{0}\}$. The set

$$\mathbb{RP}^2 = \{[\mathbf{u}] \mid \mathbf{u} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}\}$$

of all directions of $\mathbb{R}^3$ is called the *real projective plane*. The definition of collinearity for two vectors of $\mathbb{R}^3$ is the same as for vectors with two components. Two vectors $\mathbf{u}$ and $\mathbf{v}$ are collinear if and only if $\mathbf{v} \in [\mathbf{u}]$ or $\mathbf{u} \in [\mathbf{v}]$. Thus the zero vector is collinear with any other vector of $\mathbb{R}^3$, since $\mathbf{0} = 0 \cdot \mathbf{u} \in [\mathbf{u}]$. If $\mathbf{u}$ and $\mathbf{v}$ are both nonzero, the two conditions are equivalent, and both are equivalent to the equality $[\mathbf{u}] = [\mathbf{v}]$. Determinants of pairs of vectors provide the following criterion:

2.4.2 Proposition. *Two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ are collinear if and only if*

$$q_i(\mathbf{u}) \wedge q_i(\mathbf{v}) = 0 \quad (2.12)$$

for every $i \in \{1, 2, 3\}$.

More explicitly:

$$\begin{cases} q_1(\mathbf{u}) \wedge q_1(\mathbf{v}) = \begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix} = u_2v_3 - u_3v_2 = 0, \\ q_2(\mathbf{u}) \wedge q_2(\mathbf{v}) = \begin{vmatrix} u_1 & v_1 \\ u_3 & v_3 \end{vmatrix} = u_1v_3 - u_3v_1 = 0, \\ q_3(\mathbf{u}) \wedge q_3(\mathbf{v}) = \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} = u_1v_2 - u_2v_1 = 0. \end{cases} \quad (2.13)$$

Proof. Indeed, if $\mathbf{u}$ and $\mathbf{v}$ are collinear, all their corresponding components are proportional and, therefore, all determinants (2.13) vanish. Conversely, suppose that (2.12) holds for every $i \in \{1, 2, 3\}$. If $\mathbf{u} = \mathbf{0}$ there is nothing to prove. If $\mathbf{u} \neq \mathbf{0}$ there exists at least one index $1 \leq i \leq 3$ such that the i -th component u_i of $\mathbf{u}$ is nonzero. To fix ideas, suppose that $u_3 \neq 0$. Then the equation $q_1(\mathbf{u}) \wedge q_1(\mathbf{v}) = 0$, the first of (2.13), implies, by Proposition 2.2.4, the existence of $\lambda \in \mathbb{K}$ such that

$$\begin{cases} v_2 = \lambda \cdot u_2, \\ v_3 = \lambda \cdot u_3. \end{cases} \quad (2.14)$$

The second of (2.13), using the second of (2.14), gives:

$$0 = u_1v_3 - u_3v_1 = \lambda u_1u_3 - u_3v_1 = u_3(\lambda u_1 - v_1),$$

whence, dividing the first and last member by $u_3 \neq 0$, one obtains $v_1 = \lambda \cdot u_1$, sufficient to conclude that $\mathbf{v} = \lambda \cdot \mathbf{u}$, that is $\mathbf{v} \in [\mathbf{u}]$ as claimed. ■

2.4.3 Let

$$[\mathbf{u}, \mathbf{v}] := \{\lambda \mathbf{u} + \mu \mathbf{v} \mid \lambda, \mu \in \mathbb{R}\} \quad (2.15)$$

be the set of all linear combinations of $\mathbf{u}$ and $\mathbf{v}$ with real coefficients. If $\mathbf{v} = \mathbf{0}$, then

$$[\mathbf{u}, \mathbf{0}] = [\mathbf{u}].$$

If $\mathbf{u}$ and $\mathbf{v}$ are both nonzero and collinear, one has the equality:

$$[\mathbf{u}, \mathbf{v}] = [\mathbf{u}] = [\mathbf{v}].$$

If $[\mathbf{u}] \neq [\mathbf{v}]$, the set (2.15) is called the *vector plane* generated by $\mathbf{u}$ and $\mathbf{v}$.

If $[\mathbf{u}, \mathbf{v}]$ is a vector plane, any pair of vectors $(\mathbf{u}', \mathbf{v}')$ such that

$$[\mathbf{u}, \mathbf{v}] = [\mathbf{u}', \mathbf{v}']$$

is called a *basis* of the vector plane $[\mathbf{u}, \mathbf{v}]$. In particular $(\mathbf{u}, \mathbf{v})$ is a *basis* of $[\mathbf{u}, \mathbf{v}]$.

2.4.4 Three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ are said to be *coplanar* if they belong to the same vector plane, that is, if at least one of them is a linear combination of the other two. If $[\mathbf{u}] = [\mathbf{v}] = [\mathbf{w}]$, then $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are obviously coplanar. If instead, say, $[\mathbf{u}] \neq [\mathbf{v}]$,⁴ then $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are coplanar if and only if $\mathbf{w}$ belongs to the vector plane $[\mathbf{u}, \mathbf{v}]$ generated by $\mathbf{u}$ and $\mathbf{v}$. If three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$, pairwise non-collinear, are coplanar, then

$$W := [\mathbf{u}, \mathbf{v}] = [\mathbf{u}, \mathbf{w}] = [\mathbf{v}, \mathbf{w}].$$

and $(\mathbf{u}, \mathbf{v})$, $(\mathbf{u}, \mathbf{w})$ and $(\mathbf{v}, \mathbf{w})$ are three bases (but not the only ones) of W .

2.4.5 Exercise. Prove that if $[\mathbf{u}] \neq [\mathbf{v}]$, then every $\mathbf{w} \in [\mathbf{u}, \mathbf{v}]$ can be written as a linear combination of $\mathbf{u}$ and $\mathbf{v}$ in a *unique* way.

2.4.6 Definition. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be three vectors of $\mathbb{R}^3$ such that:

$$[\mathbf{u}, \mathbf{v}, \mathbf{w}] = \mathbb{R}^3$$

(that is: such that every vector of $\mathbb{R}^3$ can be written as a linear combination of $\mathbf{u}, \mathbf{v}, \mathbf{w}$). In this case the ordered triple $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ is called a *basis* of $\mathbb{R}^3$.

2.4.7 Linear independence. If the ordered triple $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ is a *basis* of $\mathbb{R}^3$ then $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are *linearly independent vectors* in the sense that the equality

$$\lambda \mathbf{u} + \mu \mathbf{v} + \nu \mathbf{w} = \mathbf{0}$$

implies $\lambda = \mu = \nu = 0$. Indeed if, for example, $\lambda \neq 0$, one would have

$$\mathbf{u} = -\frac{\mu}{\lambda} \mathbf{v} - \frac{\nu}{\lambda} \mathbf{w},$$

that is, $\mathbf{u}$ would belong to the vector subspace $[\mathbf{v}, \mathbf{w}]$ and the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ would be coplanar, contradicting the hypothesis of linear independence.

2.5 Determinants of triples of vectors

2.5.1 What follows presupposes the reading of paragraph 2.2, concerning the definition of determinant of a pair of columns of $\mathbb{R}^2$, as well as a review of the functions $q_i : \mathbb{R}^3 \rightarrow \mathbb{R}^2$, which “forget” the i -th component of a column of $\mathbb{R}^3$ — see 2.4.

⁴Which, by Proposition 2.4.2, is equivalent to $q_i(\mathbf{u}) \wedge q_i(\mathbf{v}) \neq 0$, for at least one $i \in \{1, 2, 3\}$.

2.5.2 The determinant $\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}$ of an ordered triple $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ of vectors of $\mathbb{R}^3$ is the scalar defined by:

$$\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w} := u_1 \cdot q_1(\mathbf{v}) \wedge q_1(\mathbf{w}) - u_2 \cdot q_2(\mathbf{v}) \wedge q_2(\mathbf{w}) + u_3 \cdot q_3(\mathbf{v}) \wedge q_3(\mathbf{w}). \quad (2.16)$$

More explicitly:

$$\begin{aligned} \mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w} : &= \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} := u_1 \begin{vmatrix} v_2 & w_2 \\ v_3 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & w_1 \\ v_3 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & w_1 \\ v_2 & w_2 \end{vmatrix} = \\ &= u_1(v_2w_3 - v_3w_2) - u_2(v_1w_3 - v_3w_1) + u_3(v_1w_2 - v_2w_1) \end{aligned} \quad (2.17)$$

where, for every $1 \leq i \leq 3$, u_i, v_i, w_i denote the i -th components of the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ respectively. The reader will verify, in particular, that:

$$\mathbf{i} \wedge \mathbf{j} \wedge \mathbf{k} = 1.$$

2.5.3 Linearity with respect to one argument. The determinant of three vectors is *linear with respect to the first argument*:

$$(\lambda \mathbf{u}_1 + \mu \mathbf{u}_2) \wedge \mathbf{v} \wedge \mathbf{w} = \lambda \cdot \mathbf{u}_1 \wedge \mathbf{v} \wedge \mathbf{w} + \mu \cdot \mathbf{u}_2 \wedge \mathbf{v} \wedge \mathbf{w}, \quad (2.18)$$

for every quadruple $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. Indeed, applying definition (2.16):

$$\begin{aligned} &(\lambda \mathbf{u}_1 + \mu \mathbf{u}_2) \wedge \mathbf{v} \wedge \mathbf{w} = \\ &= (\lambda \mathbf{u}_1 + \mu \mathbf{u}_2)(1) \cdot q_1(\mathbf{v}) \wedge q_1(\mathbf{w}) - (\lambda \mathbf{u}_1 + \mu \mathbf{u}_2)(2) \cdot q_2(\mathbf{v}) \wedge q_2(\mathbf{w}) \\ &\quad + (\lambda \mathbf{u}_1 + \mu \mathbf{u}_2)(3) \cdot q_3(\mathbf{v}) \wedge q_3(\mathbf{w}) \end{aligned} \quad (2.19)$$

where for every $1 \leq i \leq 3$, the expression $(\lambda \mathbf{u}_1 + \mu \mathbf{u}_2)(i)$ denotes the i -th component of the linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ with coefficients λ and μ . Denoting by $\mathbf{u}_j(i)$ the i -th component of the vector $\mathbf{u}_j$, for $j = 1, 2$, and invoking the definition of linear combination of columns, expression (2.19) equals the sum:

$$\begin{aligned} &(\lambda \mathbf{u}_1(1) + \mu \mathbf{u}_2(1)) \cdot q_1(\mathbf{v}) \wedge q_1(\mathbf{w}) - (\lambda \mathbf{u}_1(2) + \mu \mathbf{u}_2(2)) \cdot q_2(\mathbf{v}) \wedge q_2(\mathbf{w}) + \\ &\quad + (\lambda \mathbf{u}_1(3) + \mu \mathbf{u}_2(3)) \cdot q_3(\mathbf{v}) \wedge q_3(\mathbf{w}) \end{aligned}$$

which, through a rearrangement of the terms:

$$\begin{aligned} &= \lambda [\mathbf{u}_1(1) \cdot q_1(\mathbf{v}) \wedge q_1(\mathbf{w}) - \mathbf{u}_1(2) \cdot q_2(\mathbf{v}) \wedge q_2(\mathbf{w}) + \mathbf{u}_1(3) \cdot q_3(\mathbf{v}) \wedge q_3(\mathbf{w})] \\ &\quad + \mu [\mathbf{u}_2(1) \cdot q_1(\mathbf{v}) \wedge q_1(\mathbf{w}) - \mathbf{u}_2(2) \cdot q_2(\mathbf{v}) \wedge q_2(\mathbf{w}) + \mathbf{u}_2(3) \cdot q_3(\mathbf{v}) \wedge q_3(\mathbf{w})] \end{aligned} \quad (2.20)$$

$$= \lambda \cdot \mathbf{u}_1 \wedge \mathbf{v} \wedge \mathbf{w} + \mu \cdot \mathbf{u}_2 \wedge \mathbf{v} \wedge \mathbf{w},$$

as desired.

2.5.4 Observation. In the sum (2.20), what does the expression, for example:

$$\mathbf{u}(1) \cdot q_1(\mathbf{v}) \wedge q_1(\mathbf{v})$$

mean? the determinant $q_1(\mathbf{v}) \wedge q_1(\mathbf{v})$ multiplied by $\mathbf{u}(1)$ or $(\mathbf{u}(1) \cdot q_1(\mathbf{v})) \wedge q_1(\mathbf{w})$? The two ways of understanding it are equivalent, thanks to the linearity of the determinant with respect to the first argument, and therefore the writing is anything but ambiguous!

2.5.5 Exercise. The reader will verify that the determinant $\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}$ changes sign when exchanging $\mathbf{u}$ with $\mathbf{v}$ and $\mathbf{u}$ with $\mathbf{w}$. It is enough to observe that by exchanging u_i with v_i or u_i with w_i in (2.17), one obtains (2.17) with changed sign.

2.5.6 Linearity with respect to every argument. Antisymmetry once again implies linearity of the determinant of three vectors with respect to every argument. By way of example, let us verify it with respect to the third. One has:

$$\mathbf{u} \wedge \mathbf{v} \wedge (\lambda \mathbf{w}_1 + \mu \mathbf{w}_2) = -(\lambda \mathbf{w}_1 + \mu \mathbf{w}_2) \wedge \mathbf{v} \wedge \mathbf{u} =$$

whence, applying (2.18), already verified, and again antisymmetry:

$$= -\lambda \cdot \mathbf{w}_1 \wedge \mathbf{v} \wedge \mathbf{u} - \mu \cdot \mathbf{w}_2 \wedge \mathbf{v} \wedge \mathbf{u} = \lambda \cdot \mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}_1 + \mu \cdot \mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}_2$$

as was to be proved. Summing up, the determinant function $\mathbb{R}^3 \wedge \mathbb{R}^3 \wedge \mathbb{R}^3 \rightarrow \mathbb{R}$ is a

tri-linear antisymmetric form such that $\mathbf{i} \wedge \mathbf{j} \wedge \mathbf{k} = 1$.

2.5.7 Observation. As in the case of 2×2 determinants, linearity with respect to each argument implies the vanishing of the determinant of three vectors of $\mathbb{R}^3$ whenever at least one column is zero. Similarly, antisymmetry implies the vanishing of the determinant of three vectors whenever two columns coincide. The verifications are carried out exactly as in 2.2.3 i) and in 2.2.3 ii), and the reader is invited to go through them again in this case. In particular, for every $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, one has the equations:

$$\begin{cases} u_1 \cdot q_1(\mathbf{u}) \wedge q_1(\mathbf{v}) - u_2 \cdot q_2(\mathbf{u}) \wedge q_2(\mathbf{v}) + u_3 \cdot q_3(\mathbf{u}) \wedge q_3(\mathbf{v}) = 0 \\ v_1 \cdot q_1(\mathbf{u}) \wedge q_1(\mathbf{v}) - v_2 \cdot q_2(\mathbf{u}) \wedge q_2(\mathbf{v}) + v_3 \cdot q_3(\mathbf{u}) \wedge q_3(\mathbf{v}) = 0 \end{cases} \quad (2.21)$$

which are the expansion (2.16) of the identities:

$$\mathbf{u} \wedge \mathbf{u} \wedge \mathbf{v} = 0 \quad \text{and} \quad \mathbf{v} \wedge \mathbf{u} \wedge \mathbf{v} = 0,$$

in which at least two arguments of the determinant are equal.

2.5.8 Proposition. *Three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ are coplanar if and only if*

$$\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w} = 0.$$

Proof. If the three vectors are coplanar, at least one, say $\mathbf{w}$, is of the form $\lambda\mathbf{u} + \mu\mathbf{v}$ for suitable $\lambda, \mu \in \mathbb{R}$. Then

$$\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w} = \mathbf{u} \wedge \mathbf{v} \wedge (\lambda\mathbf{u} + \mu\mathbf{v}) = \lambda \cdot \mathbf{u} \wedge \mathbf{v} \wedge \mathbf{u} + \mu \cdot \mathbf{u} \wedge \mathbf{v} \wedge \mathbf{v} = 0$$

by 2.5.7. Conversely, suppose that

$$\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w} = 0.$$

If $[\mathbf{v}] = [\mathbf{w}]$, the three vectors are certainly coplanar, since, for example, $\mathbf{w} \in [\mathbf{u}, \mathbf{v}]$. Let us then suppose that $[\mathbf{v}] \neq [\mathbf{w}]$. In expression (2.16) there then exists at least one $1 \leq i \leq 3$ such that $q_i(\mathbf{v}) \wedge q_i(\mathbf{w}) \neq 0$, otherwise $\mathbf{v}$ and $\mathbf{w}$ would be parallel. It is no restriction to suppose that $i = 1$. Let $\lambda, \mu \in \mathbb{R}$ be the solutions of the linear system

$$q_1(\mathbf{u}) = \lambda q_1(\mathbf{v}) + \mu q_1(\mathbf{w}) \tag{2.22}$$

and let us prove that they imply

$$\mathbf{u} = \lambda\mathbf{v} + \mu\mathbf{w}. \tag{2.23}$$

Equation (2.22) is equivalent to $u_2 = \lambda v_2 + \mu w_2$ and $u_3 = \lambda v_3 + \mu w_3$, therefore to prove (2.23) it is enough to prove that $u_1 = \lambda v_1 + \mu w_1$. The equation $\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w} = 0$ and (2.16) imply the equality:

$$u_1 \cdot q_1(\mathbf{v}) \wedge q_1(\mathbf{w}) = u_2 \cdot q_2(\mathbf{v}) \wedge q_2(\mathbf{w}) - u_3 \cdot q_3(\mathbf{v}) \wedge q_3(\mathbf{w}) \tag{2.24}$$

that is, using (2.22):

$$= (\lambda v_2 + \mu w_2) \cdot q_2(\mathbf{v}) \wedge q_2(\mathbf{w}) - (\lambda v_3 + \mu w_3) \cdot q_3(\mathbf{v}) \wedge q_3(\mathbf{w}) =$$

whence, factoring out λ and μ :

$$\begin{aligned} &= \lambda(v_2 \cdot q_2(\mathbf{v}) \wedge q_2(\mathbf{w}) - v_3 \cdot q_3(\mathbf{v}) \wedge q_3(\mathbf{w})) + \\ &+ \mu(w_2 \cdot q_2(\mathbf{v}) \wedge q_2(\mathbf{w}) - w_3 \cdot q_3(\mathbf{v}) \wedge q_3(\mathbf{w})). \end{aligned}$$

Using again (2.21) one gets:

$$\begin{aligned} &= \lambda v_1 \cdot q_1(\mathbf{v}) \wedge q_1(\mathbf{w}) + \mu w_1 \cdot q_1(\mathbf{v}) \wedge q_1(\mathbf{w}) = \\ &= (\lambda v_1 + \mu w_1) \cdot q_1(\mathbf{v}) \wedge q_1(\mathbf{w}). \end{aligned} \quad (2.25)$$

And thus, from the equality of the left-hand side of (2.24) and the last member of (2.25), one finally obtains:

$$u_1 \cdot q_1(\mathbf{v}) \wedge q_1(\mathbf{w}) = (\lambda v_1 + \mu w_1) \cdot q_1(\mathbf{v}) \wedge q_1(\mathbf{w}),$$

which implies (2.23) in force of the hypothesis $q_1(\mathbf{v}) \wedge q_1(\mathbf{w}) \neq 0$.

2.5.9 Esempio. The three vectors

$$\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 4 \\ 5 \\ 10 \end{pmatrix}$$

are coplanar, because:

$$\begin{aligned} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \wedge \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \wedge \begin{pmatrix} 4 \\ 5 \\ 10 \end{pmatrix} &= \begin{pmatrix} 1 \\ 4 \end{pmatrix} \wedge \begin{pmatrix} 5 \\ 10 \end{pmatrix} - 2 \cdot \begin{pmatrix} 2 \\ 4 \end{pmatrix} \wedge \begin{pmatrix} 4 \\ 10 \end{pmatrix} + 3 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} \wedge \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \\ &= (10 - 20) - 2 \cdot (20 - 16) + 3 \cdot (10 - 4) = -10 - 8 + 18 = 0. \end{aligned}$$

Let the reader find as an exercise a way to express any one of the three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ as a linear combination of the other two.

2.6 Cramer's rule for 3×3 systems

2.6.1 Generalizing the situation described in 2.3, consider the system of three equations in three unknowns

$$\begin{cases} u_1x + v_1y + w_1z = b_1, \\ u_2x + v_2y + w_2z = b_2, \\ u_3x + v_3y + w_3z = b_3. \end{cases} \quad (2.26)$$

The array

$$(\mathbf{u}, \mathbf{v}, \mathbf{w}) := \begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{pmatrix},$$

is also called the *coefficient matrix* of system (2.26), which can be written equivalently as

$$x\mathbf{u} + y\mathbf{v} + z\mathbf{w} = \mathbf{b}, \quad (2.27)$$

with (hopefully) obvious meaning of the symbols: $\mathbf{b}$ is the column of the “known terms” and (x, y, z) are the unknowns. Equality (2.27) implies, obviously, the following three equalities:

$$\begin{cases} (x\mathbf{u} + y\mathbf{v} + z\mathbf{w}) \wedge \mathbf{v} \wedge \mathbf{w} = \mathbf{b} \wedge \mathbf{v} \wedge \mathbf{w} \\ \mathbf{u} \wedge (x\mathbf{u} + y\mathbf{v} + z\mathbf{w}) \wedge \mathbf{w} = \mathbf{u} \wedge \mathbf{b} \wedge \mathbf{w} \\ \mathbf{u} \wedge \mathbf{v} \wedge (x\mathbf{u} + y\mathbf{v} + z\mathbf{w}) = \mathbf{u} \wedge \mathbf{v} \wedge \mathbf{b} \end{cases}$$

Applying trilinearity and the fact that the determinant of triples of vectors with two equal arguments vanishes, one gets:

$$\begin{cases} x\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w} = \mathbf{b} \wedge \mathbf{v} \wedge \mathbf{w} \\ y\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w} = \mathbf{u} \wedge \mathbf{b} \wedge \mathbf{w} \\ z\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w} = \mathbf{u} \wedge \mathbf{v} \wedge \mathbf{b} \end{cases} \quad (2.28)$$

If the determinant $\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}$ of the coefficient matrix is nonzero, the unique solution of the system is:

$$(x, y, z) = \left(\frac{\mathbf{b} \wedge \mathbf{v} \wedge \mathbf{w}}{\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}}, \frac{\mathbf{u} \wedge \mathbf{b} \wedge \mathbf{w}}{\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}}, \frac{\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{b}}{\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}} \right),$$

which expresses *Cramer's rule*⁵ for the solution of systems of three linear equations (that is, first-degree equations) in three unknowns. If $\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w} = 0$, the three columns $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are coplanar.

2.6.2 Consequence. An ordered triple $\mathcal{B} := (\mathbf{u}, \mathbf{v}, \mathbf{w})$ of vectors of $\mathbb{R}^3$ is a *basis* of $\mathbb{R}^3$ if and only if its determinant is different from zero. Indeed if the vectors of $\mathcal{B}$ were linearly independent they would not be coplanar and the determinant would not be zero. They would therefore generate $\mathbb{R}^3$, since every $\mathbf{b} \in \mathbb{R}^3$ could be written (uniquely) as a linear combination of $\mathbf{u}, \mathbf{v}, \mathbf{w}$ with coefficients equal to the (unique) solutions of system (2.26). Conversely, if $\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w} \neq 0$, the linear system (2.26) has a solution for every choice of the column $\mathbf{b}$, that is every $\mathbf{b} \in \mathbb{R}^3$ is a unique linear combination of $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

⁵In honor of the Swiss mathematician Gabriel Cramer, (Geneva, 1704 – Bagnols-sur-Céze, 1752).

2.6.3 Esempio. The vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ of Example 2.5.9 are not a basis of $\mathbb{R}^3$ because they are coplanar. For example $\mathbf{e}_1 \wedge \mathbf{u} \wedge \mathbf{v} = -10 \neq 0$. Therefore $\mathbf{e}_1, \mathbf{u}, \mathbf{v}$ are not coplanar and, consequently, $\mathbf{e}_1$ cannot be expressed as a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

2.6.4 Definition. A basis $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ is said to be positively oriented if and only if

$$\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w} > 0$$

and negatively oriented if and only if

$$\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w} < 0.$$

2.6.5 Esempio. The canonical basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ of $\mathbb{R}^3$ is positively oriented. Instead $(\mathbf{e}_1, \mathbf{u}, \mathbf{v})$ with $\mathbf{u}, \mathbf{v}$ as in Example 2.5.9 is negatively oriented. Then $(\mathbf{e}_1, \mathbf{v}, \mathbf{u})$ is positively oriented (because when exchanging two arguments the determinant changes sign).

2.7 The cross product

2.7.1 Definition. The standard Euclidean cross product (or vector product, or “cross product” from the English cross product) on $\mathbb{R}^3$ is the unique map

$$\left\{ \begin{array}{lll} \times : \mathbb{R}^3 \times \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ (\mathbf{u}, \mathbf{v}) & \longmapsto & \mathbf{u} \times \mathbf{v} \end{array} \right.$$

linear with respect to the first argument, antisymmetric and such that:

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}. \quad (2.29)$$

2.7.2 Condition (2.29) is often summarized by the diagram:

$$\begin{array}{ccc} \mathbf{i} & \longrightarrow & \mathbf{j} \\ \swarrow & & \swarrow \\ & \mathbf{k} & \end{array}$$

which suggests that the first of (2.29), in which the vectors appear in alphabetical order, is sufficient to reconstruct the remaining ones, following the clockwise direction.

2.7.3 Linearity with respect to the first argument means the equality:

$$(\lambda \mathbf{u} + \mu \mathbf{v}) \times \mathbf{w} = \lambda \cdot \mathbf{u} \times \mathbf{w} + \mu \cdot \mathbf{v} \times \mathbf{w}, \quad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3 \quad \text{and} \quad \forall \lambda, \mu \in \mathbb{R} \quad (2.30)$$

while antisymmetry means:

$$\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}, \quad \forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^3. \quad (2.31)$$

The alert reader is already aware of the fact that (2.30), together with (2.31), imply *linearity with respect to the second argument*, namely

$$\mathbf{u} \times (\lambda \mathbf{v} + \mu \mathbf{w}) = \lambda \cdot \mathbf{u} \times \mathbf{v} + \mu \cdot \mathbf{u} \times \mathbf{w} \quad (2.32)$$

Indeed

$$\begin{aligned} \mathbf{u} \times (\lambda \mathbf{v} + \mu \mathbf{w}) &= -(\lambda \mathbf{v} + \mu \mathbf{w}) \times \mathbf{u} && \text{(using (2.31))} \\ &= -\lambda \cdot \mathbf{v} \times \mathbf{u} - \mu \cdot \mathbf{w} \times \mathbf{u} && \text{(using (2.30))} \\ &= \lambda \cdot \mathbf{u} \times \mathbf{v} + \mu \cdot \mathbf{u} \times \mathbf{w}. && \text{(using again (2.31))} \end{aligned}$$

Since (2.31) holds for every pair of vectors $(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^3 \times \mathbb{R}^3$, it holds in particular for the pair $(\mathbf{u}, \mathbf{u})$, with $\mathbf{u}$ arbitrary, whence the equality:

$$\mathbf{u} \times \mathbf{u} = -\mathbf{u} \times \mathbf{u} \iff 2 \cdot \mathbf{u} \times \mathbf{u} = \mathbf{0}$$

which, since $2 \neq 0$ in the field of real numbers, implies:

$$\mathbf{u} \times \mathbf{u} = \mathbf{0}, \quad \forall \mathbf{u} \in \mathbb{R}^3. \quad (2.33)$$

2.7.4 We now come to writing an explicit expression for the vector product, by explicitly listing its components. If u_i and v_i denote, as usual, the i -th component, respectively, of two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, for every $1 \leq i \leq 3$, one has:

$$\mathbf{u} \times \mathbf{v} = (u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k}) \times (v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k})$$

whence, invoking linearity with respect to the first argument:

$$\mathbf{u} \times \mathbf{v} = u_1 \mathbf{i} \times (v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}) + u_2 \mathbf{j} \times (v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}) + u_3 \mathbf{k} \times (v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k})$$

Linearity with respect to the second argument and antisymmetry (in particular the fact that $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$), therefore imply:

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= u_1 v_1 \mathbf{i} \times \mathbf{i} + u_1 v_2 \mathbf{i} \times \mathbf{j} + u_1 v_3 \mathbf{i} \times \mathbf{k} + u_2 v_1 \mathbf{j} \times \mathbf{i} + u_2 v_2 \mathbf{j} \times \mathbf{j} + u_2 v_3 \mathbf{j} \times \mathbf{k} + \\ &+ u_3 v_1 \mathbf{k} \times \mathbf{i} + u_3 v_2 \mathbf{k} \times \mathbf{j} + u_3 v_3 \mathbf{k} \times \mathbf{k} = \\ &= (u_2 v_3 - u_3 v_2) \mathbf{j} \times \mathbf{k} - (u_1 v_3 - u_3 v_1) \mathbf{k} \times \mathbf{i} + (u_1 v_2 - u_2 v_1) \mathbf{i} \times \mathbf{j} \\ &= (u_2 v_3 - u_3 v_2) \mathbf{i} + (u_3 v_1 - u_1 v_3) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k} \end{aligned} \quad (2.34)$$

having used in the last equality the expressions (2.29). Equivalently, in a more practical way:

$$\mathbf{u} \times \mathbf{v} = \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix}.$$

2.7.5 One will notice that (2.34) can be rewritten using the function $q_i : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ that “forgets” the i -th component (see (2.9)):

$$\mathbf{u} \times \mathbf{v} = q_1(\mathbf{u}) \wedge q_1(\mathbf{v})\mathbf{i} - q_2(\mathbf{u}) \wedge q_2(\mathbf{v})\mathbf{j} + q_3(\mathbf{u}) \wedge q_3(\mathbf{v})\mathbf{k}$$

which also suggests that the vector product can be computed by means of the expansion of a formal determinant:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & u_1 & v_1 \\ \mathbf{j} & u_2 & v_2 \\ \mathbf{k} & u_3 & v_3 \end{vmatrix}. \quad (2.35)$$

In particular

$$\langle \mathbf{u} \times \mathbf{v}, \mathbf{w} \rangle = q_1(\mathbf{u}) \wedge q_1(\mathbf{v})w_1 - q_2(\mathbf{u}) \wedge q_2(\mathbf{v})w_2 + q_3(\mathbf{u}) \wedge q_3(\mathbf{v})w_3 = \mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}. \quad (2.36)$$

The expression $\langle \mathbf{u} \times \mathbf{v}, \mathbf{w} \rangle$ is known as the *scalar triple product* and in fact coincides with the determinant of the three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$. Using the explicit expression (2.34) or, alternatively, (2.35), one can prove the following properties:

2.7.6 Properties of the vector product. *For every $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ and every $\lambda, \mu \in \mathbb{R}$ one has:*

- a) $\mathbf{u} \times \mathbf{v} = 0$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are parallel;
- b) $\mathbf{u} \times \mathbf{v}$ is a vector orthogonal both to $\mathbf{u}$ and to $\mathbf{v}$;
- c) One has the equality:

$$\langle \mathbf{u} \times \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \times \mathbf{w} \rangle; \quad (2.37)$$

- d) One has the formula:

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \langle \mathbf{u}, \mathbf{w} \rangle \mathbf{v} - \langle \mathbf{v}, \mathbf{w} \rangle \mathbf{u}. \quad (2.38)$$

Proof. If $\mathbf{v} = \lambda \mathbf{u}$ one has $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \lambda \mathbf{u} = \lambda \mathbf{u} \times \mathbf{u} = \mathbf{0}$ and this proves a). To verify that $\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$ it is enough to compute its scalar product with $\mathbf{u}$ and $\mathbf{v}$, obtaining:

$$\langle \mathbf{u} \times \mathbf{v}, \mathbf{u} \rangle = \mathbf{u} \wedge \mathbf{v} \wedge \mathbf{u} = 0 \quad \text{and} \quad \langle \mathbf{u} \times \mathbf{v}, \mathbf{v} \rangle = \mathbf{u} \wedge \mathbf{v} \wedge \mathbf{v} = 0$$

since the determinant of three vectors is zero if two arguments are equal. Let us now come to c).

$$\begin{aligned} \langle \mathbf{u} \times \mathbf{v}, \mathbf{w} \rangle &= \mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w} = && \text{(last equality (2.36) for the scalar triple product)} \\ &= -\mathbf{v} \wedge \mathbf{u} \wedge \mathbf{w} = && \text{(antisymmetry of the determinant,} \\ &&& \text{after exchanging } \mathbf{u} \text{ and } \mathbf{v}) \\ &= \mathbf{v} \wedge \mathbf{w} \wedge \mathbf{u} = && \text{(exchange of } \mathbf{w} \text{ and } \mathbf{u} \text{ in the determinant} \\ &&& \text{with consequent change of sign)} \\ &= \langle \mathbf{v} \times \mathbf{w}, \mathbf{u} \rangle = && \text{(again (2.36))} \\ &= \langle \mathbf{u}, \mathbf{v} \times \mathbf{w} \rangle && \text{(by symmetry (??)–(bs1) of the} \\ &&& \text{scalar product)} \end{aligned}$$

and c) is also proved. As for d), there are two methods: a direct one and an indirect one. The direct one consists in computing separately the left-hand side and the right-hand side and verifying that they are equal. The second consists in observing that it is enough to verify the proposition for

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{u}$$

where $\mathbf{u}$ and $\mathbf{v}$ are any two distinct vectors of the canonical basis. ■

2.7.7 Exercise. Prove that the vector product is not associative, that is, that there exists at least one triple of vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ such that

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} \neq \mathbf{u} \times (\mathbf{v} \times \mathbf{w}).$$

2.7.8 Solution of 2.7.7. It is enough to take $\mathbf{u} = \mathbf{i}$, $\mathbf{v} = \mathbf{i}$, $\mathbf{w} = \mathbf{j}$.

2.7.9 Observation (important). The vector space $\mathbb{R}^3$, with respect to the vector product “ $\times$ ”, is a *Lie algebra* ⁶. This means that $\times$ is bilinear, antisymmetric and satisfies the *Jacobi identity* ⁷:

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} + (\mathbf{v} \times \mathbf{w}) \times \mathbf{u} + (\mathbf{w} \times \mathbf{u}) \times \mathbf{v} = \mathbf{0}, \quad (2.39)$$

for every choice of $\mathbf{u}, \mathbf{v}, \mathbf{w}$. The reader is invited to prove this identity in at least one of the many possible ways, for example using (2.38).

2.7.10 Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$. The *sine* of the angle between $(\mathbf{u}, \mathbf{v})$ is defined by the equality:

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin(\mathbf{u}, \mathbf{v}).$$

It is undefined if at least one of the two vectors $\mathbf{u}$ or $\mathbf{v}$ is zero.

2.7.11 Volume. The *Euclidean volume* of three vectors in $\mathbb{R}^3$ is *by definition*

$$\text{Vol}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = |\langle \mathbf{u} \times \mathbf{v}, \mathbf{w} \rangle|. \quad (2.40)$$

Imagining the vectors as *arrows*, the volume intuitively corresponds to the volume of the parallelepiped having edges of lengths $|\mathbf{u}|, |\mathbf{v}|, |\mathbf{w}|$. For example, if $\mathbf{u}, \mathbf{v}, \mathbf{w}$ were pairwise orthogonal, one would have:

2.7.12 Solved Exercise. Prove formula (??).

Solution. If at least one of the three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is zero, the equality is trivial. Instead, if they are all nonzero, they form a basis of $\mathbb{R}^3$ and, therefore, the volume is certainly different from zero. First of all, $\mathbf{w}$ has the same direction as $\mathbf{u} \times \mathbf{v}$, since it is orthogonal to the vector plane $[\mathbf{u}, \mathbf{v}]$. In this case, as follows from Schwarz’s inequality, one has:

$$|\langle \mathbf{u} \times \mathbf{v}, \mathbf{w} \rangle| = |\mathbf{u} \times \mathbf{v}| \cdot |\mathbf{w}|$$

One must then evaluate the modulus $|\mathbf{u} \times \mathbf{v}|$ when $\langle \mathbf{u}, \mathbf{v} \rangle = 0$, that is, when they are orthogonal. To this end it is enough

2.7.13 The area of a parallelogram can be regarded as the volume of a parallelepiped of height 1 having the given parallelogram as base. If one of the two vectors is zero or the two vectors are collinear, the area is obviously zero.

$$\text{Area}(\mathbf{u}, \mathbf{v}) = \left\langle \mathbf{u} \times \mathbf{v}, \frac{\mathbf{u} \times \mathbf{v}}{|\mathbf{u} \times \mathbf{v}|} \right\rangle = |\mathbf{u} \times \mathbf{v}|, \quad (2.41)$$

⁶Named after the great Norwegian mathematician Sophus Lie (17 December 1842 - 18 February 1899).

⁷named after the great German mathematician Carl Gustav Jacob Jacobi (Potsdam, 10 December 1804 – Berlin, 18 February 1851).

which is precisely the volume of the parallelepiped $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ with $\mathbf{w}$ of modulus 1 and orthogonal to $[\mathbf{u}, \mathbf{v}]$ (it is enough to take, as was done, the vector product $\mathbf{u} \times \mathbf{v}$ and divide by its modulus).

A trivial manipulation of (2.41), based on the properties of the scalar product, gives:

$$\text{Area}(\mathbf{u}, \mathbf{v}) = \frac{\langle \mathbf{u} \times \mathbf{v}, \mathbf{u} \times \mathbf{v} \rangle}{|\mathbf{u} \times \mathbf{v}|} = \frac{|\mathbf{u} \times \mathbf{v}|^2}{|\mathbf{u} \times \mathbf{v}|} = |\mathbf{u} \times \mathbf{v}|,$$

as was to be verified.

2.7.14 Esempio. Let the vectors

$$\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

be given. Then

$$\text{Area}(\mathbf{u}, \mathbf{v}) = |\mathbf{u} \times \mathbf{v}| = \left| \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right| = \left| \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} \right| = \sqrt{11}$$

2.7.15 Concluding observation. The definition 2.7.1 chosen for the vector product has the undoubted advantage of being *ready for use*; however, it is not the one preferred by the author. Indeed the vector product of $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ can be elegantly and intrinsically defined as the unique vector of $\mathbb{R}^3$ such that

$$\langle \mathbf{u} \times \mathbf{v}, \mathbf{w} \rangle = \mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}.$$

Such a definition, using the bilinearity properties of the scalar product and the trilinearity and antisymmetry of the determinant, immediately shows that the function $\mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $(\mathbf{u}, \mathbf{v}) \mapsto \mathbf{u} \times \mathbf{v}$ is bilinear, antisymmetric and such that $\mathbf{i} \times \mathbf{j} = \mathbf{k}$. Moreover the equality:

$$\mathbf{u} \times \mathbf{v} = \langle \mathbf{u} \times \mathbf{v}, \mathbf{i} \rangle \mathbf{i} + \langle \mathbf{u} \times \mathbf{v}, \mathbf{j} \rangle \mathbf{j} + \langle \mathbf{u} \times \mathbf{v}, \mathbf{k} \rangle \mathbf{k}$$

immediately gives

$$\begin{aligned} \langle \mathbf{u} \times \mathbf{v}, \mathbf{i} \rangle &= \mathbf{u} \wedge \mathbf{v} \wedge \mathbf{i} = u_2 v_3 - u_3 v_2 \\ \langle \mathbf{u} \times \mathbf{v}, \mathbf{j} \rangle &= \mathbf{u} \wedge \mathbf{v} \wedge \mathbf{j} = u_3 v_1 - u_1 v_3 \\ \langle \mathbf{u} \times \mathbf{v}, \mathbf{k} \rangle &= \mathbf{u} \wedge \mathbf{v} \wedge \mathbf{k} = u_1 v_2 - u_2 v_1 \end{aligned}$$

in agreement with expression (2.34). The more direct way 2.7.1 adopted for defining the vector product nevertheless has its own interesting content. It is based, in fact, on the procedure by which one defines an algebra by means of the *structure constants*, which are exactly the coefficients of $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in the expression of $\mathbf{u} \times \mathbf{v}$ as a linear combination of $\mathbf{i}, \mathbf{j}, \mathbf{k}$, when $\mathbf{u}, \mathbf{v}$ are any two vectors of the canonical basis $(\mathbf{i}, \mathbf{j}, \mathbf{k})$.

2.8 Exercises

2.8.1 Consider the vectors

$$\mathbf{u} := \vec{i} - \vec{j} + 3\vec{k}, \quad \mathbf{v} := \vec{j} - 2\vec{k}, \quad \mathbf{w} := 3\vec{i} - 6\vec{j} - 3\vec{k}.$$

1. Compute

$$\mathbf{u} \times \mathbf{v}, \quad \mathbf{v} \times \mathbf{w}, \quad \text{and} \quad \mathbf{w} \times \mathbf{u}$$

Those who know how to use the software MATLAB[®] may check the results using the instruction

`cross`

which, for an ordered pair $(\mathbf{u}, \mathbf{v})$ of vectors, returns

$$\text{cross}(\mathbf{u}, \mathbf{v}) = \mathbf{u} \times \mathbf{v}.$$

2. Find the volume of the parallelepiped generated by $\mathbf{u}, \mathbf{v}, \mathbf{w}$;
3. Compute

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} \quad \text{and} \quad \mathbf{u} \times (\mathbf{v} \times \mathbf{w}).$$

Do the results coincide?

4. Compute $\langle 3\mathbf{u} - 4\mathbf{v} - \mathbf{w}, \mathbf{u} \times \mathbf{v} \rangle$.
5. Determine the locus of vectors $\mathbf{x} := x_1\vec{i} + x_2\vec{j} + x_3\vec{k}$ orthogonal to $\mathbf{w}$.
6. Compute the norms of each vector $\mathbf{u}, \mathbf{v}, \mathbf{w}$;
7. Prove that $\mathbf{w}$ is collinear with $\mathbf{u} \times \mathbf{v}$.

Soluzione: [?, Exercise 1.8.1].

2.8.2 Given the vectors of $\mathbb{R}^3$

$$\mathbf{u} := \vec{i} - 2\vec{j} + 3\vec{k}, \quad \mathbf{v} := 2\vec{i} - 4\vec{j} + 5\vec{k}, \quad \mathbf{w} := 4\vec{i} - 2\vec{j} + 5\vec{k}$$

compute:

1. $\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w} := \det(\mathbf{u}, \mathbf{v}, \mathbf{w})$;
2. $(2\mathbf{u} - \mathbf{v}) \wedge (\mathbf{u} + 5\mathbf{w}) \wedge (\mathbf{u} + \mathbf{v} + \mathbf{w})$;
3. $\vec{i} \wedge \vec{j} \wedge \vec{k}$.

Soluzione: [?, 1.8.2].

2.8.3 Consider the vectors

$$\mathbf{u} := 2\vec{i} - \vec{j} + \vec{k}, \quad \mathbf{v} := \vec{i} + \vec{j} - 2\vec{k}.$$

1. Compute the cosine $\cos(\mathbf{u}, \mathbf{v})$ and the sine $\sin(\mathbf{u}, \mathbf{v})$. Verify that

$$\sin^2(\mathbf{u}, \mathbf{v}) + \cos^2(\mathbf{u}, \mathbf{v}) = 1$$

2. Compute the area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$;
3. Compute the orthogonal projection of the vector $\mathbf{u}$ along the direction of $\mathbf{v}$ and the orthogonal projection of the vector $\mathbf{v}$ along the direction of $\mathbf{u}$.

Soluzione: [?, 1.8.3].

2.8.4 Prove by direct verification that

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \langle \mathbf{u}, \mathbf{w} \rangle \mathbf{v} - \langle \mathbf{v}, \mathbf{w} \rangle \mathbf{u}.$$

Exhibit a counterexample proving that the vector product is not associative. Verify the equality:

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} + (\mathbf{v} \times \mathbf{w}) \times \mathbf{u} + (\mathbf{w} \times \mathbf{u}) \times \mathbf{v} = 0$$

called the *Jacobi identity*.

Soluzione: [?, 1.8.4].

2.8.5

i) Find the area of the parallelogram determined by the vectors

$$\mathbf{u} := \begin{pmatrix} 3 \\ 5 \end{pmatrix} \quad \text{and} \quad \mathbf{v} := \begin{pmatrix} 7 \\ 1 \end{pmatrix}.$$

ii) Let $\mathbf{w}_1 := \begin{pmatrix} 3 \\ 8 \end{pmatrix}$, $\mathbf{w}_2 := \begin{pmatrix} 4 \\ 2 \end{pmatrix}$, $\mathbf{w}_3 := \begin{pmatrix} -1 \\ 2 \end{pmatrix}$. Determine $x, y \in \mathbb{R}$ such that

$$x\mathbf{u} + y\mathbf{v} = \mathbf{w}_i$$

for $i = 1, 2, 3$.

Soluzione: [?, 1.8.5]

2.8.6 Compute the volume of the parallelepiped generated by the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ where

$$1. \quad \mathbf{u} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} 3 \\ 4 \\ 8 \end{pmatrix};$$

$$2. \quad \mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 0 \\ 2 \\ 75 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix};$$

$$3. \quad \mathbf{u} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}, \quad \mathbf{w} = \mathbf{u} \times \mathbf{v};$$

Soluzione: It is a matter of evaluating determinants. 1) The volume is 0. What does this mean geometrically?; 2) The volume is 6. One does it mentally; 3) The volume is 75.

2.8.7 Let the vectors be given:

$$\mathbf{b}_1 := \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{b}_2 := \begin{pmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{pmatrix}, \quad \mathbf{b}_3 := \begin{pmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}.$$

1. Verify that each $\mathbf{b}_i$ ($1 \leq i \leq 3$) has norm one and that $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ are pairwise orthogonal. One says that they form an *orthonormal* basis.

2. Decompose the vector

$$\mathbf{u} := \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

along the vectors $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$. What are the components of $\mathbf{u}$ with respect to the system $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$?

3. Find the components of $\mathbf{b}_1, \mathbf{b}_2$ and $\mathbf{b}_3$ with respect to the system $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ (also called “basis”).

Soluzione: [?, 3.5.1]

2.8.8 Let the vectors

$$\mathbf{b}_1 := \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}, \quad \mathbf{b}_2 := \begin{pmatrix} 2 \\ 6 \\ -3 \end{pmatrix}.$$

be given.

1. Prove that $[\mathbf{b}_1] \perp [\mathbf{b}_2]$ (recall that $[\mathbf{u}]$ is the direction of $\mathbf{u}$, that is $\{\lambda \mathbf{u} \mid \lambda \in \mathbb{R}\}$)
2. Find two vectors of norm 1 having the same orientation as $\mathbf{b}_1$ and $\mathbf{b}_2$;
3. Find a vector $\mathbf{w}$ of norm 1 orthogonal to $\mathbf{b}_1$ and $\mathbf{b}_2$;
4. Decompose the vector

$$\mathbf{u} := \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$$

along the orthonormal triple

$$\left(\frac{\mathbf{b}_1}{|\mathbf{b}_1|}, \frac{\mathbf{b}_2}{|\mathbf{b}_2|}, \vec{w} \right)$$

5. Find the orthogonal projection of $\mathbf{u}$ along the direction $[\vec{w}]$.

Soluzione: [?, 3.5.2]

2.8.9 Find the locus of vectors $\mathbf{w}$ orthogonal to $\mathbf{u} := \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$ and $\mathbf{v} := \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$.

Soluzione: [?, 1.8.14].