

Chapter 1

Preview

In this chapter we sketch the study of the simplest nontrivial example of a vector space, namely the space of ordered n -tuples of real numbers written as columns. This space is denoted $\mathbb{R}^n$. The set of columns with n real entries forms a vector space because they can be added componentwise and scaled by real numbers by multiplying each component by the scalar. The two simple operations just mentioned enjoy specific properties that will be studied more in the following chapters and it is precisely these properties that endow the set $\mathbb{R}^n$ with the structure called a vector space. For this reason the ordered n -tuples of real numbers are called *vectors* and their entries *components*. The definition of the sum of two columns in $\mathbb{R}^n$ elevates the intuitive definition of sum based on the parallelogram rule to a rigorous definition.

Moreover $\mathbb{R}^n$ is a Euclidean vector space because an inner product can be defined on it. From a physical point of view, the inner product can be thought of as an experiment that measures the interaction of two physical states. This is evident in quantum mechanics, where the inner product of a vector space is a notion of central importance. From a slightly more geometric point of view, the inner product intuitively measures how far two vectors are from being orthogonal. Two vectors are orthogonal if their inner product is zero. Thanks to the inner product one can also measure the magnitude (or norm) of vectors: the norm of a vector u is defined as the square root of the inner product of u with itself.

A fundamental tool is the determinant of an n -tuple of vectors of $\mathbb{R}^n$ (see Chapter 10, entirely dedicated to it), which can also be thought of as a measure of how far they are from being linearly dependent. For example, the determinant of two vectors in $\mathbb{R}^2$ can be thought of as a measure of how far the two vectors are from being parallel. Two vectors are parallel to the same direction if and only if their determinant is zero. The notion of determinant allows, among other things, to solve

linear systems of two equations in two unknowns conveniently, by means of a method due to Gabriel Cramer. Such systems, in turn, may be interpreted as the classical intuitive decomposition of a vector along two fixed directions.

1.1 The vector space $\mathbb{R}^n$

1.1.1 Let

$$\mathbb{R}^n := \left\{ \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \mid u_i \in \mathbb{R} \right\}$$

be the set of columns with n real entries. We define an equality relation by

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \iff u_1 = v_1, u_2 = v_2, \dots, u_n = v_n.$$

The sum of two columns is defined componentwise as

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{pmatrix},$$

and the product of a column by a scalar $\lambda \in \mathbb{R}$ is

$$\lambda \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} \lambda u_1 \\ \lambda u_2 \\ \vdots \\ \lambda u_n \end{pmatrix}.$$

The entries of a column are also called *components*.

1.1.2 Notation. The i -th entry of the vector

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \in \mathbb{R}^n$$

will often be denoted by $\mathbf{u}(i)$, i.e. $\mathbf{u}(i) = u_i$ for each $1 \leq i \leq n$. This notation treats each column u as what it really is: a function from the set $\{1, 2, \dots, n\}$ to the real numbers. For example the column

$$u = \begin{pmatrix} 1 \\ \sqrt{5} \\ 3 \end{pmatrix} \in \mathbb{R}^3$$

is an equivalent way to write the function $u : \{1, 2, 3\} \rightarrow \mathbb{R}$ with $u(1) = 1$, $u(2) = \sqrt{5}$ and $u(3) = 3$.

1.1.3 Linear combinations. If $u, v \in \mathbb{R}^n$ and $\lambda, \mu \in \mathbb{R}$, the expression $\lambda u + \mu v$ is called a *linear combination* of u and v with coefficients λ and μ . It is the unique column whose i -th entry is the linear combination of the i -th entries, i.e.

$$(\lambda u + \mu v)(i) = \lambda u(i) + \mu v(i) \quad \text{for every } 1 \leq i \leq n.$$

Explicitly, if $u(i) = u_i$ and $v(i) = v_i$ for $i \in \{1, \dots, n\}$, then

$$\lambda u + \mu v = \begin{pmatrix} \lambda u_1 + \mu v_1 \\ \lambda u_2 + \mu v_2 \\ \vdots \\ \lambda u_n + \mu v_n \end{pmatrix}.$$

1.1.4 The zero vector and opposites. The column with all entries equal to zero

$$\mathbf{0} := \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

is called the *zero vector*. One immediately checks that $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$, so the zero column is the identity element for addition. The opposite of a vector u is the column

$$-\mathbf{u} := \begin{pmatrix} -u_1 \\ -u_2 \\ \vdots \\ -u_n \end{pmatrix}$$

whose components are the negatives of those of u . Componentwise addition yields $u + (-u) = (-u) + u = \mathbf{0}$.

1.1.5 Vector space structure. The set $\mathbb{R}^n$ equipped with the operations above is called the vector space of columns with n real entries and its elements are called vectors (columns). In particular, with respect to addition alone, $\mathbb{R}^n$ is an abelian group: addition is associative and commutative, there is a neutral element (the zero column) and every column $\mathbf{u}$ has an opposite $-\mathbf{u}$.

1.2 The canonical basis of $\mathbb{R}^n$

1.2.1 Let

$$E_n := (\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)$$

be the ordered family of coordinate columns of $\mathbb{R}^n$: for each $1 \leq j \leq n$ the column $\mathbf{e}_j$ has all entries zero except the j -th entry which equals 1. Formally

$$\mathbf{e}_j(i) = \delta_{ij} = \begin{cases} 0, & i \neq j, \\ 1, & i = j, \end{cases}$$

where δ_{ij} is the Kronecker delta.

1.2.2 Expansion in the canonical basis Every vector $\mathbf{u} \in \mathbb{R}^n$ can be written as a unique linear combination of the elements of E_n . Indeed,

$$\mathbf{u} = u_1 \mathbf{e}_1 + u_2 \mathbf{e}_2 + \dots + u_n \mathbf{e}_n.$$

If a set $\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ spans $\mathbb{R}^n$ we write $\mathbb{R}^n = [\mathbf{u}_1, \dots, \mathbf{u}_m]$. If this is true then necessarily $m \geq n$. In particular $\mathbb{R}^n = [\mathbf{e}_1, \dots, \mathbf{e}_n]$. Since each u has a unique representation with respect to E_n , we say that E_n is a basis of $\mathbb{R}^n$, the canonical basis. The coefficients in the expansion of u with respect to E_n are precisely the components of $\mathbf{u}$.

1.3 Spaces of linear combinations. Directions

1.3.1 Let $h \geq 1$ be an integer and set

$$[\mathbf{u}_1, \dots, \mathbf{u}_h] := \{\lambda_1 \mathbf{u}_1 + \dots + \lambda_h \mathbf{u}_h \mid \lambda_i \in \mathbb{R}\}.$$

It is easy to check that $[\mathbf{u}_1, \dots, \mathbf{u}_h]$ is a subspace of $\mathbb{R}^n$, i.e. it is closed under linear combinations. A nonempty subset $W \subset \mathbb{R}^n$ is a vector subspace if $\mathbf{u}, \mathbf{v} \in W$ implies $\lambda \mathbf{u} + \mu \mathbf{v} \in W$ for all $\lambda, \mu \in \mathbb{R}$. The subspaces $\{0\}$ and $\mathbb{R}^n$ are called trivial. A subspace W is *generated* by $u_1, \dots, u_h$ iff $W = [u_1, \dots, u_h]$; in that case the u_i are called generators of W .

1.3.2 Directions (or vector lines). If $h = 1$ and $\mathbf{u} \in \mathbb{R}^n$, then $[\mathbf{u}] = \{\lambda \mathbf{u} \mid \lambda \in \mathbb{R}\}$. If $u \neq 0$, the subspace $[\mathbf{u}]$ is called a *one-dimensional subspace* or *direction* of $\mathbb{R}^n$. Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are *collinear* iff $[\mathbf{u}] = [\mathbf{v}]$, which is equivalent to the existence of a nonzero scalar λ with $\mathbf{v} = \lambda \mathbf{u}$. The set of all directions of $\mathbb{R}^{n+1}$ is called the real projective n -space and denoted $\mathbb{RP}^n$.

1.3.3 3.4.3 The set of all directions of $\mathbb{R}^2$ is called the real projective line $\mathbb{RP}^1$, that of $\mathbb{R}^3$ the real projective plane $\mathbb{RP}^2$ (also denoted $P(\mathbb{R}^3)$).

1.3.4 3.4.4 Planes If $\mathbf{u}, \mathbf{v}$ are two noncollinear vectors in $\mathbb{R}^n$, i.e. $[\mathbf{u}] \neq [\mathbf{v}]$, then the subspace

$$[\mathbf{u}, \mathbf{v}] := \{\lambda \mathbf{u} + \mu \mathbf{v} \mid \lambda, \mu \in \mathbb{R}\}$$

is called a *plane* (a two-dimensional subspace). The set of all planes of $\mathbb{R}^n$ is denoted $G(2, \mathbb{R}^n)$ and is called the Grassmannian of 2-planes in $\mathbb{R}^n$.

1.4 3.5 The inner product

1.4.1 The space $\mathbb{R}^n$ is Euclidean in the sense that there is an inner product

$$\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$$

defined by

$$\langle u, v \rangle = \sum_{i=1}^n u(i)v(i) = u(1)v(1) + \cdots + u(n)v(n). \quad (1.1)$$

1.4.2 Example. Let

$$u = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}, \quad w = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

Then

$$\langle \mathbf{u}, \mathbf{v} \rangle = 1 \cdot 1 + (-1) \cdot 4 + 3 \cdot 1 = 0, \quad \langle \mathbf{v}, \mathbf{w} \rangle = 1 \cdot 1 + 4 \cdot 1 + 1 \cdot 2 = 7.$$

From now on we denote by $\mathbb{R}_{\geq 0}$ the set of nonnegative real numbers.

1.4.3 Proposition. The map $\langle \cdot, \cdot \rangle$ is a bilinear symmetric positive-definite form, i.e. for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and $\lambda, \mu \in \mathbb{R}$:

$$(bs1) \quad \langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle;$$

$$(bs2) \quad \langle \lambda \mathbf{u} + \mu \mathbf{v}, \mathbf{w} \rangle = \lambda \langle \mathbf{u}, \mathbf{w} \rangle + \mu \langle \mathbf{v}, \mathbf{w} \rangle;$$

$$(bs3) \quad \langle \mathbf{u}, \lambda \mathbf{v} + \mu \mathbf{w} \rangle = \lambda \langle \mathbf{u}, \mathbf{v} \rangle + \mu \langle \mathbf{u}, \mathbf{w} \rangle;$$

$$(bs4) \quad \langle \mathbf{u}, \mathbf{u} \rangle \in \mathbb{R}_{\geq 0} \text{ and } \langle \mathbf{u}, \mathbf{u} \rangle = 0 \text{ implies } \mathbf{u} = \mathbf{0}.$$

Proof. Symmetry (bs1) follows from commutativity of real multiplication. Linearity (bs2) follows from the distributive law and the definition (1.1). Then (bs3) follows from (bs1) and (bs2). Finally, (bs4) holds because

$$\langle u, u \rangle = \sum_{j=1}^n u(j)^2$$

is a sum of nonnegative real numbers; if the sum is zero then every component must be zero, hence $u = 0$.

1.4.4 Orthogonality. Two vectors $u, v \in \mathbb{R}^n$ are said to be orthogonal (with respect to $\langle \cdot, \cdot \rangle$) iff $\langle u, v \rangle = 0$.

1.4.5 Example. The vectors u and v in the previous example are orthogonal. The zero vector is orthogonal to every vector: indeed, for any u ,

$$\langle \mathbf{0}, \mathbf{v} \rangle = \langle \mathbf{0} \cdot \mathbf{u}, \mathbf{v} \rangle = \mathbf{0} \cdot \langle \mathbf{u}, \mathbf{v} \rangle = 0.$$

1.4.6 Corollary. The inner product (1.1) is nondegenerate: if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ for every $\mathbf{v} \in \mathbb{R}^n$ then $\mathbf{u} = \mathbf{0}$.

Proof. Take $\mathbf{v} = \mathbf{u}$; then $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ and by (bs4) $u = 0$.

1.4.7 Corollary. For $u, v \in \mathbb{R}^n$ we have $u = v$ iff

$$\langle u, w \rangle = \langle v, w \rangle \quad \text{for every } w \in \mathbb{R}^n.$$

Proof. If $u = v$ the equality holds trivially. Conversely, subtracting gives $\langle u - v, w \rangle = 0$ for all w , and by the previous corollary $u - v = 0$.

1.5 The norm of a vector

1.5.1 For $\mathbf{u} \in \mathbb{R}^n$ the scalar $|\mathbf{u}| := \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$ is called the norm (or length) of u . Using bilinearity one obtains for every $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and $\lambda, \mu \in \mathbb{R}$ the identity

$$|\lambda \mathbf{u} + \mu \mathbf{v}|^2 = \lambda^2 |\mathbf{u}|^2 + 2\lambda\mu \langle \mathbf{u}, \mathbf{v} \rangle + \mu^2 |\mathbf{v}|^2.$$

1.5.2 3.6.2 The Cauchy–Schwarz inequality**1.5.3 Theorem (Cauchy–Schwarz inequality)** For every $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq |\mathbf{u}| |\mathbf{v}|.$$

Proof. The inequality is trivial if $\mathbf{u} = \mathbf{0}$. Assume $\mathbf{u} \neq \mathbf{0}$ and consider the real function $f(x) = |x\mathbf{u} + \mathbf{v}|^2$. By (1.1) and positivity we get the quadratic inequality

$$|\mathbf{u}|^2 x^2 + 2\langle \mathbf{u}, \mathbf{v} \rangle x + |\mathbf{v}|^2 \geq 0 \quad \text{for all } x \in \mathbb{R}.$$

A quadratic polynomial is nonnegative for all real x iff its discriminant is nonpositive, so

$$0 \geq (2\langle \mathbf{u}, \mathbf{v} \rangle)^2 - 4|\mathbf{u}|^2 |\mathbf{v}|^2 \iff \langle \mathbf{u}, \mathbf{v} \rangle^2 \leq |\mathbf{u}|^2 |\mathbf{v}|^2,$$

and taking square roots gives the result. ■

1.5.4 Angles. The cosine of the angle between nonzero vectors u and v is defined by

$$\langle \mathbf{u}, \mathbf{v} \rangle = |\mathbf{u}| |\mathbf{v}| \cos(\mathbf{u}, \mathbf{v}).$$

Cauchy–Schwarz ensures $|\cos(u, v)| \leq 1$, so the cosine is well defined when u and v are nonzero. The vectors form an acute angle iff $\cos(u, v) > 0$, an obtuse angle iff $\cos(u, v) < 0$, and a right angle iff $\cos(u, v) = 0$.

1.5.5 Proposition: norm properties. The norm satisfies:

N1) $|\mathbf{u}| \geq 0$ and $|\mathbf{u}| = 0$ iff $\mathbf{u} = \mathbf{0}$;

N2) $|\lambda \mathbf{u}| = |\lambda| |\mathbf{u}|$ for all $\lambda \in \mathbb{R}$;

N3) (Triangle inequality) $|u + v| \leq |u| + |v|$ for all $u, v \in \mathbb{R}^n$.

Proof. Properties (1) and (2) are immediate. For the triangle inequality square both sides and use the identity

$$|u + v|^2 = |u|^2 + 2\langle u, v \rangle + |v|^2 \leq |u|^2 + 2|u||v| + |v|^2 = (|u| + |v|)^2,$$

where we applied Cauchy–Schwarz to bound $2\langle u, v \rangle \leq 2|u||v|$.

1.5.6 Unit vectors. A unit vector or *versor*) in $\mathbb{R}^n$ is a vector with norm 1. The canonical basis vectors e_i are unit vectors since $\langle e_i, e_i \rangle = 1$.

1.5.7 Example. For $\theta \in \mathbb{R}$ the vector

$$\mathbf{u} = \begin{pmatrix} \cos \theta \\ -\sin \theta \end{pmatrix} \in \mathbb{R}^2$$

is a unit vector since $\langle \mathbf{u}, \mathbf{u} \rangle = \cos^2 \theta + \sin^2 \theta = 1$.

1.5.8 Example. If $u \in \mathbb{R}^n$ is nonzero then $\mathbf{u}/|\mathbf{u}|$ and $-\mathbf{u}/|\mathbf{u}|$ are the two unit vectors having the same direction as $\mathbf{u}$.

1.5.9 Orthogonal complement For $\mathbf{u} \in \mathbb{R}^n$ define

$$\mathbf{u}^\perp := \{\mathbf{v} \in \mathbb{R}^n \mid \langle \mathbf{u}, \mathbf{v} \rangle = 0\}.$$

Then $\mathbf{u}^\perp$ is a subspace of $\mathbb{R}^n$. Note $0^\perp = \mathbb{R}^n$ and if $\mathbf{u} \neq 0$ then $\mathbf{u}^\perp$ is a hyperplane orthogonal to u .

1.5.10 Description of the orthogonal hyperplane. Writing $\mathbf{v} = \sum_{i=1}^n \mathbf{v}(i)\mathbf{e}_i$ we have $\mathbf{v} \in \mathbf{u}^\perp$ iff

$$0 = \langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i=1}^n \mathbf{u}(i)\mathbf{v}(i) = \mathbf{u}(1)\mathbf{v}(1) + \cdots + \mathbf{u}(n)\mathbf{v}(n),$$

which is the linear equation satisfied by the components of v .

1.5.11 Examples

1. For $n = 1$ the real numbers are identified with $\mathbb{R}^1$ endowed with the product $\langle u, v \rangle = uv$. Then

$$\mathbf{u}^\perp = \{\mathbf{0}\} \text{ if } \mathbf{u} \neq 0, \quad \mathbf{0}^\perp = \mathbb{R}.$$

2. For $u \in \mathbb{R}^2 \setminus \{0\}$ the orthogonal subspace $u^\perp$ is a one-dimensional line. The equation is

$$\mathbf{u}(1)\mathbf{v}(1) + \mathbf{u}(2)\mathbf{v}(2) = 0.$$

3. For $\mathbf{u} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$ the set $\mathbf{u}^\perp$ is a plane; one can explicitly find two linearly independent vectors spanning it by solving for one component in terms of the others.

1.6 Orthogonal projections

Let $\mathbf{u} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and $\mathbf{v} \in \mathbb{R}^n$. The orthogonal projection of $\mathbf{v}$ onto the direction $[\mathbf{u}]$ is defined by

$$\pi_{[\mathbf{u}]}(\mathbf{v}) = \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{|\mathbf{u}|^2} \mathbf{u}, \quad (1.2)$$

The projection is linear and does not depend on replacing $\mathbf{u}$ by $-\mathbf{u}$.

The vector $\mathbf{v} - \pi_{[\mathbf{u}]}(\mathbf{v})$ is orthogonal to $\pi_{[\mathbf{u}]}(\mathbf{v})$; indeed one checks

$$\langle \pi_{[\mathbf{u}]}(\mathbf{v}), \mathbf{v} - \pi_{[\mathbf{u}]}(\mathbf{v}) \rangle = 0.$$

1.6.1 Proposition. Let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be nonzero vectors of $\mathbb{R}^n$ pairwise orthogonal, i.e. $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ for $i \neq j$. Then for every $\mathbf{u} \in \mathbb{R}^n$ the following holds:

$$\mathbf{u} = \sum_{i=1}^n \frac{\langle \mathbf{u}, \mathbf{v}_i \rangle}{|\mathbf{v}_i|^2} \mathbf{v}_i = \pi_{\mathbf{v}_1}(\mathbf{u}) + \dots + \pi_{\mathbf{v}_n}(\mathbf{u}).$$

In particular $\mathbb{R}^n = [\mathbf{v}_1, \dots, \mathbf{v}_n]$, so $(\mathbf{v}_1, \dots, \mathbf{v}_n)$ is an orthogonal basis and the coefficients in the expansion are unique and given by the displayed formula.

Proof. If $\mathbf{u} = \sum_{i=1}^n \lambda_i \mathbf{v}_i$ then taking inner product with $\mathbf{v}_j$ and using orthogonality yields

$$\langle \mathbf{u}, \mathbf{v}_j \rangle = \lambda_j \langle \mathbf{v}_j, \mathbf{v}_j \rangle = \lambda_j |\mathbf{v}_j|^2,$$

hence $\lambda_j = \langle \mathbf{u}, \mathbf{v}_j \rangle / |\mathbf{v}_j|^2$, which gives the formula.

1.6.2 Example. If $\mathbf{u}$ is any nonzero vector in $\mathbb{R}^2$ then every $\mathbf{w} \in \mathbb{R}^2$ can be decomposed as

$$\mathbf{w} = \pi_{[\mathbf{u}]}(\mathbf{w}) + \pi_{[\mathbf{u}]^\perp}(\mathbf{w}),$$

i.e. as the sum of its projection onto $[\mathbf{u}]$ and onto the perpendicular direction.

1.6.3 Orthonormal bases An orthonormal basis of $\mathbb{R}^n$ is an ordered n -tuple $(\mathbf{b}_1, \dots, \mathbf{b}_n)$ of mutually orthogonal unit vectors, i.e. $\langle \mathbf{b}_i, \mathbf{b}_j \rangle = \delta_{ij}$. In this case the expansion of $\mathbf{u}$ simplifies to

$$\mathbf{u} = \sum_{i=1}^n \langle \mathbf{u}, \mathbf{b}_i \rangle \mathbf{b}_i.$$

1.6.4 Example. The canonical basis E_n is orthonormal. Indeed $\langle \mathbf{u}, \mathbf{e}_i \rangle = u(i)$, so the i -th component of $\mathbf{u}$ is obtained by scalar multiplying with $\mathbf{e}_i$.

1.6.5 Example. (An explicit orthonormal triple in $\mathbb{R}^3$ and an expansion example can be constructed; omitted here for brevity.)

1.6.6 Proposition. If $B = (b_1, \dots, b_n)$ is an orthonormal basis of $\mathbb{R}^n$ and $u, v \in \mathbb{R}^n$ have components $\mathbf{u}_i = \langle \mathbf{u}, \mathbf{b}_i \rangle$ and $\mathbf{v}_i = \langle \mathbf{v}, \mathbf{b}_i \rangle$, then

$$\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i=1}^n u_i v_i.$$

Proof. Write $\mathbf{u} = \sum_i u_i b_i$ and $\mathbf{v} = \sum_j v_j b_j$ and use bilinearity and orthonormality to obtain the result. ■

1.6.7 Warning. The orthonormality hypothesis is essential. If the basis is not orthogonal the simple coordinate dot-product formula need not hold.

1.6.8 Gram–Schmidt orthogonalization. Given a basis $(\mathbf{u}_1, \dots, \mathbf{u}_n)$ of $\mathbb{R}^n$, one can orthonormalize it by the Gram–Schmidt process producing an orthonormal basis $(\mathbf{b}_1, \dots, \mathbf{b}_n)$ such that $[\mathbf{b}_1, \dots, \mathbf{b}_i] = [\mathbf{u}_1, \dots, \mathbf{u}_i]$ for each i . The algorithm is classical: set $\mathbf{b}_1 = \mathbf{u}_1/|\mathbf{u}_1|$, then iterate by subtracting projections and normalizing.