

Exercise

Let $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \in \mathbb{R}^{2 \times 2}$ and

consider the linear map

$$f_A : \mathbb{R}^{2 \times 2} \longrightarrow \mathbb{R}^{2 \times 2}$$

given by

$$f_A(B) = AB$$

- i) Determine the image, its basis and its dimension;
- ii) Determine the kernel, its basis and its dimension;
- iii) Determine eigenvalues, eigenspaces and the characteristic polynomial;
- iv) Determine an associated matrix to f_A .

Solution

To fix the idea let us explicitly write the linear map proposed in the exercise

Let $B = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$ a generic matrix of $\mathbb{R}^{2 \times 2}$

$$\begin{aligned} \text{Then } f_A(B) = AB &= \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x & y \\ z & t \end{pmatrix} = \\ &= \begin{pmatrix} x+z & y+t \\ 2x+2z & 2y+2t \end{pmatrix} \quad (*) \end{aligned}$$

i) Image the image $\text{Im}(f_A)$ is the vector space generated by the images of some basis vectors of $\mathbb{R}^{2 \times 2}$.

The natural basis of $\mathbb{R}^{2 \times 2}$ is

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$\uparrow \quad \quad \uparrow \quad \quad \uparrow \quad \quad \uparrow$
 $E_{11} \quad E_{12} \quad E_{21} \quad E_{22}$

Indeed

$$\mathbb{R}^{2 \times 2} = [E_{11}, E_{12}, E_{21}, E_{22}]$$

Therefore

$$\begin{aligned}\operatorname{Im} f_A &= [f_A(E_{11}), f_A(E_{12}), f_A(E_{21}), f_A(E_{22})] = \\&= [A \cdot E_{11}, A \cdot E_{12}, A \cdot E_{21}, A \cdot E_{22}] \\&= \left[\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \right] \\&= \left[\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \right]\end{aligned}$$

It follows that

$\dim \operatorname{Im}(f_A) = 2$ and $\left\{ \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \right\}$
is one basis.

ii) The formula

$$4 = \dim \mathbb{R}^{2 \times 2} = \dim \operatorname{Im}(f_A) + \dim \operatorname{Ker}(f_A)$$

gives $\dim \operatorname{Ker}(f_A) = 2$.

$$\begin{pmatrix} x & y \\ z & t \end{pmatrix} \in \ker(f_A) \Leftrightarrow f_A \begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Leftrightarrow$$

$$\begin{pmatrix} x+z & y+t \\ 2(x+z) & 2(y+t) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Leftrightarrow$$

$$z = -x \quad t = -y$$

Thus:

$$\begin{aligned} \ker(f_A) &= \left\{ \begin{pmatrix} x & y \\ -x & -y \end{pmatrix} \mid x, y \in \mathbb{R} \right\} = \\ &= \left[\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \right] \end{aligned}$$

confirming that $\dim_{\mathbb{R}} \ker(f_A) = 2$

NOTICE $\ker(f_A)$ is the eigenspace of the eigenvalue 0.

So $mg(0) = \dim \ker(f_A) = 2$
and $ma(0) \geq 2$

ii) Eigenspaces . One eigenvalue is zero, as computed.

Now $\lambda \in \text{Spec}(f_A) \Leftrightarrow \exists B \neq 0, B \in \mathbb{R}^{2 \times 2}$

$$AB = \lambda B$$

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x & y \\ z & t \end{pmatrix} = \lambda \begin{pmatrix} x & y \\ z & t \end{pmatrix} \Leftrightarrow$$

$$\begin{pmatrix} x+y & xy+t \\ 2(x+y) & 2y+2t \end{pmatrix} = \begin{pmatrix} \lambda x & \lambda y \\ \lambda z & \lambda t \end{pmatrix} \Leftrightarrow$$

$$\begin{cases} x+z = \lambda x \\ 2x+2z = \lambda z \end{cases} \wedge \begin{cases} y+t = \lambda y \\ 2y+2t = \lambda t \end{cases}$$

From the first system we get

$$\begin{cases} (1-\lambda)x + y = 0 \\ 2x + (2-\lambda)z = 0 \end{cases}$$

which has non zero solutions $\Leftrightarrow$

$$\begin{vmatrix} 1-\lambda & 1 \\ 2 & 2-\lambda \end{vmatrix} = 0 \Leftrightarrow$$

$$\lambda(\lambda-3) = 0$$

$$\begin{matrix} \nearrow \lambda = 0 \\ \searrow \lambda = 3 \end{matrix}$$

Known

The same can be argued by the system

$$\begin{cases} y + t = \lambda y \\ 2y + 2t = \lambda t \end{cases}$$

Let us compute the eigenspace of f_A with respect to β

$$\begin{cases} (1 - \lambda)x + z = 0 \\ 2x + (2 - \lambda)z = 0 \end{cases} \quad \text{for } \lambda = 3 \text{ gives}$$

$$\begin{cases} -2x + z = 0 \\ 2x - z = 0 \end{cases} \quad \Leftrightarrow z = 2x$$

Thus $\begin{pmatrix} x & 0 \\ 2x & 0 \end{pmatrix}$ is an eigenvector with respect to β .

Using the other system one finds

$\begin{pmatrix} 0 & y \\ 0 & 2y \end{pmatrix}$ is eigenvector.

The general Eigenspace is

$$\begin{pmatrix} x & y \\ 2x & 2y \end{pmatrix} = x \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} + y \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}$$

Thus $\dim \ker(f_A - 3\text{id}_A) = 2$

$$\text{mg}(3) = 2 \Rightarrow \text{ma}(3) \geq 2$$

Since $\text{ma}(0) + \text{ma}(3) = 4 \Rightarrow$

$\text{ma}(0) = 2$ and $\text{ma}(3) = 2$. Therefore

$$P_{f_A}(T) = T^2 (T - 3)^2$$

$\uparrow$
 characteristic
 Polynomial.

iv) The associated matrix to f_A in the basis $\mathcal{E} = (E_{11}, E_{12}, E_{21}, E_{22})$ is given by

$$M_{\mathcal{E}}^{\mathcal{E}} = \left[(A E_{11})_{\mathcal{E}}, (A E_{12})_{\mathcal{E}}, (A E_{21})_{\mathcal{E}}, (A E_{22})_{\mathcal{E}} \right] =$$

=

$$= \left[\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}_E, \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}_E, \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}_E, \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}_E \right] =$$

$$= \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{pmatrix}$$

End of exercise If you want, please
compute the characteristic polynomial out
of the associated matrix.