

5.1 Linear systems.

5.1.1 Let $A \in \mathbb{K}^{m \times n}$. Then A can be viewed as a linear map

$$A : \mathbb{K}^n \longrightarrow \mathbb{K}^m.$$

Since $A = (A\mathbf{e}_1, \dots, A\mathbf{e}_n)$, and since $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ is a basis of $\mathbb{K}^n$, it follows that $\text{Im}(A) = [A \cdot \mathbf{e}_1, \dots, A \cdot \mathbf{e}_n]$, that is, the columns of A generate the image. Let now $\mathbf{w} \in \mathbb{K}^m$ and let

$$A^{-1}\mathbf{w} := \{\mathbf{v} \in \mathbb{K}^n \mid A\mathbf{v} = \mathbf{w}\}$$

The set $A^{-1}(\mathbf{w})$ is called the set of preimages of $\mathbf{w}$ (not to be confused with the inverse of the matrix A , which might not exist, at the vector $\mathbf{w}$). Note that this set may be empty, in the case where $\mathbf{w} \notin \text{Im}(A)$. Observe that if $\mathbf{w} \neq \mathbf{0}$, then $A^{-1}(\mathbf{w})$ is not a vector subspace of $\mathbb{K}^n$. Indeed, if it were, $\mathbf{0}_{\mathbb{K}^n}$ should be one of its elements, but this is impossible, since $A\mathbf{0}_{\mathbb{K}^n} = \mathbf{0}_{\mathbb{K}^m} \neq \mathbf{w}$. Conversely, $A^{-1}(\mathbf{0})$ is a vector subspace, traditionally called the *kernel* of the matrix A . More details will follow.

5.1.2 The problem we wish to study is that of determining the preimages of a given vector $\mathbf{b}$ of $\mathbb{K}^m$ through the linear map $A : \mathbb{K}^n \longrightarrow \mathbb{K}^m$ defined by a matrix $A \in \mathbb{K}^{m \times n}$. This amounts to solving the equation

$$A \cdot \mathbf{x} = \mathbf{b} \tag{5.1}$$

with matrix coefficients and vector unknowns. If one sets:

$$A := \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

then finding the solution of equation (4.1) is equivalent to solving the system of m linear equations in n unknowns:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots + \vdots + \ddots + \vdots = \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases} \tag{5.2}$$

which can be written in *explicit matrix form* as:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \tag{5.3}$$

or, in explicit vector form:

$$x_1 C_1(A) + x_2 C_2(A) + \dots + x_n C_n(A) = \mathbf{b} \quad (5.4)$$

or, more explicitly:

$$x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \dots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

5.1.3 Proposition. *The linear system $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b} \in \text{Im}(A)$*

Proof. The proof is obvious. If $\mathbf{b} \in \text{Im}(A)$, by definition of image there exists at least one $\mathbf{x}_0 \in \mathbb{K}^n$ such that $A\mathbf{x}_0 = \mathbf{b}$, that is, there exists at least one solution of the linear system $A\mathbf{x} = \mathbf{b}$. Conversely, if the linear system $A\mathbf{x} = \mathbf{b}$ has a solution, then, by definition of [image](#), $\mathbf{b} \in \text{Im}(A)$. ■

Let $\ker(A) := \{\mathbf{x} \in \mathbb{K}^n \mid A\mathbf{x} = \mathbf{0}_{\mathbb{K}^m}\}$. In other words, $\ker(A)$ is the set of solutions of the *associated homogeneous linear system* $A\mathbf{x} = \mathbf{0}_{\mathbb{K}^m}$. Suppose that $\mathbf{b} \in \text{Im}(A)$. Then $A^{-1}\mathbf{b} \neq \emptyset$. Then $A^{-1}(\mathbf{b})$ is the set of solutions of the system $A\mathbf{x} = \mathbf{b}$. If $\mathbf{x}_0 \in A^{-1}(\mathbf{b})$, that is, if $A\mathbf{x}_0 = \mathbf{b}$, then $\mathbf{x}_0$ will also be called a *particular solution* of equation (4.1).

5.1.4 Definition. *If $A^{-1}(\mathbf{b}) \neq \emptyset$, the dimension of the set of solutions is equal, by definition, to:*

$$\dim A^{-1}(\mathbf{b}) = \dim \ker(A)$$

If $\dim A^{-1}(\mathbf{b}) = n - r$, one also says that the system has ∞^{n-r} solutions.

5.1.5 Proposition. *Suppose that $A^{-1}(\mathbf{b}) \neq \emptyset$ and let $\mathbf{x}_0 \in A^{-1}(\mathbf{b})$. Then:*

$$A^{-1}(\mathbf{b}) := \mathbf{x}_0 + \ker(A) := \{\mathbf{x}_0 + \mathbf{u} \mid \mathbf{u} \in \ker(A)\}. \quad (5.5)$$

Proof. The proof proceeds by double inclusion. First of all, if $\mathbf{x} \in \mathbf{x}_0 + \ker(A)$, then $\mathbf{x} = \mathbf{x}_0 + \mathbf{u}$ for some $\mathbf{u} \in \ker(A)$. Therefore

$$A\mathbf{x} = A(\mathbf{x}_0 + \mathbf{u}) = A\mathbf{x}_0 + A\mathbf{u} = \mathbf{b} + \mathbf{0}_{\mathbb{K}^m}$$

since $\mathbf{u} \in \ker(A)$. This proves that $\mathbf{x}_0 + \ker(A) \subseteq A^{-1}(\mathbf{b})$. Conversely, suppose that $\mathbf{x} \in A^{-1}(\mathbf{b})$. Then $\mathbf{b} = A\mathbf{x} = A\mathbf{x}_0$, that is, $A\mathbf{x} - A\mathbf{x}_0 = A(\mathbf{x} - \mathbf{x}_0) = \mathbf{0}_{\mathbb{K}^m}$, that is,

$\mathbf{u} := \mathbf{x} - \mathbf{x}_0 \in \ker(A)$. This proves that $A^{-1}(\mathbf{b}) \subseteq \mathbf{x}_0 + \ker(A)$, that is, equality (4.5). ■

An element of $\ker(A)$ is also called a *solution of the associated homogeneous system* of A . Therefore proposition 4.1.5 says that *the general solution of the system* $A\mathbf{x} = \mathbf{b}$ is the sum of a particular solution and the general solution of the associated homogeneous system.

5.1.6 Observation. Let $C^\infty(\mathbb{R})$ be the vector space of real-valued functions of a real variable that are infinitely differentiable. Let $D := d/dx$ be differentiation with respect to x . From analysis courses one knows that if $f, g \in C^\infty(\mathbb{R})$ and $\lambda, \mu \in \mathbb{R}$, then

$$D(\lambda \cdot f + \mu \cdot g) = \lambda \cdot Df + \mu \cdot Dg.$$

Therefore D is an endomorphism of $C^\infty(\mathbb{R})$. Since the composition of endomorphisms is an endomorphism and the sum of endomorphisms is an endomorphism, it follows that every expression of the form

$$P(D) := D^n + a_1 D^{n-1} + \dots + a_n \mathbf{1}$$

is an endomorphism of $C^\infty(\mathbb{R})$ called a “differential operator of order n ”. Solving the equation

$$P(D)f = g$$

means solving the associated differential equation:

$$f^{(n)} + a_1 f^{(n-1)} + \dots + a_n f = g.$$

From the theory of differential equations with constant coefficients (see also the wonderful [?]) one knows that the general solution of the differential equation is given by a particular solution plus the general solution of the associated homogeneous equation. In other words, if $p = p(x)$ is a particular solution, the set of solutions is:

$$P(D)^{-1}(p) = \ker P(D) + p.$$

5.1.7 Let now $A\mathbf{x} = \mathbf{b}$ be a linear system. The matrix:

$$(A|\mathbf{b}) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array} \right)$$

is called the *augmented matrix of the system* (the bar is inserted purely for convenience, in order also to distinguish visually the coefficient matrix from the column of constant terms).

5.1.8 Theorem (Rouché-Capelli) *The linear system (4.1) has solutions if and only if the rank of the matrix A coincides with the rank of the augmented matrix $(A|\mathbf{b})$. In the latter case, the dimension of the space of solutions is $\dim A^{-1}(\mathbf{b}) = \dim \ker(A) = n - r$, where r is the rank of A .*

Proof. Indeed, if the system is solvable, then $\mathbf{b} \in \text{Im}(A)$ and, therefore, $\mathbf{b}$ is a linear combination of the columns of A . Hence, since $[C_1(A), \dots, C_n(A)] = [C_1(A), \dots, C_n(A), \mathbf{b}]$, that is,

$$\text{rk}(A) = \dim_{\mathbb{K}}[C_1(A), \dots, C_n(A)] = \dim_{\mathbb{K}}[C_1(A), \dots, C_n(A), \mathbf{b}] = \text{rk}(A|\mathbf{b}).$$

Conversely, if $\text{rk}(A) = \text{rk}(A|\mathbf{b})$, then $\mathbf{b}$ is a linear combination of the columns of A , which generate the image, and therefore $\mathbf{b}$ lies in the image, that is, the system has solutions. ■

5.1.9 Observation. Note that for the homogeneous linear system $A\mathbf{x} = \mathbf{0}$, which always has solutions (at the very least the zero solution), the Rouché-Capelli condition is trivially satisfied.



5.1.10 If $A\mathbf{x} = \mathbf{b}$ is a Cramer system, that is, if $A \in \mathbb{K}^{n \times n}$ (A , that is, is a square matrix), the solution certainly exists if A is invertible. Indeed, if A is invertible, the columns of A form a basis of $\mathbb{K}^n$, and therefore every vector $\mathbf{b}$ is a linear combination of those vectors. The condition for the matrix not to be invertible is $\det(A) = 0$. If $\det(A) \neq 0$, then the system has $\infty^{n-n} = \infty^0$ solutions. The solution is unique. Indeed, if $\mathbf{x}$ and $\mathbf{x}'$ are solutions, then $\mathbf{x} - \mathbf{x}' \in \ker(A)$. Since $\ker(A) = \mathbf{0}$, it follows that $\mathbf{x} = \mathbf{x}'$. ■

5.1.11 Solved Exercise. Study, as the parameters $h, k \in \mathbb{R}$ vary, the solutions of the linear system

$$\begin{cases} x_1 - x_2 + x_3 + 3x_4 = 1 \\ x_1 + kx_2 + + kx_4 = k \\ x_2 - x_3 + hx_4 = 0 \end{cases}$$

Solution. The augmented matrix of the system is:

$$(A(h, k) | \mathbf{b}(k)) = \left(\begin{array}{cccc|c} 1 & -1 & 1 & 3 & 1 \\ 1 & k & 0 & k & k \\ 0 & 1 & -1 & h & 0 \end{array} \right)$$

The notation serves to recall the explicit dependence of the coefficient matrix and the constant column on the parameters h and k . We can now begin a row reduction, as learned in ??, proposing this first elementary transformation:

$R_3 \rightarrow R_3 + R_1$, obtaining:

$$(A_1(h, k) | \mathbf{b}_1(k)) := \left(\begin{array}{cccc|c} 1 & -1 & 1 & 3 & 1 \\ 1 & k & 0 & k & k \\ 1 & 0 & 0 & h+3 & 1 \end{array} \right)$$

It is immediately seen that if $k \neq 0$, for every $h \in \mathbb{R}$, the matrix $A_1(h, k)$ is reduced and, therefore, $\text{rk}(A) = \text{rk}A_1(h, k) = \text{rk}(A_1(h, k) | \mathbf{b}_1(k)) = 3$. By the Rouché–Capelli theorem, therefore, the system is compatible and has $\infty^{4-3} = \infty^1$, a way of saying that there is a one-parameter family of solutions:

$$\begin{cases} x_1 = 1 - (3 + h)t \\ x_2 = \frac{1}{k}(k - kt - 1 + 3t + ht) \\ x_3 = ht + \frac{1}{k}(k - kt - 1 + 3t + ht) \\ x_4 = t \end{cases}$$

If $k = 0$, the matrix $(A_1(h, 0), \mathbf{b}_1(0))$ becomes:

$$(A_1(h, 0) | \mathbf{b}_1(0)) = \left(\begin{array}{cccc|c} 1 & -1 & 1 & 3 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & h+3 & 1 \end{array} \right)$$

Performing the further elementary row transformation:

$$R_3(A_1(h, 0), \mathbf{b}_1(0)) \rightarrow R_3(A_1(h, 0), \mathbf{b}_1(0)) - R_2(A_1(h, 0), \mathbf{b}_1(0))$$

one obtains:

$$(A_2(h, 0), \mathbf{b}_2(0)) = \left(\begin{array}{cccc|c} 1 & -1 & 1 & 3 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & h+3 & 1 \end{array} \right)$$

For $h \neq -3$, the matrix $A_2(h, 0)$ is reduced (up to interchanging the third and second rows); $\text{rk}(A) = 3$ and $\text{rk}(A|B) = 3$.

Therefore, by the Rouché–Capelli Theorem, the system is compatible and has $\infty^{4-3} = \infty^1$ solutions depending on one parameter:

$$\begin{cases} x_1 = 0 \\ x_2 = t \\ x_3 = 1 + t - \frac{3}{h+3} \\ x_4 = \frac{1}{h+3} \end{cases}$$

If $k = 0 \wedge h = -3$, the augmented matrix becomes:

$$(A_2(0, -3), \mathbf{b}_2(0)) = \left(\begin{array}{cccc|c} 1 & -1 & 1 & 3 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right)$$

The matrix is reduced; $\text{rk}(A_2(0, -3)) = 2$ and $\text{rk}(A_2(0, -3)|\mathbf{b}_2(0)) = 3$. Therefore, by the Rouché – Capelli theorem, the system has no solutions.

5.1.12 To readers, male or female, who might wish to complain, arguing that one example is too little, we reply: after all, these exercises are all the same...And, in any case, others follow just below.