

ExALaa. Let the linear map $f : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ be defined by

$$f \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} x + z - t \\ y + z \\ x - y - t \end{pmatrix}.$$

- (a) Find the matrix M_f of f with respect to the standard bases of $\mathbb{R}^4$ and $\mathbb{R}^3$, respectively.
- (b) Determine the dimension of the kernel of f .
- (c) Determine the dimension of the image of f .
- (d) Find a basis for the kernel of f .
- (e) Find a basis for the image of f .
- (f) Determine, if it exists, a vector in $\mathbb{R}^3$ having no preimages.
- (g) Given the vector

$$\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \in \mathbb{R}^3,$$

determine the set $f^{-1}(\mathbf{v})$ of its preimages.

ExALab. Let the linear map $f : \mathbb{R}^3 \rightarrow \mathbb{R}^4$ be defined by

$$f \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + z \\ y + z \\ x + y + 2z \\ 2y + 2z \end{pmatrix}.$$

- (a) Find the matrix A of f with respect to the standard bases of $\mathbb{R}^3$ and $\mathbb{R}^4$, respectively.
- (b) Determine the dimension of the kernel of A .
- (c) Find a basis for the kernel of A .
- (d) Determine the dimension of the image of A .
- (e) Find a basis for the image of A .
- (f) Determine, if it exists, a vector in $\mathbb{R}^4$ having no preimages.
- (g) Given the vector

$$\mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \in \mathbb{R}^4,$$

determine the set $A^{-1}(\mathbf{v})$ of its preimages.

ExALac. Let the linear map $f : \mathbb{R}^3 \rightarrow \mathbb{R}^4$ be defined by

$$f \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y - z \\ y + z \\ x - y + 3z \\ x + z \end{pmatrix}.$$

- (a) Find the matrix M of f with respect to the standard bases of $\mathbb{R}^3$ and $\mathbb{R}^4$, respectively.
- (b) Determine the kernel of M .
- (c) Decide whether f is injective.
- (d) Determine the dimension of the image of f .
- (e) Find a basis for the image of f .
- (f) Prove that f is not surjective and determine a vector in $\mathbb{R}^4$ having no preimages.
- (g) Given the vector

$$\mathbf{v} = \begin{pmatrix} -1 \\ 1 \\ 3 \\ 1 \end{pmatrix} \in \mathbb{R}^4,$$

determine whether the set $f^{-1}(\mathbf{v})$ of its preimages is empty or not.

ExALad. Let the linear map $f : \mathbb{R}^3 \rightarrow \mathbb{R}^4$ be defined by

$$f \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y - z \\ y + z \\ x - y + 3z \\ z \end{pmatrix}.$$

- (a) Find the matrix M of f with respect to the standard bases of $\mathbb{R}^3$ and $\mathbb{R}^4$, respectively.
- (b) What is the dimension of the kernel of f ?
- (c) Decide whether f is injective.
- (d) Determine the dimension of the image of f .
- (e) Find a basis for the image of f .
- (f) Prove that f is not surjective and determine a vector in $\mathbb{R}^4$ having no preimages.
- (g) Given the vector

$$\mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^4,$$

determine the set $f^{-1}(\mathbf{v})$ of its preimages.

ExALba. Let f be the endomorphism of $\mathbb{R}^3$ associated, with respect to the standard bases, with the matrix

$$A = \begin{pmatrix} 2 & -4 & -8 \\ -1 & 2 & 4 \\ 1 & -2 & -4 \end{pmatrix}$$

- (a) Find the dimension of $\ker(f)$.
- (b) Find a basis of $\ker(f)$.
- (c) Find the dimension of $\Im(f)$, the image of f .
- (d) Find a basis of $\Im(f)$.
- (e) Verify that every vector in the image of f lies in the kernel of f , and deduce that $f \circ f = 0$.
- (f) Does there exist $P \in Gl_3(\mathbb{R})$ such that $P^T A P$ is diagonal?
- (g) Does there exist $P \in Gl_3(\mathbb{R})$ such that $P^{-1} A P$ is diagonal?

ExALbb. Let f be the endomorphism of $\mathbb{R}^3$ associated, with respect to the standard bases, with the matrix

$$A = \begin{pmatrix} 2 & -4 & -8 \\ -1 & 2 & 4 \\ -1 & 2 & 4 \end{pmatrix}$$

- (a) Find the dimension of $\ker(f)$.
- (b) Find a basis of $\ker(f)$.
- (c) Find the dimension of $\Im(f)$, the image of f .
- (d) Find a basis of $\Im(f)$.
- (e) Find the eigenvalues of the matrix, determining their algebraic and geometric multiplicities.
- (f) Does there exist $P \in Gl_3(\mathbb{R})$ such that $P^T A P$ is diagonal?
- (g) Does there exist $P \in Gl_3(\mathbb{R})$ such that $P^{-1} A P$ is diagonal?

ExGALfa. Let

$$A := \begin{pmatrix} 5 & 2 & 2 & 2 \\ 2 & 5 & 2 & 2 \\ 2 & 2 & 5 & 2 \\ 2 & 2 & 2 & 5 \end{pmatrix}$$

- (a) Decide whether the matrix A is diagonalizable;
- (b) Prove that the matrix $A - 11 \cdot \mathbf{1}_{4 \times 4}$ is not invertible, and that $3 \in \text{Spec}(A)$;
- (c) Find a diagonal form of the matrix A , if it exists;
- (d) Determine the characteristic polynomial of A ;

- (e) Prove that the vector $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4$ is orthogonal to $\ker(A - 3\mathbf{1}_{4 \times 4})$;
- (f) Diagonalize the matrix A ;
- (g) Say whether there exists an invertible matrix P , and explain why, such that $P^T A P$ is diagonal.

ExGALfb. Let

$$A := \begin{pmatrix} 5 & 1 & 1 & 1 \\ 1 & 5 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 5 \end{pmatrix}$$

- (a) Determine $\ker(A)$;
- (b) Prove that the matrix $A - 6 \cdot \mathbf{1}_{4 \times 4}$ is not invertible, and that $4 \in \text{Spec}(A)$;
- (c) Find a diagonal form of the matrix A , if it exists;
- (d) Prove that the vector $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4$ is orthogonal to $\ker(A - 3\mathbf{1}_{4 \times 4})$;
- (e) Determine $P \in GL_4(\mathbb{R})$ such that $P^{-1} A P$ is diagonal;
- (f) Say whether there exists an invertible matrix P , and explain why, such that $P^T A P$ is diagonal.

ExGALfc. Let

$$A := \begin{pmatrix} 5 & 1 & 1 & 1 \\ 1 & 5 & 1 & 1 \\ 1 & 1 & 5 & 1 \\ -3 & -3 & -3 & 1 \end{pmatrix}$$

- (a) Determine $\ker(A)$;
- (b) Prove that the matrix $B := A - 4 \cdot \mathbf{1}_{4 \times 4}$ is not invertible;
- (c) Determine the characteristic polynomial of A ;
- (d) Say whether A is diagonalizable, briefly explaining why;
- (e) Find a basis for each eigenspace;
- (f) Prove that B^4 is the zero matrix;
- (g) Say whether there exists an invertible matrix P , and explain why, such that $P^T A P$ is diagonal.

ExGALfd. Let

$$A := \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ -3 & -3 & -3 & -2 \end{pmatrix}$$

- (a) Determine $\Im(A)$;
- (b) Prove that the matrix $B := A - \cdot \mathbf{1}_{4 \times 4}$ is not invertible;
- (c) Determine the characteristic polynomial of A ;

- (d) Say whether A is diagonalizable, briefly explaining why;
- (e) Find a basis for each eigenspace of A ;
- (f) Prove that B^4 is the zero matrix;
- (g) Does there exist $P \in Gl_4(\mathbb{R})$ such that $P^{-1}AP$ is diagonal?
- (h) Say whether there exists an invertible matrix P , and explain why, such that P^TAP is diagonal.

ExALbc Let the matrix

$$A = \begin{pmatrix} 2 & -4 & -8 \\ -1 & 2 & 4 \\ -1 & 2 & 4 \end{pmatrix}$$

be regarded as the endomorphism of $\mathbb{R}^3$ defined by $\mathbf{u} \mapsto A \cdot \mathbf{u}$.

- (a) Find the dimension of $\ker A$.
- (b) Find a basis of $\ker A$.
- (c) Determine the dimension of $\Im A$, the image of A .
- (d) Find a basis of $\Im A$.
- (e) Verify that every vector in the image of A lies in the kernel of A , and deduce that $A^n = 0$ for every $n \geq 2$.
- (f) Does there exist $P \in Gl_3(\mathbb{R})$ such that P^TAP is diagonal?
- (g) Does there exist $P \in Gl_3(\mathbb{R})$ such that $P^{-1}AP$ is diagonal?

ExALbd Let the matrix

$$A = \begin{pmatrix} -1 & 2 & 4 \\ -1 & 2 & 4 \\ 2 & -4 & -8 \end{pmatrix}$$

be regarded as the endomorphism of $\mathbb{R}^3$ defined by $\mathbf{u} \mapsto A \cdot \mathbf{u}$.

- (a) Find the dimension of $\ker A$.
- (b) Find a basis of $\ker A$.
- (c) Determine the dimension of $\Im A$, the image of A .
- (d) Find a basis of $\Im A$.
- (e) Determine the eigenvalues of A and their algebraic and geometric multiplicities.
- (f) Is the matrix A diagonalizable? Why?
- (g) Does there exist $P \in Gl_3(\mathbb{R})$ such that P^TAP is diagonal?

ExALca. Let f be the endomorphism of $\mathbf{R}^3$ associated, with respect to the standard bases, with the matrix

$$M = \begin{pmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ 1 & 0 & 1 \end{pmatrix}.$$

- (a) Find the dimension and a basis of the kernel of f , and a Cartesian equation for it.
 - (b) Determine the eigenvalues and eigenvectors of M .
 - (c) Determine the eigenspaces of f .
 - (d) Find, if possible, a basis $\mathcal{B}$ of $\mathbf{R}^3$ that diagonalizes M .
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ExCALCaa. Let the function

$$f(x, y) = 9x - (y - x)^2 - 2y^3$$

be given, and let S_f be the graph surface of f , with Cartesian equation $F(x, y, z) = 0$, where $F(x, y, z) := f(x, y) - z$.

- (a) Compute the gradient of f and the gradient of F .
- (b) Find the tangent plane to S_f at the point $(1, 1, 7)$.
- (c) Determine any local maximum, local minimum, and saddle points of the function f .
- (d) Give a parametric representation of the line passing through the point with coordinates $(1, 1, 7)$ and orthogonal to the tangent plane to S_f at that point.

ExCALCba. Let the function

$$g(x, y, z) = (\sqrt{x})^2 y^2 + x^2 - 3 + z$$

and the curve $\gamma(t) = (t, t^2, t^3)$, for $1 \leq t \leq 2$, be given.

- (a) Determine the domain of g .
- (b) Determine any local maximum, local minimum, and saddle points of the function g .
- (c) Determine $\dot{\gamma}(0)$, the tangent vector to γ at $t = 0$.
- (d) Compute the line integral:

$$\int_{\gamma} \frac{g ds}{\sqrt{1 + 4x^2 + 9y^2}}.$$

ExCALCbb. Let the function

$$g(x, y, z) = x^3 + y^2 - (\sqrt{x+2})^2 - z$$

and the curve $\gamma(t) = (t, t + t^2, 1 + t^3)$, for $-1 \leq t \leq 1$, be given.

- (a) Determine the domain of g .
- (b) Determine any local maximum, local minimum, and saddle points of the function g .

- (c) Determine $\dot{\gamma}(0)$, the tangent vector to γ at $t = 0$.
- (d) Compute the line integral:

$$\int_{\gamma} \frac{gds}{\sqrt{1+4x^2+9y^2}}.$$

ExCALCca. Let the function

$$f(x, y) = xy^2 + y^3 - x$$

be given, and let S_f be the graph surface of f , defined by the equation $F(x, y, z) = 0$ with $F(x, y, z) := f(x, y) - z$.

- (a) Determine any local maximum, local minimum, and saddle points of the function f .
- (b) Find the tangent plane to S_f at the point $P_0(1, 1, 1)$.
- (c) Determine, in Cartesian form, the normal line to S_f at the point $P_0(1, 1, 1)$.
- (d) Given the function $\gamma : [0, 1] \rightarrow \mathbf{R}^3$ defined by $\gamma(t) = (1, 0, 2t)$, compute $\|\gamma'(t)\|$ and the integral $\int_{\gamma} F$.

ExGALdb. Exercise. Let the map $f : \mathbb{R}^4 \rightarrow \mathbb{R}^2$ be defined by

$$f \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} ax_1^2 + 3x_2 + 2x_3 - x_4 \\ 2x_1 + 3x_2 - x_3 - 4x_4 - b \end{pmatrix}$$

- (a) Determine the values of $a, b \in \mathbb{R}$ for which f is linear.
- (b) Set $a = 0$ and determine the values of b for which

$$f^{-1} \begin{pmatrix} 0 \\ 0 \end{pmatrix} := \{\mathbf{u} \in \mathbb{R}^4 \mid f(\mathbf{u}) = 0\}$$

is a vector subspace.

ExGALdc. Let the matrices

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \quad \text{and} \quad B := \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$$

be given.

- (a) Verify that 4 is an eigenvalue of A and 8 is an eigenvalue of B ;
- (b) Compute the difference $A \cdot B - B \cdot A$;
- (c) Decide whether A and B are diagonalizable, explaining why;

- (d) Find a matrix P that diagonalizes A and compute $P^{-1}BP$;
- (e) Determine the eigenspaces of A and those of B .

ExGALdd. Exercise. Consider the function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ given by

$$f(x, y, z) = 3x^2 + 3y^2 + 3z^2 - 2x - 4y - 2z$$

- (a) Evaluate $f(1, 2, 3)$.
- (b) Find the coordinates of at least one point belonging to the sphere $3x^2 + 3y^2 + 3z^2 - 2x - 4y - 2z - 26 = 0$.
- (c) Determine the center and radius of the spheres

$$f^{-1}(d) = 0$$

as $d > 0$ varies.

- (d) Determine d such that $f^{-1}(d) = \left(\frac{1}{2}, \frac{2}{3}, \frac{1}{3}\right)$.

ExGALea. Consider the matrix

$$A := \begin{pmatrix} 2 & 1 & 2 \\ 4 & 2 & 4 \\ -4 & -2 & -4 \end{pmatrix}$$

- (a) Determine a linear map $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ such that $\ker f = \ker A$;
- (b) Say whether f is surjective;
- (c) Determine the dimension and a basis of $\ker f$;
- (d) Find all eigenvalues of A ;
- (e) Determine all eigenspaces of A ;
- (f) Say whether there exists a matrix P such that P^TAP is diagonal, explaining why;
- (g) Say whether there exists an invertible matrix P such that $P^{-1}AP$ is diagonal, explaining why.

Exercise

ExGALeb. Consider the matrix

$$A := \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$$

- (a) Determine a linear map $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ such that $\ker f = \ker A$;
- (b) Say whether f is surjective;
- (c) Determine the dimension and a basis of $\ker f$;
- (d) Find all eigenvalues of A ;

- (e) Determine all eigenspaces of A ;
- (f) Say whether there exists a matrix P such that $P^T A P$ is diagonal, explaining why;
- (g) Describe the conic

$$(x, y, z) A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

ExGALec. Consider the matrix

$$A := \begin{pmatrix} 6 & 4 & 1 \\ 6 & 11 & 2 \\ -9 & -15 & -3 \end{pmatrix}$$

- (a) Determine a linear map $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ such that $\ker f = \ker A$.
- (b) State whether f is surjective.
- (c) Prove that 2 is an eigenvalue of A .
- (d) Determine the geometric multiplicity of 2.
- (e) Find all eigenvalues of A .
- (f) Determine all eigenspaces of A .
- (g) State whether there exists a matrix P such that $P^T A P$ is diagonal, explaining why.