Schubert Calculus on a Grassmann Algebra

Letterio Gatto

Politecnico di Torino

Brolo, September 12th–15th, 2007

This slide show
(and more on this subject)
is available at the URL:

This slide show
(and more on this subject)
is available at the URL:

http://calvino.polito.it/~gatto

This slide show
(and more on this subject)
is available at the URL:

http://calvino.polito.it/~gatto

Summary

Summary

Summary

Summary

Summary

Summary

Summary

Summary

Summary

Summary

References for this talk. . .

Schubert Calculus via Hasse–Schmidt Derivations,

References for this talk. . .

1. ——, *Schubert Calculus via Hasse–Schmidt Derivations*,

References for this talk. . .

1. ——, *Schubert Calculus via Hasse–Schmidt Derivations*, Asian J. Math., **9**, No. 3, (2005), 315–322.

References for this talk. . .

1. ——, *Schubert Calculus via Hasse–Schmidt Derivations*, Asian J. Math., **9**, No. 3, (2005), 315–322.
2. ——, *Schubert Calculus: An Algebraic Introduction*, Lecture Notes, 25° Colóquio Brasileiro de Matemática, IMPA, Rio de Janeiro, 2005.

References for this talk. . .

1. ——, *Schubert Calculus via Hasse–Schmidt Derivations*, Asian J. Math., **9**, No. 3, (2005), 315–322.
2. ——, *Schubert Calculus: An Algebraic Introduction*, Lecture Notes, 25° Colóquio Brasileiro de Matemática, IMPA, Rio de Janeiro, 2005.
3. Taíse Santiago, *Schubert Calculus on a Grassmann Algebra*, Doctoral Thesis, Politecnico di Torino, 2006.

References for this talk. . .

1. ——, *Schubert Calculus via Hasse–Schmidt Derivations*, Asian J. Math., **9**, No. 3, (2005), 315–322.
2. ——, *Schubert Calculus: An Algebraic Introduction*, Lecture Notes, 25° Colóquio Brasileiro de Matemática, IMPA, Rio de Janeiro, 2005.
3. Taíse Santiago, *Schubert Calculus on a Grassmann Algebra*, Doctoral Thesis, Politecnico di Torino, 2006.
4. ——, Taíse Santiago, *Schubert Calculus on a Grassmann Algebra*, Canad. Math. Bull. (to appear).

References for this talk...

1. ——, *Schubert Calculus via Hasse–Schmidt Derivations*, Asian J. Math., **9**, No. 3, (2005), 315–322.
2. ——, *Schubert Calculus: An Algebraic Introduction*, Lecture Notes, 25° Colóquio Brasileiro de Matemática, IMPA, Rio de Janeiro, 2005.
3. Taíse Santiago, *Schubert Calculus on a Grassmann Algebra*, Doctoral Thesis, Politecnico di Torino, 2006.
4. ——, Taíse Santiago, *Schubert Calculus on a Grassmann Algebra*, Canad. Math. Bull. (to appear).
5. Taíse Santiago, *Catalan Traffic and Integrals on the Grassmannian of Lines*, Discrete Math., 2007 (To appear) `math.AG/ 0704.23761v1`.

References for this talk. . .

1. ——, *Schubert Calculus via Hasse–Schmidt Derivations*, Asian J. Math., **9**, No. 3, (2005), 315–322.
2. ——, *Schubert Calculus: An Algebraic Introduction*, Lecture Notes, 25° Colóquio Brasileiro de Matemática, IMPA, Rio de Janeiro, 2005.
3. Taíse Santiago, *Schubert Calculus on a Grassmann Algebra*, Doctoral Thesis, Politecnico di Torino, 2006.
4. ——, Taíse Santiago, *Schubert Calculus on a Grassmann Algebra*, Canad. Math. Bull. (to appear).
5. Taíse Santiago, *Catalan Traffic and Integrals on the Grassmannian of Lines*, Discrete Math., 2007 (To appear) `math.AG/ 0704.23761v1`.
6. Taíse Santiago, *Degree of Grassmannian of Lines*, Acc. Sc. Torino, Atti Sc. Fis., **140**, 2006, 7–14.

References for this talk. . .

1. ——, *Schubert Calculus via Hasse–Schmidt Derivations*, Asian J. Math., **9**, No. 3, (2005), 315–322.
2. ——, *Schubert Calculus: An Algebraic Introduction*, Lecture Notes, 25° Colóquio Brasileiro de Matemática, IMPA, Rio de Janeiro, 2005.
3. Taíse Santiago, *Schubert Calculus on a Grassmann Algebra*, Doctoral Thesis, Politecnico di Torino, 2006.
4. ——, Taíse Santiago, *Schubert Calculus on a Grassmann Algebra*, Canad. Math. Bull. (to appear).
5. Taíse Santiago, *Catalan Traffic and Integrals on the Grassmannian of Lines*, Discrete Math., 2007 (To appear) `math.AG/ 0704.23761v1`.
6. Taíse Santiago, *Degree of Grassmannian of Lines*, Acc. Sc. Torino, Atti Sc. Fis., **140**, 2006, 7–14.
7. D. Laksov, A. Thorup, *A Determinantal Formula for the Exterior Powers of the Polynomial Ring*, Indiana Univ. Math. J., **56**, No. 2, 2007.

References for this talk. . .

1. ——, *Schubert Calculus via Hasse–Schmidt Derivations*, Asian J. Math., **9**, No. 3, (2005), 315–322.
2. ——, *Schubert Calculus: An Algebraic Introduction*, Lecture Notes, 25° Colóquio Brasileiro de Matemática, IMPA, Rio de Janeiro, 2005.
3. Taíse Santiago, *Schubert Calculus on a Grassmann Algebra*, Doctoral Thesis, Politecnico di Torino, 2006.
4. ——, Taíse Santiago, *Schubert Calculus on a Grassmann Algebra*, Canad. Math. Bull. (to appear).
5. Taíse Santiago, *Catalan Traffic and Integrals on the Grassmannian of Lines*, Discrete Math., 2007 (To appear) `math.AG/ 0704.23761v1`.
6. Taíse Santiago, *Degree of Grassmannian of Lines*, Acc. Sc. Torino, Atti Sc. Fis., **140**, 2006, 7–14.
7. D. Laksov, A. Thorup, *A Determinantal Formula for the Exterior Powers of the Polynomial Ring*, Indiana Univ. Math. J., **56**, No. 2, 2007.
8. ——, Taíse Santiago, *Equivariant Schubert Calculus*, `arXiv:math.AG/ 070344501v1`.

References for this talk. . .

1. ——, *Schubert Calculus via Hasse–Schmidt Derivations*, Asian J. Math., **9**, No. 3, (2005), 315–322.
2. ——, *Schubert Calculus: An Algebraic Introduction*, Lecture Notes, 25° Colóquio Brasileiro de Matemática, IMPA, Rio de Janeiro, 2005.
3. Taíse Santiago, *Schubert Calculus on a Grassmann Algebra*, Doctoral Thesis, Politecnico di Torino, 2006.
4. ——, Taíse Santiago, *Schubert Calculus on a Grassmann Algebra*, Canad. Math. Bull. (to appear).
5. Taíse Santiago, *Catalan Traffic and Integrals on the Grassmannian of Lines*, Discrete Math., 2007 (To appear) `math.AG/ 0704.23761v1`.
6. Taíse Santiago, *Degree of Grassmannian of Lines*, Acc. Sc. Torino, Atti Sc. Fis., **140**, 2006, 7–14.
7. D. Laksov, A. Thorup, *A Determinantal Formula for the Exterior Powers of the Polynomial Ring*, Indiana Univ. Math. J., **56**, No. 2, 2007.
8. ——, Taíse Santiago, *Equivariant Schubert Calculus*, `arXiv:math.AG/ 070344501v1`.
9. D. Laksov, *The formalism of Equivariant Schubert Calculus*, Preprint, KTH, 2006.

Set Up

Set Up

- Let $A := A_0 \oplus A_1 \oplus \ldots$ be a graded ring with $A_0 = \mathbb{Z}$;

Set Up

- Let $A := A_0 \oplus A_1 \oplus \ldots$ be a graded ring with $A_0 = \mathbb{Z}$;
- X an indeterminate over A;

Set Up

- Let $A := A_0 \oplus A_1 \oplus \ldots$ be a graded ring with $A_0 = \mathbb{Z}$;
- X an indeterminate over A;
- $\mathrm{p} := X^n - e_1X^{n-1} + \ldots + (-1)^n e_n \in A[X]$, a monic polynomial of degree n such that $e_i \in A_i$;

Set Up

- Let $A := A_0 \oplus A_1 \oplus \ldots$ be a graded ring with $A_0 = \mathbb{Z}$;
- X an indeterminate over A;
- $\mathrm{p} := X^n - e_1 X^{n-1} + \ldots + (-1)^n e_n \in A[X]$, a monic polynomial of degree n such that $e_i \in A_i$;
- $M := XA[X]$ and $M(\mathrm{p}) = M/\mathrm{p}M$.

Set Up

- Let $A := A_0 \oplus A_1 \oplus \ldots$ be a graded ring with $A_0 = \mathbb{Z}$;
- X an indeterminate over A;
- $\mathrm{p} := X^n - e_1X^{n-1} + \ldots + (-1)^n e_n \in A[X]$, a monic polynomial of degree n such that $e_i \in A_i$;
- $M := XA[X]$ and $M(\mathrm{p}) = M/\mathrm{p}M$.

The A-module $M(\mathrm{p})$ is generated by $\epsilon^i := X^i + \mathrm{p}M$.

Set Up

- Let $A := A_0 \oplus A_1 \oplus \ldots$ be a graded ring with $A_0 = \mathbb{Z}$;
- X an indeterminate over A;
- $\mathrm{p} := X^n - e_1 X^{n-1} + \ldots + (-1)^n e_n \in A[X]$, a monic polynomial of degree n such that $e_i \in A_i$;
- $M := XA[X]$ and $M(\mathrm{p}) = M/\mathrm{p}M$.

The A-module $M(\mathrm{p})$ is generated by $\epsilon^i := X^i + \mathrm{p}M$. In particular:

Set Up

- Let $A := A_0 \oplus A_1 \oplus \ldots$ be a graded ring with $A_0 = \mathbb{Z}$;
- X an indeterminate over A;
- $\mathrm{p} := X^n - e_1X^{n-1} + \ldots + (-1)^n e_n \in A[X]$, a monic polynomial of degree n such that $e_i \in A_i$;
- $M := XA[X]$ and $M(\mathrm{p}) = M/\mathrm{p}M$.

The A-module $M(\mathrm{p})$ is generated by $\epsilon^i := X^i + \mathrm{p}M$. In particular:

$$(\epsilon^1, \ldots, \epsilon^n),$$

Set Up

- Let $A := A_0 \oplus A_1 \oplus \ldots$ be a graded ring with $A_0 = \mathbb{Z}$;
- X an indeterminate over A;
- $\mathrm{p} := X^n - e_1X^{n-1} + \ldots + (-1)^n e_n \in A[X]$, a monic polynomial of degree n such that $e_i \in A_i$;
- $M := XA[X]$ and $M(\mathrm{p}) = M/\mathrm{p}M$.

The A-module $M(\mathrm{p})$ is generated by $\epsilon^i := X^i + \mathrm{p}M$. In particular:

$$(\epsilon^1, \ldots, \epsilon^n),$$

is an A-basis of $M(\mathrm{p})$. One has, e.g.:

Set Up

- Let $A := A_0 \oplus A_1 \oplus \ldots$ be a graded ring with $A_0 = \mathbb{Z}$;
- X an indeterminate over A;
- $\mathrm{p} := X^n - e_1 X^{n-1} + \ldots + (-1)^n e_n \in A[X]$, a monic polynomial of degree n such that $e_i \in A_i$;
- $M := XA[X]$ and $M(\mathrm{p}) = M/\mathrm{p}M$.

The A-module $M(\mathrm{p})$ is generated by $\epsilon^i := X^i + \mathrm{p}M$. In particular:

$$(\epsilon^1, \ldots, \epsilon^n),$$

is an A-basis of $M(\mathrm{p})$. One has, e.g.:

$$\epsilon^{n+1} = e_1 \epsilon^n - e_2 \epsilon^{n-1} + \ldots - (-1)^n e_n \epsilon^1$$

Exterior Algebra, Exterior Powers.

Exterior Algebra, Exterior Powers.

Let

Exterior Algebra, Exterior Powers.

Let

$$\bigwedge M(\mathrm{p}) := \bigoplus_{k\geq 0} \bigwedge^k M(\mathrm{p})$$

Exterior Algebra, Exterior Powers.

Let

$$\bigwedge M(\mathrm{p}) := \bigoplus_{k\geq 0} \bigwedge^k M(\mathrm{p})$$

be the exterior algebra of $M(\mathrm{p})$.

Exterior Algebra, Exterior Powers.

Let

$$\bigwedge M(\mathrm{p}) := \bigoplus_{k\geq 0} \bigwedge^k M(\mathrm{p})$$

be the exterior algebra of $M(\mathrm{p})$.

Then:

Exterior Algebra, Exterior Powers.

Let

$$\bigwedge M(\mathrm{p}) := \bigoplus_{k\geq 0} \bigwedge^k M(\mathrm{p})$$

be the exterior algebra of $M(\mathrm{p})$.

Then:

$$(\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k})_{1\leq i_1<\ldots<i_k\leq n}$$

Exterior Algebra, Exterior Powers.

Let

$$\bigwedge M(\mathrm{p}) := \bigoplus_{k\geq 0} \bigwedge^k M(\mathrm{p})$$

be the exterior algebra of $M(\mathrm{p})$.

Then:

$$(\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k})_{1\leq i_1<\ldots<i_k\leq n}$$

is a basis of $\bigwedge^k M(\mathrm{p})$, the k^{th} exterior power of $M(\mathrm{p})$.

Example.

Example.

Let $M := A\epsilon^1 \oplus A\epsilon^2 \oplus A\epsilon^3 \oplus A\epsilon^4$. Then

Example.

Let $M := A\epsilon^1 \oplus A\epsilon^2 \oplus A\epsilon^3 \oplus A\epsilon^4$. Then

$$\underbrace{\epsilon^1 \wedge \epsilon^2,}_{w=0}$$

Example.

Let $M := A\epsilon^1 \oplus A\epsilon^2 \oplus A\epsilon^3 \oplus A\epsilon^4$. Then

$$\underbrace{\epsilon^1 \wedge \epsilon^2,}_{w=0}\ \underbrace{\epsilon^1 \wedge \epsilon^3,}_{w=1}$$

Example.

Let $M := A\epsilon^1 \oplus A\epsilon^2 \oplus A\epsilon^3 \oplus A\epsilon^4$. Then

$$\underbrace{\epsilon^1 \wedge \epsilon^2}_{w=0}, \underbrace{\epsilon^1 \wedge \epsilon^3}_{w=1}, \underbrace{\epsilon^1 \wedge \epsilon^4}_{w=2},$$

Example.

Let $M := A\epsilon^1 \oplus A\epsilon^2 \oplus A\epsilon^3 \oplus A\epsilon^4$. Then

$$\underbrace{\epsilon^1 \wedge \epsilon^2}_{w=0},\ \underbrace{\epsilon^1 \wedge \epsilon^3}_{w=1},\ \underbrace{\epsilon^1 \wedge \epsilon^4}_{w=2},\ \underbrace{\epsilon^2 \wedge \epsilon^3}_{w=2},$$

Example.

Let $M := A\epsilon^1 \oplus A\epsilon^2 \oplus A\epsilon^3 \oplus A\epsilon^4$. Then

$$\underbrace{\epsilon^1 \wedge \epsilon^2}_{w=0}, \ \underbrace{\epsilon^1 \wedge \epsilon^3}_{w=1}, \ \underbrace{\epsilon^1 \wedge \epsilon^4}_{w=2}, \ \underbrace{\epsilon^2 \wedge \epsilon^3}_{w=2}, \ \underbrace{\epsilon^2 \wedge \epsilon^4}_{w=3},$$

Example.

Let $M := A\epsilon^1 \oplus A\epsilon^2 \oplus A\epsilon^3 \oplus A\epsilon^4$. Then

$$\underbrace{\epsilon^1 \wedge \epsilon^2}_{w=0}, \underbrace{\epsilon^1 \wedge \epsilon^3}_{w=1}, \underbrace{\epsilon^1 \wedge \epsilon^4}_{w=2}, \underbrace{\epsilon^2 \wedge \epsilon^3}_{w=2}, \underbrace{\epsilon^2 \wedge \epsilon^4}_{w=3}, \underbrace{\epsilon^3 \wedge \epsilon^4}_{w=4},$$

Example.

Let $M := A\epsilon^1 \oplus A\epsilon^2 \oplus A\epsilon^3 \oplus A\epsilon^4$. Then

$$\underbrace{\epsilon^1 \wedge \epsilon^2}_{w=0},\ \underbrace{\epsilon^1 \wedge \epsilon^3}_{w=1},\ \underbrace{\epsilon^1 \wedge \epsilon^4}_{w=2},\ \underbrace{\epsilon^2 \wedge \epsilon^3}_{w=2},\ \underbrace{\epsilon^2 \wedge \epsilon^4}_{w=3},\ \underbrace{\epsilon^3 \wedge \epsilon^4}_{w=4},$$

is a basis of $\bigwedge^2 M$: $w = (i_1 - 1) + (i_2 - 2)$ is the *weight* of $\epsilon^{i_1} \wedge \epsilon^{i_2}$

A Basic Fact $\left(\frac{d^i}{dx^i}(f^{(j)}) = f^{(i+j)}\right)$

A Basic Fact $\left(\frac{d^i}{dx^i}(f^{(j)}) = f^{(i+j)}\right)$

There is one and only one A-algebra homomorphism

A Basic Fact $\left(\frac{d^i}{dx^i}(f^{(j)}) = f^{(i+j)}\right)$

There is one and only one A-algebra homomorphism

$$D_t := \sum_{i\geq 0} D_i t^i : \bigwedge M(\mathrm{p}) \longrightarrow \bigwedge M(\mathrm{p})[[t]]$$

$$(D_i \in End_A(\bigwedge M(\mathrm{p}))$$

A Basic Fact $\left(\frac{d^i}{dx^i}(f^{(j)}) = f^{(i+j)}\right)$

There is one and only one A-algebra homomorphism

$$D_t := \sum_{i\geq 0} D_i t^i : \bigwedge M(\mathrm{p}) \longrightarrow \bigwedge M(\mathrm{p})[[t]]$$

$$(D_i \in End_A(\bigwedge M(\mathrm{p}))$$

such that

A Basic Fact $\left(\frac{d^i}{dx^i}(f^{(j)}) = f^{(i+j)}\right)$

There is one and only one A-algebra homomorphism

$$D_t := \sum_{i\geq 0} D_i t^i : \bigwedge M(\mathrm{p}) \longrightarrow \bigwedge M(\mathrm{p})[[t]]$$

$$(D_i \in End_A(\bigwedge M(\mathrm{p}))$$

such that

$$D_i \epsilon^j = \epsilon^{i+j}$$

Schubert calculus on a Grassmann Algebra.

Schubert calculus on a Grassmann Algebra.

The explicit way to phrase that D_t is an A-algebra homomorphism is

Schubert calculus on a Grassmann Algebra.

The explicit way to phrase that D_t is an A-algebra homomorphism is

$$D_t(\alpha \wedge \beta) = D_t\alpha \wedge D_t\beta$$

Schubert calculus on a Grassmann Algebra.

The explicit way to phrase that D_t is an A-algebra homomorphism is

$$D_t(\alpha \wedge \beta) = D_t\alpha \wedge D_t\beta$$

the fundamental equation of Schubert Calculus

Schubert calculus on a Grassmann Algebra.

The explicit way to phrase that D_t is an A-algebra homomorphism is

$$D_t(\alpha \wedge \beta) = D_t\alpha \wedge D_t\beta$$

the fundamental equation of Schubert Calculus
(on a Grassmann Algebra)

Schubert calculus on a Grassmann Algebra.

The explicit way to phrase that D_t is an A-algebra homomorphism is

$$D_t(\alpha \wedge \beta) = D_t\alpha \wedge D_t\beta$$

the fundamental equation of Schubert Calculus
(on a Grassmann Algebra)

We are used to such a kind of equations!

Schubert calculus on a Grassmann Algebra.

The explicit way to phrase that D_t is an A-algebra homomorphism is

$$D_t(\alpha \wedge \beta) = D_t\alpha \wedge D_t\beta$$

the fundamental equation of Schubert Calculus
(on a Grassmann Algebra)

We are used to such a kind of equations!

If $f, g \in C^\infty(\mathbb{R})$ then:

Schubert calculus on a Grassmann Algebra.

The explicit way to phrase that D_t is an A-algebra homomorphism is

$$D_t(\alpha \wedge \beta) = D_t\alpha \wedge D_t\beta$$

the fundamental equation of Schubert Calculus
(on a Grassmann Algebra)

We are used to such a kind of equations!

If $f, g \in C^\infty(\mathbb{R})$ then:

$$D_t(fg) = D_t(f)D_t(g) \quad \text{where} \quad D_t = \sum_{i\geq 0} \frac{1}{i!}\frac{d^i}{dx^i}t^i$$

Schubert calculus on a Grassmann Algebra.

The explicit way to phrase that D_t is an A-algebra homomorphism is

$$D_t(\alpha \wedge \beta) = D_t\alpha \wedge D_t\beta$$

the fundamental equation of Schubert Calculus
(on a Grassmann Algebra)

We are used to such a kind of equations!

If $f, g \in C^\infty(\mathbb{R})$ then:

$$D_t(fg) = D_t(f)D_t(g) \quad \text{where} \quad D_t = \sum_{i\geq 0} \frac{1}{i!}\frac{d^i}{dx^i}t^i$$

(The Taylor expansion of the product of f and g is the product of the Taylor expansions of f and g respectively.)

Schubert calculus on a Grassmann Algebra.

The explicit way to phrase that D_t is an A-algebra homomorphism is

$$D_t(\alpha \wedge \beta) = D_t\alpha \wedge D_t\beta$$

the fundamental equation of Schubert Calculus
(on a Grassmann Algebra)

SCGA I: Leibniz Rule

SCGA I: Leibniz Rule

The fundamental equation is equivalent to:

SCGA I: Leibniz Rule

The fundamental equation is equivalent to:

$$D_h(\alpha \wedge \beta) = \sum_{i=0}^{h} D_i\alpha \wedge D_{h-i}\beta, \qquad \forall \alpha,\ \beta\ \in \bigwedge M$$

SCGA I: Leibniz Rule

The fundamental equation is equivalent to:

$$D_h(\alpha \wedge \beta) = \sum_{i=0}^{h} D_i\alpha \wedge D_{h-i}\beta, \qquad \forall \alpha,\ \beta\ \in \bigwedge M$$

which is the h^{th} order Leibniz rule ($h \geq 0$). For example:

SCGA I: Leibniz Rule

The fundamental equation is equivalent to:

$$D_h(\alpha \wedge \beta) = \sum_{i=0}^{h} D_i\alpha \wedge D_{h-i}\beta, \qquad \forall \alpha,\ \beta\ \in \bigwedge M$$

which is the h^{th} order Leibniz rule ($h \geq 0$). For example:

$$D_2(\alpha \wedge \beta) =$$

SCGA I: Leibniz Rule

The fundamental equation is equivalent to:

$$D_h(\alpha \wedge \beta) = \sum_{i=0}^{h} D_i\alpha \wedge D_{h-i}\beta, \qquad \forall \alpha,\ \beta\ \in \bigwedge M$$

which is the h^{th} order Leibniz rule $(h \geq 0)$. For example:

$$D_2(\alpha \wedge \beta) = D_2\alpha \wedge \beta +$$

SCGA I: Leibniz Rule

The fundamental equation is equivalent to:

$$D_h(\alpha \wedge \beta) = \sum_{i=0}^{h} D_i\alpha \wedge D_{h-i}\beta, \qquad \forall \alpha,\ \beta\ \in \bigwedge M$$

which is the h^{th} order Leibniz rule ($h \geq 0$). For example:

$$D_2(\alpha \wedge \beta) = D_2\alpha \wedge \beta + D_1\alpha \wedge D_1\beta +$$

SCGA I: Leibniz Rule

The fundamental equation is equivalent to:

$$D_h(\alpha \wedge \beta) = \sum_{i=0}^{h} D_i\alpha \wedge D_{h-i}\beta, \qquad \forall \alpha,\ \beta\ \in \bigwedge M$$

which is the h^{th} order Leibniz rule ($h \geq 0$). For example:

$$D_2(\alpha \wedge \beta) = D_2\alpha \wedge \beta + D_1\alpha \wedge D_1\beta + \alpha \wedge D_2\beta$$

SCGA I: Leibniz Rule

The fundamental equation is equivalent to:

$$D_h(\alpha \wedge \beta) = \sum_{i=0}^{h} D_i\alpha \wedge D_{h-i}\beta, \qquad \forall \alpha,\ \beta\ \in \bigwedge M$$

which is the h^{th} order Leibniz rule ($h \geq 0$). For example:

$$D_2(\alpha \wedge \beta) = D_2\alpha \wedge \beta + D_1\alpha \wedge D_1\beta + \alpha \wedge D_2\beta$$

One more example:

SCGA I: Leibniz Rule

The fundamental equation is equivalent to:

$$D_h(\alpha \wedge \beta) = \sum_{i=0}^{h} D_i\alpha \wedge D_{h-i}\beta, \qquad \forall \alpha,\ \beta\ \in \bigwedge M$$

which is the h^{th} order Leibniz rule ($h \geq 0$). For example:

$$D_2(\alpha \wedge \beta) = D_2\alpha \wedge \beta + D_1\alpha \wedge D_1\beta + \alpha \wedge D_2\beta$$

One more example:

$$D_2(\epsilon^1 \wedge \epsilon^2) =$$

SCGA I: Leibniz Rule

The fundamental equation is equivalent to:

$$D_h(\alpha \wedge \beta) = \sum_{i=0}^{h} D_i\alpha \wedge D_{h-i}\beta, \qquad \forall \alpha,\ \beta\ \in \bigwedge M$$

which is the h^{th} order Leibniz rule ($h \geq 0$). For example:

$$D_2(\alpha \wedge \beta) = D_2\alpha \wedge \beta + D_1\alpha \wedge D_1\beta + \alpha \wedge D_2\beta$$

One more example:

$$D_2(\epsilon^1 \wedge \epsilon^2) = \ D_2\epsilon^1 \wedge \epsilon^2 +$$

SCGA I: Leibniz Rule

The fundamental equation is equivalent to:

$$D_h(\alpha \wedge \beta) = \sum_{i=0}^{h} D_i\alpha \wedge D_{h-i}\beta, \qquad \forall \alpha,\ \beta \ \in \bigwedge M$$

which is the h^{th} order Leibniz rule ($h \geq 0$). For example:

$$D_2(\alpha \wedge \beta) = D_2\alpha \wedge \beta + D_1\alpha \wedge D_1\beta + \alpha \wedge D_2\beta$$

One more example:

$$D_2(\epsilon^1 \wedge \epsilon^2) = \ D_2\epsilon^1 \wedge \epsilon^2 + D_1\epsilon^1 \wedge D_1\epsilon^2 +$$

SCGA I: Leibniz Rule

The fundamental equation is equivalent to:

$$D_h(\alpha \wedge \beta) = \sum_{i=0}^{h} D_i\alpha \wedge D_{h-i}\beta, \qquad \forall \alpha,\ \beta \ \in \bigwedge M$$

which is the h^{th} order Leibniz rule ($h \geq 0$). For example:

$$D_2(\alpha \wedge \beta) = D_2\alpha \wedge \beta + D_1\alpha \wedge D_1\beta + \alpha \wedge D_2\beta$$

One more example:

$$D_2(\epsilon^1 \wedge \epsilon^2) = \ D_2\epsilon^1 \wedge \epsilon^2 + D_1\epsilon^1 \wedge D_1\epsilon^2 + \epsilon^1 \wedge D_2\epsilon^2 =$$

SCGA I: Leibniz Rule

The fundamental equation is equivalent to:

$$D_h(\alpha \wedge \beta) = \sum_{i=0}^{h} D_i\alpha \wedge D_{h-i}\beta, \qquad \forall\alpha,\ \beta\ \in \bigwedge M$$

which is the h^{th} order Leibniz rule ($h \geq 0$). For example:

$$D_2(\alpha \wedge \beta) = D_2\alpha \wedge \beta + D_1\alpha \wedge D_1\beta + \alpha \wedge D_2\beta$$

One more example:

$$D_2(\epsilon^1 \wedge \epsilon^2) = D_2\epsilon^1 \wedge \epsilon^2 + D_1\epsilon^1 \wedge D_1\epsilon^2 + \epsilon^1 \wedge D_2\epsilon^2 =$$
$$= \epsilon^3 \wedge \epsilon^2 +$$

SCGA I: Leibniz Rule

The fundamental equation is equivalent to:

$$D_h(\alpha \wedge \beta) = \sum_{i=0}^{h} D_i\alpha \wedge D_{h-i}\beta, \qquad \forall \alpha,\ \beta\ \in \bigwedge M$$

which is the h^{th} order Leibniz rule ($h \geq 0$). For example:

$$D_2(\alpha \wedge \beta) = D_2\alpha \wedge \beta + D_1\alpha \wedge D_1\beta + \alpha \wedge D_2\beta$$

One more example:

$$D_2(\epsilon^1 \wedge \epsilon^2) = D_2\epsilon^1 \wedge \epsilon^2 + D_1\epsilon^1 \wedge D_1\epsilon^2 + \epsilon^1 \wedge D_2\epsilon^2 =$$

$$= \epsilon^3 \wedge \epsilon^2 + \epsilon^2 \wedge \epsilon^3 +$$

SCGA I: Leibniz Rule

The fundamental equation is equivalent to:

$$D_h(\alpha \wedge \beta) = \sum_{i=0}^{h} D_i\alpha \wedge D_{h-i}\beta, \qquad \forall \alpha,\ \beta\ \in \bigwedge M$$

which is the h^{th} order Leibniz rule ($h \geq 0$). For example:

$$D_2(\alpha \wedge \beta) = D_2\alpha \wedge \beta + D_1\alpha \wedge D_1\beta + \alpha \wedge D_2\beta$$

One more example:

$$D_2(\epsilon^1 \wedge \epsilon^2) = D_2\epsilon^1 \wedge \epsilon^2 + D_1\epsilon^1 \wedge D_1\epsilon^2 + \epsilon^1 \wedge D_2\epsilon^2 =$$
$$= \epsilon^3 \wedge \epsilon^2 + \epsilon^2 \wedge \epsilon^3 + \epsilon^1 \wedge \epsilon^4 =$$

SCGA I: Leibniz Rule

The fundamental equation is equivalent to:

$$D_h(\alpha \wedge \beta) = \sum_{i=0}^{h} D_i\alpha \wedge D_{h-i}\beta, \qquad \forall \alpha,\ \beta\ \in \bigwedge M$$

which is the h^{th} order Leibniz rule ($h \geq 0$). For example:

$$D_2(\alpha \wedge \beta) = D_2\alpha \wedge \beta + D_1\alpha \wedge D_1\beta + \alpha \wedge D_2\beta$$

One more example:

$$D_2(\epsilon^1 \wedge \epsilon^2) = D_2\epsilon^1 \wedge \epsilon^2 + D_1\epsilon^1 \wedge D_1\epsilon^2 + \epsilon^1 \wedge D_2\epsilon^2 =$$
$$= \epsilon^3 \wedge \epsilon^2 + \epsilon^2 \wedge \epsilon^3 + \epsilon^1 \wedge \epsilon^4 = \epsilon^1 \wedge \epsilon^4$$

SCGA II: Integration by Parts $\left(\int f \cdot dg = fg - \int df \cdot g\right)$

SCGA II: Integration by Parts $\left(\int f \cdot dg = fg - \int df \cdot g\right)$

Leibniz's rule implies

SCGA II: Integration by Parts $\left(\int f \cdot dg = fg - \int df \cdot g\right)$

Leibniz's rule implies

Integration by Parts

SCGA II: Integration by Parts $\left(\int f \cdot dg = fg - \int df \cdot g\right)$

Leibniz's rule implies

Integration by Parts

$$D_h\alpha \wedge \epsilon^i = D_h(\alpha \wedge \epsilon^i) - D_{h-1}(\alpha \wedge \epsilon^{i+1})$$

SCGA II: Integration by Parts $\left(\int f \cdot dg = fg - \int df \cdot g\right)$

Leibniz's rule implies

Integration by Parts

$$D_h\alpha \wedge \epsilon^i = D_h(\alpha \wedge \epsilon^i) - D_{h-1}(\alpha \wedge \epsilon^{i+1})$$

For example:

SCGA II: Integration by Parts $\left(\int f \cdot dg = fg - \int df \cdot g\right)$

Leibniz's rule implies

Integration by Parts

$$D_h\alpha \wedge \epsilon^i = D_h(\alpha \wedge \epsilon^i) - D_{h-1}(\alpha \wedge \epsilon^{i+1})$$

For example:

$\epsilon^2 \wedge \epsilon^5 =$

SCGA II: Integration by Parts $\left(\int f \cdot dg = fg - \int df \cdot g\right)$

Leibniz's rule implies

Integration by Parts

$$D_h\alpha \wedge \epsilon^i = D_h(\alpha \wedge \epsilon^i) - D_{h-1}(\alpha \wedge \epsilon^{i+1})$$

For example:

$$\epsilon^2 \wedge \epsilon^5 = \epsilon^2 \wedge D_1\epsilon^4 =$$

SCGA II: Integration by Parts $\left(\int f \cdot dg = fg - \int df \cdot g\right)$

Leibniz's rule implies

Integration by Parts

$$D_h\alpha \wedge \epsilon^i = D_h(\alpha \wedge \epsilon^i) - D_{h-1}(\alpha \wedge \epsilon^{i+1})$$

For example:

$$\epsilon^2 \wedge \epsilon^5 = \epsilon^2 \wedge D_1\epsilon^4 = D_1(\epsilon^2 \wedge \epsilon^4)$$

SCGA II: Integration by Parts $\left(\int f \cdot dg = fg - \int df \cdot g\right)$

Leibniz's rule implies

Integration by Parts

$$D_h\alpha \wedge \epsilon^i = D_h(\alpha \wedge \epsilon^i) - D_{h-1}(\alpha \wedge \epsilon^{i+1})$$

For example:

$$\epsilon^2 \wedge \epsilon^5 = \epsilon^2 \wedge D_1\epsilon^4 = D_1(\epsilon^2 \wedge \epsilon^4) - D_1\epsilon^2 \wedge \epsilon^4 =$$

SCGA II: Integration by Parts $\left(\int f \cdot dg = fg - \int df \cdot g\right)$

Leibniz's rule implies

Integration by Parts

$$D_h\alpha \wedge \epsilon^i = D_h(\alpha \wedge \epsilon^i) - D_{h-1}(\alpha \wedge \epsilon^{i+1})$$

For example:

$$\epsilon^2 \wedge \epsilon^5 = \epsilon^2 \wedge D_1\epsilon^4 = D_1(\epsilon^2 \wedge \epsilon^4) - D_1\epsilon^2 \wedge \epsilon^4 = D_1(\epsilon^2 \wedge \epsilon^4) - \epsilon^3 \wedge \epsilon^4$$

$\mathcal{A}^*(\bigwedge M(\mathrm{p}))$

$\mathcal{A}^*(\bigwedge M(\mathrm{p}))$

Let $A[\mathbf{T}]$ be the polynomial ring $A[T_1, T_2, \ldots]$.

$\mathcal{A}^*(\bigwedge M(\mathrm{p}))$

Let $A[\mathbf{T}]$ be the polynomial ring $A[T_1, T_2, \ldots]$.

Give degree i to the monomial T_i.

$\mathcal{A}^*(\bigwedge M(\mathrm{p}))$

Let $A[\mathbf{T}]$ be the polynomial ring $A[T_1, T_2, \ldots]$.

Give degree i to the monomial T_i.

Let

$$\mathrm{ev}_D : A[\mathbf{T}] \longrightarrow End_A(\bigwedge M(\mathrm{p}))$$

be the natural map $T_i \mapsto D_i$ and

$\mathcal{A}^*(\bigwedge M(\mathrm{p}))$

Let $A[\mathbf{T}]$ be the polynomial ring $A[T_1, T_2, \ldots]$.

Give degree i to the monomial T_i.

Let

$$\mathrm{ev}_D : A[\mathbf{T}] \longrightarrow End_A(\bigwedge M(\mathrm{p}))$$

be the natural map $T_i \mapsto D_i$ and

$$\mathcal{A}^*(\bigwedge M(\mathrm{p})) := Im(\mathrm{ev}_D) \subseteq End_A(\bigwedge M(\mathrm{p})).$$

$\mathcal{A}^*(\bigwedge^k M(\mathrm{p}))$

$\mathcal{A}^*(\bigwedge^k M(\mathrm{p}))$

Denote by

$\mathcal{A}^*(\bigwedge^k M(\mathrm{p}))$

Denote by

$$\mathcal{A}^*(\bigwedge^k M(\mathrm{p}))$$

$\mathcal{A}^*(\bigwedge^k M(\mathrm{p}))$

Denote by

$$\mathcal{A}^*(\bigwedge^k M(\mathrm{p}))$$

the image of the natural restriction map

$\mathcal{A}^*(\bigwedge^k M(\mathrm{p}))$

Denote by

$$\mathcal{A}^*(\bigwedge^k M(\mathrm{p}))$$

the image of the natural restriction map

$$\begin{cases} \rho_k : & \mathcal{A}^*(\bigwedge M(\mathrm{p})) \longrightarrow End_A(\bigwedge^k M(\mathrm{p})) \\ & P(D) \longmapsto P(D)_{|_{\bigwedge^k M(\mathrm{p})}} \end{cases}$$

Giambelli's problem has a solution

Giambelli's problem has a solution

Theorem.

Giambelli's problem has a solution

Theorem. *The natural evaluation map:*

Giambelli's problem has a solution

Theorem. *The natural evaluation map:*

$$\begin{cases} \mathrm{ev}_{\epsilon^1\wedge\ldots\wedge\epsilon^k} & : & \mathcal{A}^*(\bigwedge M(\mathrm{p})) & \longrightarrow & \bigwedge^k M(\mathrm{p}) \\ & & P(D) & \longmapsto & P(D)\cdot\epsilon^1\wedge\ldots\wedge\epsilon^k \end{cases}$$

Giambelli's problem has a solution

Theorem. *The natural evaluation map:*

$$\begin{cases} \mathrm{ev}_{\epsilon^1\wedge\ldots\wedge\epsilon^k} & : & \mathcal{A}^*(\bigwedge M(\mathrm{p})) & \longrightarrow & \bigwedge^k M(\mathrm{p}) \\ & & P(D) & \longmapsto & P(D)\cdot\epsilon^1\wedge\ldots\wedge\epsilon^k \end{cases}$$

is surjective.

Giambelli's problem has a solution

Theorem. *The natural evaluation map:*

$$\begin{cases} \mathrm{ev}_{\epsilon^1\wedge\ldots\wedge\epsilon^k} & : \quad \mathcal{A}^*(\bigwedge M(\mathrm{p})) \longrightarrow \bigwedge^k M(\mathrm{p}) \\ & \qquad P(D) \longmapsto P(D)\cdot\epsilon^1\wedge\ldots\wedge\epsilon^k \end{cases}$$

is surjective.

Proof.

Giambelli's problem has a solution

Theorem. *The natural evaluation map:*

$$\begin{cases} \mathrm{ev}_{\epsilon^1\wedge\ldots\wedge\epsilon^k} & : & \mathcal{A}^*(\bigwedge M(\mathrm{p})) & \longrightarrow & \bigwedge^k M(\mathrm{p}) \\ & & P(D) & \longmapsto & P(D)\cdot\epsilon^1\wedge\ldots\wedge\epsilon^k \end{cases}$$

is surjective.

Proof. Enough to prove that for each $\epsilon^{i_1}\wedge\ldots\wedge\epsilon^{i_k}\in\bigwedge^k M(\mathrm{p})$ there exists a polynomial expression $P(D)$ such that

Giambelli's problem has a solution

Theorem. *The natural evaluation map:*

$$\begin{cases} \mathrm{ev}_{\epsilon^1\wedge\ldots\wedge\epsilon^k} & : & \mathcal{A}^*(\bigwedge M(\mathrm{p})) & \longrightarrow & \bigwedge^k M(\mathrm{p}) \\ & & P(D) & \longmapsto & P(D)\cdot\epsilon^1\wedge\ldots\wedge\epsilon^k \end{cases}$$

is surjective.

Proof. Enough to prove that for each $\epsilon^{i_1}\wedge\ldots\wedge\epsilon^{i_k}\in\bigwedge^k M(\mathrm{p})$ there exists a polynomial expression $P(D)$ such that

$$\epsilon^{i_1}\wedge\ldots\wedge\epsilon^{i_k} = P(D)\cdot\epsilon^1\wedge\ldots\wedge\epsilon^k.$$

One then concludes using integration by parts. ∎

Giambelli's problem has a solution

Theorem. *The natural evaluation map:*

$$\begin{cases} \mathrm{ev}_{\epsilon^1\wedge\ldots\wedge\epsilon^k} & : & \mathcal{A}^*(\bigwedge M(\mathrm{p})) & \longrightarrow & \bigwedge^k M(\mathrm{p}) \\ & & P(D) & \longmapsto & P(D)\cdot\epsilon^1\wedge\ldots\wedge\epsilon^k \end{cases}$$

is surjective.

Proof. Enough to prove that for each $\epsilon^{i_1}\wedge\ldots\wedge\epsilon^{i_k}\in\bigwedge^k M(\mathrm{p})$ there exists a polynomial expression $P(D)$ such that

$$\epsilon^{i_1}\wedge\ldots\wedge\epsilon^{i_k} = P(D)\cdot\epsilon^1\wedge\ldots\wedge\epsilon^k.$$

One then concludes using integration by parts. ∎

For example:

Giambelli's problem has a solution

Theorem. *The natural evaluation map:*

$$\begin{cases} \mathrm{ev}_{\epsilon^1\wedge\ldots\wedge\epsilon^k} & : & \mathcal{A}^*(\bigwedge M(\mathrm{p})) & \longrightarrow & \bigwedge^k M(\mathrm{p}) \\ & & P(D) & \longmapsto & P(D)\cdot\epsilon^1\wedge\ldots\wedge\epsilon^k \end{cases}$$

is surjective.

Proof. Enough to prove that for each $\epsilon^{i_1}\wedge\ldots\wedge\epsilon^{i_k}\in\bigwedge^k M(\mathrm{p})$ there exists a polynomial expression $P(D)$ such that

$$\epsilon^{i_1}\wedge\ldots\wedge\epsilon^{i_k} = P(D)\cdot\epsilon^1\wedge\ldots\wedge\epsilon^k.$$

One then concludes using integration by parts. ■

For example:

Recall

In a previous slide we saw that, applying Leibniz rule:

$$D_2(\epsilon^1\wedge\epsilon^2) = \epsilon^1\wedge\epsilon^4.$$

Theorem. *The natural evaluation map:*

$$\begin{cases} \mathrm{ev}_{\epsilon^1\wedge\ldots\wedge\epsilon^k} & : \quad \mathcal{A}^*(\bigwedge M(\mathrm{p})) \longrightarrow \bigwedge^k M(\mathrm{p}) \\ & \qquad P(D) \longmapsto P(D)\cdot\epsilon^1\wedge\ldots\wedge\epsilon^k \end{cases}$$

is surjective.

Proof. One must prove that for each $\epsilon^{i_1}\wedge\ldots\wedge\epsilon^{i_k} \in \bigwedge^k M(\mathrm{p})$ there exists a polynomial expression $P(D)$ such that

$$\epsilon^{i_1}\wedge\ldots\wedge\epsilon^{i_k} = P(D)\cdot\epsilon^1\wedge\ldots\wedge\epsilon^k.$$

By integration by parts. ∎

For example:

Theorem. *The natural evaluation map:*

$$\begin{cases} \mathrm{ev}_{\epsilon^1\wedge\ldots\wedge\epsilon^k} & : & \mathcal{A}^*(\bigwedge M(\mathrm{p})) & \longrightarrow & \bigwedge^k M(\mathrm{p}) \\ & & P(D) & \longmapsto & P(D)\cdot\epsilon^1\wedge\ldots\wedge\epsilon^k \end{cases}$$

is surjective.

Proof. One must prove that for each $\epsilon^{i_1}\wedge\ldots\wedge\epsilon^{i_k}\in\bigwedge^k M(\mathrm{p})$ there exists a polynomial expression $P(D)$ such that

$$\epsilon^{i_1}\wedge\ldots\wedge\epsilon^{i_k} = P(D)\cdot\epsilon^1\wedge\ldots\wedge\epsilon^k.$$

By integration by parts. ■

For example:

$\epsilon^2\wedge\epsilon^3 =$

Theorem. *The natural evaluation map:*

$$\begin{cases} \mathrm{ev}_{\epsilon^1\wedge\ldots\wedge\epsilon^k} & : \quad \mathcal{A}^*(\bigwedge M(\mathrm{p})) \longrightarrow \bigwedge^k M(\mathrm{p}) \\ & \qquad P(D) \longmapsto P(D)\cdot\epsilon^1\wedge\ldots\wedge\epsilon^k \end{cases}$$

is surjective.

Proof. One must prove that for each $\epsilon^{i_1}\wedge\ldots\wedge\epsilon^{i_k}\in\bigwedge^k M(\mathrm{p})$ there exists a polynomial expression $P(D)$ such that

$$\epsilon^{i_1}\wedge\ldots\wedge\epsilon^{i_k} = P(D)\cdot\epsilon^1\wedge\ldots\wedge\epsilon^k.$$

By integration by parts. ■

For example:

$\epsilon^2\wedge\epsilon^3 = D_1\epsilon^1\wedge\epsilon^3 =$

Theorem. *The natural evaluation map:*

$$\begin{cases} \mathrm{ev}_{\epsilon^1\wedge\ldots\wedge\epsilon^k} & : & \mathcal{A}^*(\bigwedge M(\mathrm{p})) & \longrightarrow & \bigwedge^k M(\mathrm{p}) \\ & & P(D) & \longmapsto & P(D)\cdot\epsilon^1\wedge\ldots\wedge\epsilon^k \end{cases}$$

is surjective.

Proof. One must prove that for each $\epsilon^{i_1}\wedge\ldots\wedge\epsilon^{i_k}\in\bigwedge^k M(\mathrm{p})$ there exists a polynomial expression $P(D)$ such that

$$\epsilon^{i_1}\wedge\ldots\wedge\epsilon^{i_k}=P(D)\cdot\epsilon^1\wedge\ldots\wedge\epsilon^k.$$

By integration by parts. ■

For example:

$$\epsilon^2\wedge\epsilon^3=D_1\epsilon^1\wedge\epsilon^3=D_1(\epsilon^1\wedge\epsilon^3)$$

Theorem. *The natural evaluation map:*

$$\begin{cases} \mathrm{ev}_{\epsilon^1\wedge\ldots\wedge\epsilon^k} & : \quad \mathcal{A}^*(\bigwedge M(\mathrm{p})) \longrightarrow \bigwedge^k M(\mathrm{p}) \\ & \qquad P(D) \longmapsto P(D)\cdot\epsilon^1\wedge\ldots\wedge\epsilon^k \end{cases}$$

is surjective.

Proof. One must prove that for each $\epsilon^{i_1}\wedge\ldots\wedge\epsilon^{i_k}\in\bigwedge^k M(\mathrm{p})$ there exists a polynomial expression $P(D)$ such that

$$\epsilon^{i_1}\wedge\ldots\wedge\epsilon^{i_k} = P(D)\cdot\epsilon^1\wedge\ldots\wedge\epsilon^k.$$

By integration by parts. ■

For example:

$$\epsilon^2\wedge\epsilon^3 = D_1\epsilon^1\wedge\epsilon^3 = D_1(\epsilon^1\wedge\epsilon^3) - \epsilon^1\wedge\epsilon^4 =$$

Theorem. *The natural evaluation map:*

$$\begin{cases} \mathrm{ev}_{\epsilon^1\wedge\ldots\wedge\epsilon^k} & : & \mathcal{A}^*(\bigwedge M(\mathrm{p})) & \longrightarrow & \bigwedge^k M(\mathrm{p}) \\ & & P(D) & \longmapsto & P(D)\cdot\epsilon^1\wedge\ldots\wedge\epsilon^k \end{cases}$$

is surjective.

Proof. One must prove that for each $\epsilon^{i_1}\wedge\ldots\wedge\epsilon^{i_k}\in\bigwedge^k M(\mathrm{p})$ there exists a polynomial expression $P(D)$ such that

$$\epsilon^{i_1}\wedge\ldots\wedge\epsilon^{i_k}=P(D)\cdot\epsilon^1\wedge\ldots\wedge\epsilon^k.$$

By integration by parts. ∎

For example:

$$\epsilon^2\wedge\epsilon^3 = D_1\epsilon^1\wedge\epsilon^3 = D_1(\epsilon^1\wedge\epsilon^3)-\epsilon^1\wedge\epsilon^4 = D_1(D_1(\epsilon^1\wedge\epsilon^2))$$

Theorem. *The natural evaluation map:*

$$\begin{cases} \mathrm{ev}_{\epsilon^1\wedge\ldots\wedge\epsilon^k} & : \quad \mathcal{A}^*(\bigwedge M(\mathrm{p})) \longrightarrow \bigwedge^k M(\mathrm{p}) \\ & \qquad P(D) \longmapsto P(D)\cdot\epsilon^1\wedge\ldots\wedge\epsilon^k \end{cases}$$

is surjective.

Proof. One must prove that for each $\epsilon^{i_1}\wedge\ldots\wedge\epsilon^{i_k}\in\bigwedge^k M(\mathrm{p})$ there exists a polynomial expression $P(D)$ such that

$$\epsilon^{i_1}\wedge\ldots\wedge\epsilon^{i_k} = P(D)\cdot\epsilon^1\wedge\ldots\wedge\epsilon^k.$$

By integration by parts. ∎

For example:

$$\epsilon^2\wedge\epsilon^3 = D_1\epsilon^1\wedge\epsilon^3 = D_1(\epsilon^1\wedge\epsilon^3) - \epsilon^1\wedge\epsilon^4 = D_1(D_1(\epsilon^1\wedge\epsilon^2)) - D_2(\epsilon^1\wedge\epsilon^2) =$$

Theorem. *The natural evaluation map:*

$$\begin{cases} \mathrm{ev}_{\epsilon^1\wedge\ldots\wedge\epsilon^k} & : \quad \mathcal{A}^*(\bigwedge M(\mathrm{p})) \longrightarrow \bigwedge^k M(\mathrm{p}) \\ & \quad\quad P(D) \longmapsto P(D)\cdot\epsilon^1\wedge\ldots\wedge\epsilon^k \end{cases}$$

is surjective.

Proof. One must prove that for each $\epsilon^{i_1}\wedge\ldots\wedge\epsilon^{i_k}\in\bigwedge^k M(\mathrm{p})$ there exists a polynomial expression $P(D)$ such that

$$\epsilon^{i_1}\wedge\ldots\wedge\epsilon^{i_k}=P(D)\cdot\epsilon^1\wedge\ldots\wedge\epsilon^k.$$

By integration by parts. ∎

For example:

$$\epsilon^2\wedge\epsilon^3 = D_1\epsilon^1\wedge\epsilon^3 = D_1(\epsilon^1\wedge\epsilon^3) - \epsilon^1\wedge\epsilon^4 = D_1(D_1(\epsilon^1\wedge\epsilon^2)) - D_2(\epsilon^1\wedge\epsilon^2) =$$

$$= (D_1^2 - D_2)\cdot\epsilon^1\wedge\epsilon^2 =$$

Theorem. *The natural evaluation map:*

$$\begin{cases} \mathrm{ev}_{\epsilon^1\wedge\ldots\wedge\epsilon^k} & : \quad \mathcal{A}^*(\bigwedge M(\mathrm{p})) \longrightarrow \bigwedge^k M(\mathrm{p}) \\ & \qquad\qquad P(D) \longmapsto P(D)\cdot\epsilon^1\wedge\ldots\wedge\epsilon^k \end{cases}$$

is surjective.

Proof. One must prove that for each $\epsilon^{i_1}\wedge\ldots\wedge\epsilon^{i_k}\in\bigwedge^k M(\mathrm{p})$ there exists a polynomial expression $P(D)$ such that

$$\epsilon^{i_1}\wedge\ldots\wedge\epsilon^{i_k} = P(D)\cdot\epsilon^1\wedge\ldots\wedge\epsilon^k.$$

By integration by parts. ∎

For example:

$$\epsilon^2\wedge\epsilon^3 = D_1\epsilon^1\wedge\epsilon^3 = D_1(\epsilon^1\wedge\epsilon^3) - \epsilon^1\wedge\epsilon^4 = D_1(D_1(\epsilon^1\wedge\epsilon^2)) - D_2(\epsilon^1\wedge\epsilon^2) =$$

$$= (D_1^2 - D_2)\cdot\epsilon^1\wedge\epsilon^2 = \begin{vmatrix} D_1 & D_2 \\ D_0 & D_1 \end{vmatrix}\epsilon^1\wedge\epsilon^2.$$

Computing $\mathcal{A}^*(\bigwedge^k M(\mathrm{p}))$

Computing $\mathcal{A}^*(\bigwedge^k M(\mathrm{p}))$

Hence:

Computing $\mathcal{A}^*(\bigwedge^k M(\mathrm{p}))$

Hence:

$$\mathcal{A}^*(\bigwedge^k M(\mathrm{p})) := \frac{\mathcal{A}^*(\bigwedge M(\mathrm{p}))}{\ker(\rho_k)} = \frac{\mathcal{A}^*(\bigwedge M(\mathrm{p}))}{\ker(\mathrm{ev}_{\epsilon^1\wedge\ldots\wedge\epsilon^k})}$$

Computing $\mathcal{A}^*(\bigwedge^k M(\mathrm{p}))$

Hence:

$$\mathcal{A}^*(\bigwedge^k M(\mathrm{p})) := \frac{\mathcal{A}^*(\bigwedge M(\mathrm{p}))}{\ker(\rho_k)} = \frac{\mathcal{A}^*(\bigwedge M(\mathrm{p}))}{\ker(\mathrm{ev}_{\epsilon^1\wedge\ldots\wedge\epsilon^k})}$$

The map:

Computing $\mathcal{A}^*(\bigwedge^k M(\mathrm{p}))$

Hence:

$$\mathcal{A}^*(\bigwedge^k M(\mathrm{p})) := \frac{\mathcal{A}^*(\bigwedge M(\mathrm{p}))}{\ker(\rho_k)} = \frac{\mathcal{A}^*(\bigwedge M(\mathrm{p}))}{\ker(\mathrm{ev}_{\epsilon^1\wedge\ldots\wedge\epsilon^k})}$$

The map: $\begin{cases} \Pi_k \quad : \quad \mathcal{A}^*(\bigwedge^k M(\mathrm{p})) & \longrightarrow \quad \bigwedge^k M(\mathrm{p}) \\ \qquad \rho_k(P(D)) & \longmapsto \quad P(D)\epsilon^1 \wedge \ldots \wedge \epsilon^k \end{cases}$

Computing $\mathcal{A}^*(\bigwedge^k M(\mathrm{p}))$

Hence:

$$\mathcal{A}^*(\bigwedge^k M(\mathrm{p})) := \frac{\mathcal{A}^*(\bigwedge M(\mathrm{p}))}{\ker(\rho_k)} = \frac{\mathcal{A}^*(\bigwedge M(\mathrm{p}))}{\ker(\mathrm{ev}_{\epsilon^1\wedge\ldots\wedge\epsilon^k})}$$

The map: $\begin{cases} \Pi_k : \mathcal{A}^*(\bigwedge^k M(\mathrm{p})) \longrightarrow \bigwedge^k M(\mathrm{p}) \\ \rho_k(P(D)) \mapsto P(D)\epsilon^1 \wedge \ldots \wedge \epsilon^k \end{cases}$

is said to be *Poincaré Isomorphism*

Computing $\mathcal{A}^*(\bigwedge^k M(\mathrm{p}))$

Hence:

$$\mathcal{A}^*(\bigwedge^k M(\mathrm{p})) := \frac{\mathcal{A}^*(\bigwedge M(\mathrm{p}))}{\ker(\rho_k)} = \frac{\mathcal{A}^*(\bigwedge M(\mathrm{p}))}{\ker(\mathrm{ev}_{\epsilon^1\wedge\ldots\wedge\epsilon^k})}$$

The map: $\begin{cases} \Pi_k \quad : \quad \mathcal{A}^*(\bigwedge^k M(\mathrm{p})) & \longrightarrow \quad \bigwedge^k M(\mathrm{p}) \\ \qquad\qquad \rho_k(P(D)) & \longmapsto \quad P(D)\epsilon^1 \wedge \ldots \wedge \epsilon^k \end{cases}$

is said to be *Poincaré Isomorphism*

In a sense:

$$\text{“} \quad \mathcal{A}^*(\bigwedge^k M(\mathrm{p})) = \bigwedge^k \mathcal{A}^*(M(\mathrm{p})) \quad \text{”}$$

Intersection Theory on Grassmann Bundles

Intersection Theory on Grassmann Bundles

Let

Intersection Theory on Grassmann Bundles

Let $E \longrightarrow Y$

Intersection Theory on Grassmann Bundles

Let $E \longrightarrow Y$ be a vector bundle of rank n;

Furthermore, let:

Intersection Theory on Grassmann Bundles

Let $E \longrightarrow Y$ be a vector bundle of rank n;

Furthermore, let:

$A := A^*(Y)$

Intersection Theory on Grassmann Bundles

Let $E \longrightarrow Y$ be a vector bundle of rank n;

Furthermore, let:

$A := A^*(Y)$ and

Intersection Theory on Grassmann Bundles

Let $E \longrightarrow Y$ be a vector bundle of rank n;

Furthermore, let:

$$A := A^*(Y) \quad \text{and} \quad \mathtt{p} = X^n - c_1(E)X^{n-1} + \ldots + c_n(E)$$

Intersection Theory on Grassmann Bundles

Let $E \longrightarrow Y$ be a vector bundle of rank n;

Furthermore, let:

$$A := A^*(Y) \quad \text{and} \quad \mathrm{p} = X^n - c_1(E)X^{n-1} + \ldots + c_n(E)$$

Construct $\bigwedge^k M(\mathrm{p})$ as before

Intersection Theory on Grassmann Bundles

Let $E \longrightarrow Y$ be a vector bundle of rank n;

Furthermore, let:

$$A := A^*(Y) \quad \text{and} \quad \mathrm{p} = X^n - c_1(E)X^{n-1} + \ldots + c_n(E)$$

Construct $\bigwedge^k M(\mathrm{p})$ as before

$$M := XA[X],\ M(\mathrm{p}) := M/\mathrm{p}M$$

Intersection Theory on Grassmann Bundles

Let $E \longrightarrow Y$ be a vector bundle of rank n;

Furthermore, let:

$$A := A^*(Y) \quad \text{and} \quad \mathrm{p} = X^n - c_1(E)X^{n-1} + \ldots + c_n(E)$$

Construct $\bigwedge^k M(\mathrm{p})$ as before

Intersection Theory on Grassmann Bundles

Let $E \longrightarrow Y$ be a vector bundle of rank n;

Furthermore, let:

$$A := A^*(Y) \quad \text{and} \quad \mathrm{p} = X^n - c_1(E)X^{n-1} + \ldots + c_n(E)$$

Construct $\bigwedge^k M(\mathrm{p})$ as before

It turns out that

Intersection Theory on Grassmann Bundles

Let $E \longrightarrow Y$ be a vector bundle of rank n;

Furthermore, let:

$$A := A^*(Y) \quad \text{and} \quad \mathrm{p} = X^n - c_1(E)X^{n-1} + \ldots + c_n(E)$$

Construct $\bigwedge^k M(\mathrm{p})$ as before

It turns out that (easy):

Intersection Theory on Grassmann Bundles

Let $E \longrightarrow Y$ be a vector bundle of rank n;

Furthermore, let:

$$A := A^*(Y) \quad \text{and} \quad \mathrm{p} = X^n - c_1(E)X^{n-1} + \ldots + c_n(E)$$

Construct $\bigwedge^k M(\mathrm{p})$ as before

It turns out that (easy):

$$\bigwedge^1 M(\mathrm{p}) = M(\mathrm{p}) \cong A_*(\mathbb{P}(E))$$

Intersection Theory on Grassmann Bundles

Let $E \longrightarrow Y$ be a vector bundle of rank n;

Furthermore, let:

$A := A^*(Y)$ and $\mathrm{p} = X^n - c_1(E)X^{n-1} + \ldots + c_n(E)$

Construct $\bigwedge^k M(\mathrm{p})$ as before

It turns out that (easy):

$$\bigwedge^1 M(\mathrm{p}) = M(\mathrm{p}) \cong A_*(\mathbb{P}(E))$$

which is an A-module freely generated by $\epsilon^i := \xi^{i-1} \cap [\mathbb{P}(E)]$,

where

$$\xi = c_1(O_{\mathbb{P}(E)}(-1))$$

The Main Theorem (Laksov & Thorup, 2006)

The Main Theorem (Laksov & Thorup, 2006)

The Main Theorem (Laksov & Thorup, 2006)

The following diagram

The Main Theorem (Laksov & Thorup, 2006)

The following diagram

$$
\begin{array}{ccc}
A^*(G(k,E)) \otimes A_*(G(k,E)) & \stackrel{\cap}{\longrightarrow} & A_*(G(k,E)) \\
\iota_k \otimes \delta_k^{-1} \Big\downarrow & & \delta_k \Big\uparrow \\
\mathcal{A}^*(\bigwedge^k M(\mathrm{p})) \otimes \bigwedge^k M(\mathrm{p}) & \longrightarrow & \bigwedge^k M(\mathrm{p})
\end{array}
$$

The Main Theorem (Laksov & Thorup, 2006)

The following diagram

$$\begin{array}{ccc} A^*(G(k,E))\otimes A_*(G(k,E)) & \xrightarrow{\cap} & A_*(G(k,E)) \\ \iota_k\otimes\delta_k^{-1}\Big\downarrow & & \delta_k\Big\uparrow \\ \mathcal{A}^*(\bigwedge^k M(\mathrm{p}))\otimes\bigwedge^k M(\mathrm{p}) & \longrightarrow & \bigwedge^k M(\mathrm{p}) \end{array}$$

commutes.

The Main Theorem (Laksov & Thorup, 2006)

The following diagram

$$\begin{array}{ccc} A^*(G(k,E)) \otimes A_*(G(k,E)) & \xrightarrow{\cap} & A_*(G(k,E)) \\ \iota_k \otimes \delta_k^{-1} \Big\downarrow & & \delta_k \Big\uparrow \\ \mathcal{A}^*(\bigwedge^k M(\mathrm{p})) \otimes \bigwedge^k M(\mathrm{p}) & \longrightarrow & \bigwedge^k M(\mathrm{p}) \end{array}$$

commutes.

$$\begin{pmatrix} where & \iota_k(P(\sigma)) = P(D) \\ and & \delta_k(\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}) = \Delta_{i_1,\ldots,i_k}(\sigma) \cap [G(k,E)] \end{pmatrix}$$

The Main Theorem (Laksov & Thorup, 2006)

The symmetric structure of $\bigwedge^k A[X]$

Let $S := A[X_1, \ldots, X_k]^{sym}$. Then $\bigotimes^k A[X] \to \bigwedge^k A[X]$ is S-linear

The following diagram

$$\begin{array}{ccc} A^*(G(k,E)) \otimes A_*(G(k,E)) & \xrightarrow{\cap} & A_*(G(k,E)) \\ \iota_k \otimes \delta_k^{-1} \big\downarrow & & \delta_k \big\uparrow \\ \mathcal{A}^*(\bigwedge^k M(\mathrm{p})) \otimes \bigwedge^k M(\mathrm{p}) & \longrightarrow & \bigwedge^k M(\mathrm{p}) \end{array}$$

commutes.

$$\begin{pmatrix} \textit{where} & \iota_k(P(\sigma)) = P(D) \\ \textit{and} & \delta_k(\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}) = \Delta_{i_1,\ldots,i_k}(\sigma) \cap [G(k,E)] \end{pmatrix}$$

Newton's Binomial Formulas

Newton's Binomial Formulas

What do we gain by our dictionary?

Newton's Binomial Formulas

What do we gain by our dictionary?

Working on the **Exterior Algebra** rather than on a single exterior power,

we inherit

Newton's Binomial Formulas

What do we gain by our dictionary?

Working on the **Exterior Algebra** rather than on a single exterior power,

we inherit

Newton's type binomial formulas!

Newton's Binomial Formulas

What do we gain by our dictionary?

Working on the **Exterior Algebra** rather than on a single exterior power,

we inherit

Newton's type binomial formulas!

see

Newton's Binomial Formulas

What do we gain by our dictionary?

Working on the **Exterior Algebra** rather than on a single exterior power,

we inherit

Newton's type binomial formulas!

see

⋆ J. Cordovez, ——, *Newton's Binomial Formulas in Schubert Calculus*, Preprint, Politecnico di Torino, 2007.

1st Newton's formulas $(a+b)^n$

1st Newton's formulas $(a+b)^n$

The first is gotten via a simple induction by iterating

$$D_1(\alpha \wedge \beta) = D_1\alpha \wedge \beta + \alpha \wedge D_1\beta.$$

1st Newton's formulas $(a+b)^n$

The first is gotten via a simple induction by iterating

$$D_1(\alpha \wedge \beta) = D_1\alpha \wedge \beta + \alpha \wedge D_1\beta.$$

$$D_1^m(\alpha \wedge \beta) = \sum_{j=0}^{m} \binom{m}{j} D_1^j\alpha \wedge D_1^{m-j}\beta \qquad (1)$$

1st Newton's formulas $(a+b)^n$

The first is gotten via a simple induction by iterating

$$D_1(\alpha\wedge\beta)=D_1\alpha\wedge\beta+\alpha\wedge D_1\beta.$$

$$D_1^m(\alpha\wedge\beta)=\sum_{j=0}^{m}\binom{m}{j}D_1^j\alpha\wedge D_1^{m-j}\beta \qquad (1)$$

holding for each $\alpha,\beta\in\bigwedge M(\mathrm{p})$ and each $m\geq 1$

2nd Newton's Formula

2nd Newton's Formula

Similarly, iterating $D_h(\alpha \wedge \epsilon^i) = D_h\alpha \wedge \epsilon^i + D_{h-1}(\alpha \wedge \epsilon^{i+1})$,

2nd Newton's Formula

Similarly, iterating $D_h(\alpha \wedge \epsilon^i) = D_h\alpha \wedge \epsilon^i + D_{h-1}(\alpha \wedge \epsilon^{i+1})$,

integration by parts

$$D_h\alpha \wedge \epsilon^i = D_h(\alpha \wedge \epsilon^i) - D_{h-1}(\alpha \wedge \epsilon^{i+1})$$

2nd Newton's Formula

Similarly, iterating $D_h(\alpha \wedge \epsilon^i) = D_h\alpha \wedge \epsilon^i + D_{h-1}(\alpha \wedge \epsilon^{i+1})$,

2nd Newton's Formula

Similarly, iterating $D_h(\alpha \wedge \epsilon^i) = D_h\alpha \wedge \epsilon^i + D_{h-1}(\alpha \wedge \epsilon^{i+1})$,
one gets a second Newton's type formula (Cordovez):

2nd Newton's Formula

Similarly, iterating $D_h(\alpha \wedge \epsilon^i) = D_h\alpha \wedge \epsilon^i + D_{h-1}(\alpha \wedge \epsilon^{i+1})$,
one gets a second Newton's type formula (Cordovez):

$$D_h^m(\alpha \wedge \epsilon^i) = \sum_{j=0}^{m} \binom{m}{j} D_{h-1}^j (D_h^{m-j}\alpha \wedge \epsilon^{i+j}) \tag{2}$$

2nd Newton's Formula

Similarly, iterating $D_h(\alpha \wedge \epsilon^i) = D_h\alpha \wedge \epsilon^i + D_{h-1}(\alpha \wedge \epsilon^{i+1})$,
one gets a second Newton's type formula (Cordovez):

$$D_h^m(\alpha \wedge \epsilon^i) = \sum_{j=0}^{m} \binom{m}{j} D_{h-1}^j(D_h^{m-j}\alpha \wedge \epsilon^{i+j}) \qquad (2)$$

holding for each $\alpha \in \bigwedge M(\mathrm{p})$ and each $h, m \geq 0$

2nd Newton's Formula

Similarly, iterating $D_h(\alpha \wedge \epsilon^i) = D_h\alpha \wedge \epsilon^i + D_{h-1}(\alpha \wedge \epsilon^{i+1})$, one gets a second Newton's type formula (Cordovez):

$$D_h^m(\alpha \wedge \epsilon^i) = \sum_{j=0}^{m} \binom{m}{j} D_{h-1}^j(D_h^{m-j}\alpha \wedge \epsilon^{i+j}) \tag{2}$$

holding for each $\alpha \in \bigwedge M(\mathrm{p})$ and each $h, m \geq 0$

(when $h = 1$ one gets precisely formula (1) for $\beta = \epsilon^i$)

2nd Newton's Formula

Similarly, iterating $D_h(\alpha \wedge \epsilon^i) = D_h\alpha \wedge \epsilon^i + D_{h-1}(\alpha \wedge \epsilon^{i+1})$,
one gets a second Newton's type formula (Cordovez):

$$D_h^m(\alpha \wedge \epsilon^i) = \sum_{j=0}^{m} \binom{m}{j} D_{h-1}^j(D_h^{m-j}\alpha \wedge \epsilon^{i+j}) \qquad (2)$$

holding for each $\alpha \in \bigwedge M(\mathrm{p})$ and each $h, m \geq 0$

(when $h = 1$ one gets precisely formula (1) for $\beta = \epsilon^i$)

Claim:
formula (2) is unspeakable in the classical formulation of Schubert Calculus

2nd Newton's Formula

Similarly, iterating $D_h(\alpha \wedge \epsilon^i) = D_h\alpha \wedge \epsilon^i + D_{h-1}(\alpha \wedge \epsilon^{i+1})$, one gets a second Newton's type formula (Cordovez):

$$D_h^m(\alpha \wedge \epsilon^i) = \sum_{j=0}^{m} \binom{m}{j} D_{h-1}^j (D_h^{m-j}\alpha \wedge \epsilon^{i+j}) \qquad (2)$$

holding for each $\alpha \in \bigwedge M(\mathrm{p})$ and each $h, m \geq 0$

(when $h = 1$ one gets precisely formula (1) for $\beta = \epsilon^i$)

Claim:

formula (2) is unspeakable in the classical formulation of Schubert Calculus

Challenge:

disprove the claim!

Schubert Calculus on Grassmann Varieties.

Schubert Calculus on Grassmann Varieties.

From now on let $A = \mathbb{Z}$ and $\mathtt{p} = X^n$.

Schubert Calculus on Grassmann Varieties.

From now on let $A = \mathbb{Z}$ and $\mathrm{p} = X^n$.

Then $M_n := M(\mathrm{p})$ is a free $\mathbb{Z}$-module of rank n generated by $(\epsilon^1, \ldots, \epsilon^n)$.

Schubert Calculus on Grassmann Varieties.

From now on let $A = \mathbb{Z}$ and $\mathrm{p} = X^n$.

Then $M_n := M(\mathrm{p})$ is a free $\mathbb{Z}$-module of rank n generated by $(\epsilon^1, \ldots, \epsilon^n)$.

In this case $\epsilon^j = 0$ if $j > n$.

Schubert Calculus on Grassmann Varieties.

From now on let $A = \mathbb{Z}$ and $\mathrm{p} = X^n$.

Then $M_n := M(\mathrm{p})$ is a free $\mathbb{Z}$-module of rank n generated by $(\epsilon^1, \ldots, \epsilon^n)$.

In this case $\epsilon^j = 0$ if $j > n$.

The weight of $\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}$ is $(i_1 - 1) + \ldots + (i_k - k)$. Then:

$$\bigwedge^k M_n = \bigoplus (\bigwedge^k M_n)_w$$

Schubert Calculus on Grassmann Varieties.

From now on let $A = \mathbb{Z}$ and $\mathrm{p} = X^n$.

Then $M_n := M(\mathrm{p})$ is a free $\mathbb{Z}$-module of rank n generated by $(\epsilon^1, \ldots, \epsilon^n)$.

In this case $\epsilon^j = 0$ if $j > n$.

The weight of $\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}$ is $(i_1 - 1) + \ldots + (i_k - k)$. Then:

$$\bigwedge^k M_n = \bigoplus (\bigwedge^k M_n)_w$$

The fundamental element

$$g_k = \epsilon^1 \wedge \ldots \wedge \epsilon^k$$

is the unique of weight 0 while

$$\pi_{k,n} = \epsilon^{n-k+1} \wedge \ldots \wedge \epsilon^n$$

is the unique of weight $k(n-k)$ (the maximum possible).

One has:

One has:

$$D_h^m(\bigwedge^k M_n)_w \subseteq (\bigwedge^k M_n)_{w+mh}$$

i.e., D_h is homogeneous of degree h.

One has:

$$D_h^m(\bigwedge^k M_n)_w \subseteq (\bigwedge^k M_n)_{w+mh}$$

i.e., D_h is homogeneous of degree h.

In particular

$$D_1^{k(n-k)}(\bigwedge^k M_n)_0 \cong (\bigwedge^k M_n)_{k(n-k)}$$

One has:

$$D_h^m(\bigwedge^k M_n)_w \subseteq (\bigwedge^k M_n)_{w+mh}$$

i.e., D_h is homogeneous of degree h.

In particular

$$D_1^{k(n-k)}(\bigwedge^k M_n)_0 \cong (\bigwedge^k M_n)_{k(n-k)}$$

i.e.

One has:

$$D_h^m(\bigwedge^k M_n)_w \subseteq (\bigwedge^k M_n)_{w+mh}$$

i.e., D_h is homogeneous of degree h.

In particular

$$D_1^{k(n-k)}(\bigwedge^k M_n)_0 \cong (\bigwedge^k M_n)_{k(n-k)}$$

i.e.

$$D_1^{k(n-k)}\epsilon^1 \wedge \ldots \wedge \epsilon^k = d_{k,n} \cdot \epsilon^{n+1-k} \wedge \ldots \wedge \epsilon^n$$

A piece of Notation

A piece of Notation

Define:

$$\int_n \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} =$$

$$= \begin{cases} 1 & \textit{if} \quad (i_1, \ldots, i_k) \quad \textit{is an even permutation of} \quad (n-k+1, \ldots, n) \\ -1 & \textit{if} \quad (i_1, \ldots, i_k) \quad \textit{is an odd permutation of} \quad (n-k+1, \ldots, n) \\ 0 & \textit{otherwise} \end{cases}$$

A piece of Notation

Define:

$$\int_n \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} =$$

$$= \begin{cases} 1 & \textit{if} \quad (i_1, \ldots, i_k) \quad \textit{is an even permutation of} \quad (n-k+1, \ldots, n) \\ -1 & \textit{if} \quad (i_1, \ldots, i_k) \quad \textit{is an odd permutation of} \quad (n-k+1, \ldots, n) \\ 0 & \textit{otherwise} \end{cases}$$

Extend $\int$ by $\mathbb{Z}$-linearity, getting $\int : \bigwedge^k M_n \to \mathbb{Z}$

A piece of Notation

Define:

$$\int_n \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} =$$

$$= \begin{cases} 1 & \textit{if} \quad (i_1, \ldots, i_k) \quad \textit{is an even permutation of} \quad (n-k+1, \ldots, n) \\ -1 & \textit{if} \quad (i_1, \ldots, i_k) \quad \textit{is an odd permutation of} \quad (n-k+1, \ldots, n) \\ 0 & \textit{otherwise} \end{cases}$$

Extend $\int$ by $\mathbb{Z}$-linearity, getting $\int : \bigwedge^k M_n \to \mathbb{Z}$

The Main Theorem

A piece of Notation

Define:

$$\int_n \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} =$$

$$= \begin{cases} 1 & \textit{if} \quad (i_1, \ldots, i_k) \quad \textit{is an even permutation of} \quad (n-k+1, \ldots, n) \\ -1 & \textit{if} \quad (i_1, \ldots, i_k) \quad \textit{is an odd permutation of} \quad (n-k+1, \ldots, n) \\ 0 & \textit{otherwise} \end{cases}$$

Extend $\int$ by $\mathbb{Z}$-linearity, getting $\int : \bigwedge^k M_n \to \mathbb{Z}$

The Main Theorem

The main Theorem

$$\begin{array}{ccc} A^*(G(k,E)) \otimes A_*(G(k,E)) & \xrightarrow{\cap} & A_*(G(k,E)) \\ \iota_k \otimes \delta_k^{-1} \Big\downarrow & & \delta_k \Big\uparrow \\ \mathcal{A}^*(\bigwedge^k M(\mathrm{p})) \otimes \bigwedge^k M(\mathrm{p}) & \longrightarrow & \bigwedge^k M(\mathrm{p}) \end{array}$$

A piece of Notation

Define:

$$\int_n \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} =$$

$$= \begin{cases} 1 & \textit{if} \quad (i_1, \ldots, i_k) \quad \textit{is an even permutation of} \quad (n-k+1, \ldots, n) \\ -1 & \textit{if} \quad (i_1, \ldots, i_k) \quad \textit{is an odd permutation of} \quad (n-k+1, \ldots, n) \\ 0 & \textit{otherwise} \end{cases}$$

Extend $\int$ by $\mathbb{Z}$-linearity, getting $\int : \bigwedge^k M_n \to \mathbb{Z}$

The Main Theorem

A piece of Notation

Define:

$$\int_n \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} =$$

$$= \begin{cases} 1 & \textit{if} \quad (i_1, \ldots, i_k) \quad \textit{is an even permutation of} \quad (n-k+1, \ldots, n) \\ -1 & \textit{if} \quad (i_1, \ldots, i_k) \quad \textit{is an odd permutation of} \quad (n-k+1, \ldots, n) \\ 0 & \textit{otherwise} \end{cases}$$

Extend $\int$ by $\mathbb{Z}$-linearity, getting $\int : \bigwedge^k M_n \to \mathbb{Z}$

The Main Theorem
implies that

$$\int_{G(k,n)} P(\sigma) \cap \Omega_{i_1 \ldots i_k} = \int P(D) \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}$$

Example

Example

Let us compute the degree of $G(2, 2+n)$ in the Plücker embedding.

Example

Let us compute the degree of $G(2, 2+n)$ in the Plücker embedding.
It is

$$\int_{G(2,n+2)} \sigma_1^{2n} \cap [G(2, n+2)] = \int_{n+2} D_1^{2n} \epsilon^1 \wedge \epsilon^2$$

Example

Let us compute the degree of $G(2, 2+n)$ in the Plücker embedding. It is

$$\int_{G(2,n+2)} \sigma_1^{2n} \cap [G(2, n+2)] = \int_{n+2} D_1^{2n} \epsilon^1 \wedge \epsilon^2$$

It is the coefficient of $\pi_{2,n+2} := \epsilon^{n+1} \wedge \epsilon^{n+2}$ in the expansion of $D_1^{2n} \epsilon^1 \wedge \epsilon^2$

Example

Let us compute the degree of $G(2, 2+n)$ in the Plücker embedding.
It is

$$\int_{G(2,n+2)} \sigma_1^{2n} \cap [G(2, n+2)] = \int_{n+2} D_1^{2n}\epsilon^1 \wedge \epsilon^2$$

It is the coefficient of $\pi_{2,n+2} := \epsilon^{n+1} \wedge \epsilon^{n+2}$ in the expansion of $D_1^{2n}\epsilon^1 \wedge \epsilon^2$

Notice that:

$$wt(\epsilon^{1+n} \wedge \epsilon^{2+n}) = 2n, \ \deg(D_1^{2n}) = 2n, \ \mathrm{wt}(\epsilon^1 \wedge \epsilon^2) = 0$$

and

$$\mathrm{rk}_{\mathbb{Z}}(\bigwedge^2 M_{n+2})_{2n} = 1.$$

Using the 1st Newton formula:

Using the 1st Newton formula:

1st Newton Formula

$$D_1^m(\alpha \wedge \beta) = \sum_{j=0}^{m} \binom{m}{j} D_1^j\alpha \wedge D_1^{m-j}\beta$$

Using the 1st Newton formula:

Using the 1st Newton formula:

$$D_1^{2n}\epsilon^1 \wedge \epsilon^2 =$$

Using the 1st Newton formula:

$$D_1^{2n}\epsilon^1 \wedge \epsilon^2 = \sum_{j=0}^{2n} \binom{2n}{j} D_1^j\epsilon^1 \wedge D_1^{2n-j}\epsilon^2 = \sum_{j=0}^{2n} \binom{2n}{j} \epsilon^{1+j} \wedge \epsilon^{2+2n-j}$$

Using the 1st Newton formula:

$$D_1^{2n}\epsilon^1 \wedge \epsilon^2 = \sum_{j=0}^{2n} \binom{2n}{j} D_1^j\epsilon^1 \wedge D_1^{2n-j}\epsilon^2 = \sum_{j=0}^{2n} \binom{2n}{j} \epsilon^{1+j} \wedge \epsilon^{2+2n-j}$$

In the sum we get contributions only from $j = n$ and $j = n + 1$:

$$D_1^{2n}\epsilon^1 \wedge \epsilon^2 = \binom{2n}{n}\epsilon^{1+n} \wedge \epsilon^{2+n} + \binom{2n}{n+1}\epsilon^{2+n} \wedge \epsilon^{1+n}$$

Using the 1st Newton formula:

$$D_1^{2n}\epsilon^1 \wedge \epsilon^2 = \sum_{j=0}^{2n} \binom{2n}{j} D_1^j \epsilon^1 \wedge D_1^{2n-j}\epsilon^2 = \sum_{j=0}^{2n} \binom{2n}{j} \epsilon^{1+j} \wedge \epsilon^{2+2n-j}$$

In the sum we get contributions only from $j = n$ and $j = n + 1$:

$$D_1^{2n}\epsilon^1 \wedge \epsilon^2 = \binom{2n}{n} \epsilon^{1+n} \wedge \epsilon^{2+n} + \binom{2n}{n+1} \epsilon^{2+n} \wedge \epsilon^{1+n}$$

$$= \left[\binom{2n}{n} - \binom{2n}{n+1} \right] \epsilon^{1+n} \wedge \epsilon^{2+n} = \frac{2n!}{n!(n+1)!} \epsilon^{1+n} \wedge \epsilon^{2+n}$$

Exactly by the same technique one finds:

$$\int_{n+k} D_1^{kn}\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} = \omega_{i_1,\ldots,i_k} = \frac{(kn-w)!\prod_{j<k}(i_k - i_j)}{(n+k-i_1)! \cdot \ldots \cdot (n+k-i_k)!} \quad (3)$$

Exactly by the same technique one finds:

$$\int_{n+k} D_1^{kn}\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} = \omega_{i_1,\ldots,i_k} = \frac{(kn-w)!\prod_{j<k}(i_k - i_j)}{(n+k-i_1)!\cdot\ldots\cdot(n+k-i_k)!} \quad (3)$$

where $w = \sum_{j=1}^{k}(i_j - j)$

Exactly by the same technique one finds:

$$\int_{n+k} D_1^{kn}\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} = \omega_{i_1,\ldots,i_k} = \frac{(kn-w)!\prod_{j<k}(i_k - i_j)}{(n+k-i_1)!\cdot \ldots \cdot (n+k-i_k)!} \quad (3)$$

where $w = \sum_{j=1}^{k}(i_j - j)$

It is the degree of the Schubert variety

$$\Omega_{i_1\ldots i_k}(F^\bullet),$$

$F^\bullet$ a given complete flag of $\mathbb{C}^n$.

Rational Space curves having flexes at prescribed points.

Rational Space curves having flexes at prescribed points.

Question (Ranestad): Find a list of the number of rational curves in $\mathbb{P}^3$ of degree $n+3$ having inflectional tangent at $2n$ general marked points.

Rational Space curves having flexes at prescribed points.

Question (Ranestad): Find a list of the number of rational curves in $\mathbb{P}^3$ of degree $n+3$ having inflectional tangent at $2n$ general marked points.

Any such a curve can be gotten by projecting a rational normal curve in $\mathbb{P}^{n+3}$ from a $\mathbb{P}^{n-1}$ which intersects the osculating plane at the marked points. Therefore the sought for number is that of the $\mathbb{P}^{n-1}$'s having such a behaviour. This amounts to compute the integral.

$$\int_{G(n,n+4)} \sigma_2^{2n} \cap [G(n, n+4)] = \int_{G(4,n+4)} \sigma_2^{2n} \cap [G(4, n+4)]$$

One usually computes σ_2^{2n} via iteration of Pieri's formula.

One usually computes σ_2^{2n} via iteration of Pieri's formula.

You may ask *Schubert*

One usually computes σ_2^{2n} via iteration of Pieri's formula.

You may ask *Schubert* [(1)]

[(1)] [S. Katz and S. A. Strømme, "Schubert", a Maple© package for intersection theory and enumerative geometry, `http://math.uib.no/schubert/`]

One usually computes σ_2^{2n} via iteration of Pieri's formula.

You may ask *Schubert* [(1)] doing it for you, but...

[(1)] [S. Katz and S. A. Strømme, "Schubert", a Maple© package for intersection theory and enumerative geometry, `http://math.uib.no/schubert/`]

One usually computes σ_2^{2n} via iteration of Pieri's formula.

You may ask *Schubert* [1] doing it for you, but...

...when $n = 12$ you get the following message

`Execution stopped: Stack limit reached.`

(Vainsencher, Økland – private communication).

---

[1] [S. Katz and S. A. Strømme, "Schubert", a Maple© package for intersection theory and enumerative geometry, `http://math.uib.no/schubert/`]

One usually computes σ_2^{2n} via iteration of Pieri's formula.

You may ask *Schubert* [1] doing it for you, but...

...when $n = 12$ you get the following message

`Execution stopped: Stack limit reached.`

(Vainsencher, Økland – private communication).

However, we have a formula:

[1] [S. Katz and S. A. Strømme, "Schubert", a Maple© package for intersection theory and enumerative geometry, `http://math.uib.no/schubert/`]

$$\int_{G(4,n+4)} \sigma_2^{2n} \cap [G(4, n+4)] = \tag{4}$$

$$\int_{G(4,n+4)} \sigma_2^{2n} \cap [G(4,n+4)] = \tag{4}$$

$$\sum_{\substack{l_1+l_2+l_3+l_4=2n \\ l_1,l_2,l_3,l_4\geq 0 \\ 0\leq m_2\leq l_2 \\ 0\leq m_3\leq l_3+l_2-m_2}} \binom{2n}{l_1,l_2,l_3,l_4}\binom{l_2}{m_2}\binom{l_3+l_2-m_2}{m_3}\cdot \omega_{I(l_1,l_2,l_3,l_4,m_1,m_2,m_3)}$$

$$\int_{G(4,n+4)} \sigma_2^{2n} \cap [G(4,n+4)] = \tag{4}$$

$$\sum_{\substack{l_1+l_2+l_3+l_4=2n \\ l_1,l_2,l_3,l_4 \geq 0 \\ 0 \leq m_2 \leq l_2 \\ 0 \leq m_3 \leq l_3+l_2-m_2}} \binom{2n}{l_1,l_2,l_3,l_4}\binom{l_2}{m_2}\binom{l_3+l_2-m_2}{m_3} \cdot \omega_{I(l_1,l_2,l_3,l_4,m_1,m_2,m_3)}$$

where

$$I(l_1,l_2,l_3,l_4,m_1,m_2,m_3)$$

is the ordered 4-tuple of positive integers

$$(1+l_1,\ 2+l_2+m_2,\ 3+l_3+m_3,\ 4+2l_4+(l_2-m_2)+(l_3-m_3)).$$

A very important Question:

A very important Question:

WHO CARE?

A very important Question:

WHO CARE?

We don't, we may produce many similar formulas!

We got formula (4) by applying 2nd Newton formula

We got formula (4) by applying 2nd Newton formula

$$D_h^m(\epsilon^i \wedge \alpha) = \sum_{j=0}^{m} \binom{m}{j} D_{h-1}^j(\epsilon^{i+j} \wedge D_h^{m-j}\alpha)$$

We got formula (4) by applying 2nd Newton formula

We got formula (4) by applying 2nd Newton formula

to

$$D_2^{2n}(\epsilon^1 \wedge \epsilon^2 \wedge \epsilon^3 \wedge \epsilon^4) = D_2^{2n}((\epsilon^1 \wedge \epsilon^2 \wedge \epsilon^3) \wedge \epsilon^4)$$

$$\sum_{l_1=0}^{2n} \binom{2n}{l_1} D_1^{l_1}(\epsilon^{1+l_1} \wedge D_2^{2n-l_1}(\epsilon^2 \wedge \epsilon^3 \wedge \epsilon^4)).$$

We got formula (4) by applying 2nd Newton formula

to

$$D_2^{2n}(\epsilon^1 \wedge \epsilon^2 \wedge \epsilon^3 \wedge \epsilon^4) = D_2^{2n}((\epsilon^1 \wedge \epsilon^2 \wedge \epsilon^3) \wedge \epsilon^4)$$

$$\sum_{l_1=0}^{2n} \binom{2n}{l_1} D_1^{l_1}(\epsilon^{1+l_1} \wedge D_2^{2n-l_1}(\epsilon^2 \wedge \epsilon^3 \wedge \epsilon^4)).$$

Then we apply the same formula to

$$D_2^{2n-l_1}(\epsilon^2 \wedge \epsilon^3 \wedge \epsilon^4) = D_2^{2n-l_1}(\epsilon^2 \wedge (\epsilon^3 \wedge \epsilon^4))$$

and then once again, and then...

. . . we wrote a trivial CoCoA (version 4.7) code to compute the a list varying n, and

...we wrote a trivial CoCoA (version 4.7) code to compute the a list varying n, and

with an Apple iBook G4, 1.2GHz, RAM, 768Mb

...we wrote a trivial CoCoA (version 4.7) code to compute the a list varying n, and

with an Apple iBook G4, 1.2GHz, RAM, 768Mb

...we wrote a trivial CoCoA (version 4.7) code to compute the a list varying n, and

with an Apple iBook G4, 1.2GHz, RAM, 768Mb

$\longleftarrow$ my iBook

. . . we wrote a trivial CoCoA (version 4.7) code to compute the a list varying n, and

with an Apple iBook G4, 1.2GHz, RAM, 768Mb

$\longleftarrow$ my iBook

in a couple of hours we got the following list:

n	♯(summands)	$R_n := \int \sigma_2^{2n} \cap [G(4, 4+n)]$	execution time
0	1	1	0,082s
1	13	0	0,242s
2	56	1	0,614s
3	142	5	1,449s
4	331	126	3,340s
5	641	3396	6,434s
6	1191	114675	12,081s
7	1981	4430712	20,053s
8	3221	190720530	32,755s
9	4866	8942188632	50,085s
10	7256	449551230102	1m 20s
11	10268	23948593282950	2m 55s
12	14418	1339757254689348	2m 44s
13	19466	78153481093195800	4m 02s
14	26156	4727142898098368085	5m 2s
15	34086	295116442188446065635	9m 9s
16	44286	18945322608397492982250	10m 46s
17	56141	1246718376589846006057200	11m 52s
18	71031	83878801924226511500933250	16m 37s
19	88071	5756860011979383129907915050	19m 55s
20	109061	402290757162008042628235950300	25m 53s
21	132783	28575935656515287427874861725000	34m 37s
22	161533	2060372706082551084572192852992530	01h 07m

Two days ago, Jan Magnus Økland sent me the list up to $n = 30$.

Two days ago, Jan Magnus Økland sent me the list up to $n = 30$.

He got it in a couple of hours...

...using Schubert2 for Macaulay (Grayson, Daniel and Stillmann)

Two days ago, Jan Magnus Økland sent me the list up to $n = 30$.

He got it in a couple of hours...

...using Schubert2 for Macaulay (Grayson, Daniel and Stillmann)

on a machine with processor speed 2,2 GHz and 16Gb Ram

Two days ago, Jan Magnus Økland sent me the list up to $n = 30$.

He got it in a couple of hours...

...using Schubert2 for Macaulay (Grayson, Daniel and Stillmann)

on a machine with processor speed 2,2 GHz and 16Gb Ram

Then, yesterday morning I tried with CoCoA $n = 23$ and $n = 24$. In a couple of hours I got

Two days ago, Jan Magnus Økland sent me the list up to $n = 30$.

He got it in a couple of hours...

...using Schubert2 for Macaulay (Grayson, Daniel and Stillmann)

on a machine with processor speed 2,2 GHz and 16Gb Ram

Then, yesterday morning I tried with CoCoA $n = 23$ and $n = 24$. In a couple of hours I got

$$R_{23} = 150602793256105806699840616089824880$$

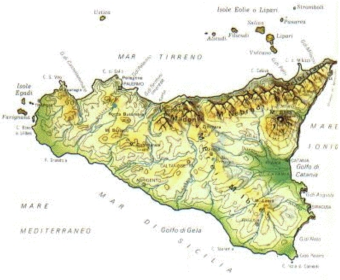
and

$$R_{24} = 111476845977869020878159294744162032 76$$

Thank You!

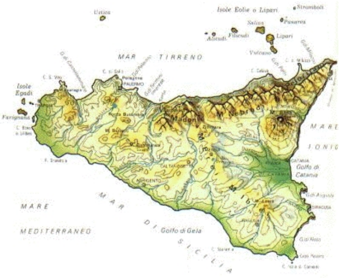
Thank You!
(Grazie)

Thank You!
(Grazie)



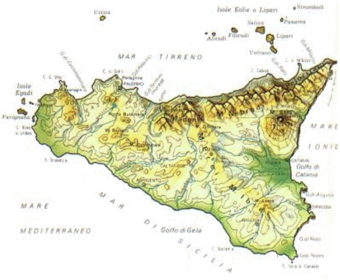
Thank You!

(Grazie)



letterio

Thank You!
(Grazie)

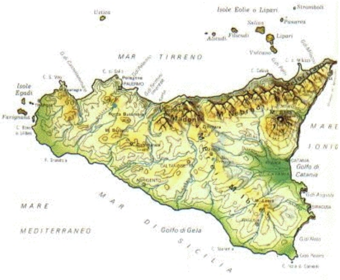


letterio

http://

Thank You!

(Grazie)



letterio

http://c

Thank You!

(Grazie)

letterio

http://ca

Thank You!
(Grazie)

letterio

http://cal

Thank You!
(Grazie)

letterio

http://cal

Thank You!

(Grazie)

letterio

http://calv

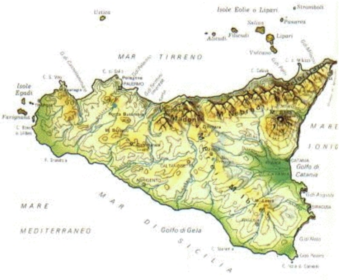
Thank You!

(Grazie)

letterio

http://calvi

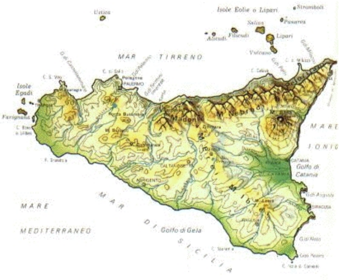
Thank You!
(Grazie)



letterio

http://calvin

Thank You!
(Grazie)



letterio

http://calvino

Thank You!
(Grazie)

letterio

http://calvino.

Thank You!

(Grazie)

letterio

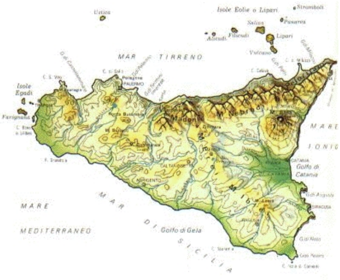
http://calvino.p

Thank You!
(Grazie)

letterio

http://calvino.po

Thank You!
(Grazie)



letterio

http://calvino.pol

Thank You!

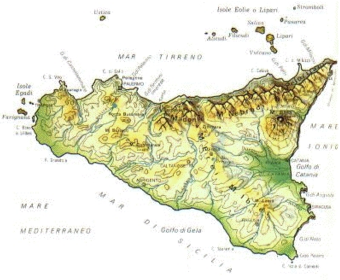
(Grazie)

letterio

http://calvino.poli

Thank You!

(Grazie)

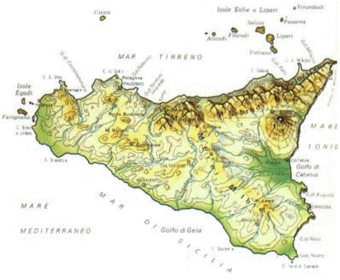


letterio

http://calvino.polit

Thank You!

(Grazie)



letterio

http://calvino.polito

Thank You!
(Grazie)

letterio

http://calvino.polito.

Thank You!

(Grazie)

letterio

http://calvino.polito.i

Thank You!
(Grazie)

letterio

http://calvino.polito.it

Thank You!

(Grazie)

letterio

http://calvino.polito.it/~

Thank You!

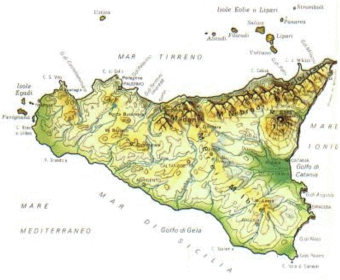
(Grazie)

letterio

http://calvino.polito.it/~g

Thank You!

(Grazie)



letterio

http://calvino.polito.it/~ga

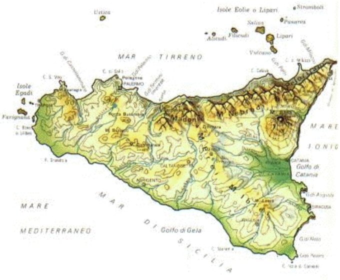
Thank You!

(Grazie)

letterio

http://calvino.polito.it/~gat

Thank You!
(Grazie)

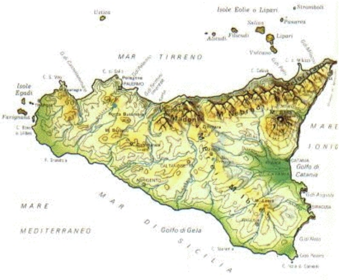


letterio

http://calvino.polito.it/~gatt

Thank You!

(Grazie)



letterio

http://calvino.polito.it/~gatto