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Title: Embedding the Grassmann Cone in a Polynomial Ring

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Abstract: I will report on some joint work with P. Salehyan and I. Scherbak.

Let $X, x_1, \ldots, x_r$ be indeterminates over the rationals. Schur polynomials in $(x_1, \ldots, x_r)$ permit to define a natural isomorphism $\phi : \bigwedge^r \mathbb{Q}[X] \to B_r := \mathbb{Q}[x_1, \ldots, x_r]$ and one says that $\eta \in \bigwedge^r \mathbb{Q}[X]$ is *decomposable* if it can be written as $p_1(X) \wedge \cdots \wedge p_r(X)$, for some $p_i \in \mathbb{Q}[X]$. The *Grassmann cone* $G_r \subseteq \bigwedge^r \mathbb{Q}[X]$ is, by definition, the locus of all decomposable tensors of $\bigwedge^r \mathbb{Q}[X]$.

The talk aims to show how to write explicit equations of $\phi(G_r(\mathbb{Q}))$, the isomorphic image of the Grassmann cone in the polynomial ring B_r, via the notion of Hasse-Schmidt derivation on a Grassmann algebra. The latter provides a method to explicitly determine a map $\Psi_r(z) : B_r \to B_{r-1}((z)) \otimes B_{r+1}((z))$ which enjoys the following property: a polynomial $p := p(x_1, \ldots, x_r) \in B_r$ belongs to $\phi(G_r)$ if and only if the residue of $\Psi_r(z)(p)$ vanishes at $z = 0$. The formula is deduced by exploiting the action of the cohomology ring of the Grassmannian $G_r(\mathbb{C}^\infty)$ on $\bigwedge^r \mathbb{Q}[X]$, which so establishes a link of the subject with Schubert calculus. Moreover, the limit for $r \to \infty$ of the equation $\mathrm{Res}_{z=0}\Psi_r(z)(p) = 0$ returns the celebrated system of infinitely many bilinear PDEs known as KP hierarchy. In fact, such system includes the generalization of the Korteweg–de Vries equation for solitary waves that, in the Seventies, the Soviet physicists Kadomtsev and Petviashvili introduced to model plasma physics.