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Intersection Formulas in Grassmann Varieties

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Flexes of Rational Curves

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Any such curve can be gotten by projecting a rational normal curve in $\mathbb{P}^{d+3}$ from a $\mathbb{P}^{d-1}$ which intersects the osculating plane at the marked points. Therefore the sought for number is that of the $\mathbb{P}^{d-1}$'s having such a behaviour.

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how may look a formula for it?

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Or alternatively:

what is the degree of the Plücker embedding of the grassmannian $G(k, \mathbb{P}^n)$*?*

The answer is:

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$$a_0 = n - p - 1, \qquad a_1 = n - p + 1, \qquad a_2 = n - p + 2, \; \ldots, \; a_p = n$$

setzt. In beiden Fällen ergiebt sich übereinstimmend:

$$(27) \qquad \frac{\underline{|(p+1)(n-p)}\;\underline{|1}\;\underline{|2}\;\underline{|3}\cdots\underline{|p}}{\underline{|n}\;\underline{|n-1}\;\underline{|n-2}\cdots\underline{|n-p}}.$$

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$$\frac{\underline{|(p+1)(n-p)}\,\underline{|1}\,\underline{|2}\,\underline{|3}\cdots\underline{|p}}{\underline{|n}\,\underline{|n-1}\,\underline{|n-2}\cdots\underline{|n-p}}. \tag{27}$$

In modern language:

$$d_{k,n} = \int \sigma_1^{(1+k)(n-k)} \cap [G(k,n)] = \frac{1!2! \cdot \ldots \cdot k!((1+k)(n-k))!}{n!(n-1)! \cdot \ldots \cdot (n-k)!} \tag{1}$$

For a proof of (1) one can look at the classical book of Hodge-Pedoe (see also Fulton's *Intersection Theory*). It is via induction: then one should figure out the formula in advance.

A special case of (1) is very popular: how many lines do intersect 4 others in $\mathbb{P}^3$? Putting $n = 3$ and $k = 1$ in (1) one gets 2.

In the same vein, another problem is:

find the number of lines intersecting 4 subspaces of codimension n in general position in $\mathbb{P}^{2n+1}$

It amounts to compute

$$\int \sigma_n^4 \cap [G(2, 2n + 2)]$$

The above number is $n + 1$: see Griffiths&Harris and/or Donagi.

General Question

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To get such an expression, our tool is *Schubert Calculus on Grassmann Algebras* (SCGA) of an A-module M (rather than on a Grassmann variety!).

What is this?

SCGA 1

SCGA(M) is the datum of an A-algebra homomorphism:

$$D_t := \sum_{i \geq 0} D_i t^i : \bigwedge M \longrightarrow \bigwedge M[[t]]$$

i.e. such that

$$D_t(\alpha \wedge \beta) = D_t\alpha \wedge D_t\beta, \quad \forall \alpha, \beta \in \bigwedge M \tag{2}$$

which we called the *fundamental equation of Schubert Calculus on a Grassmann algebra*.

Eq. (2) is equivalent to:

$$D_h(\alpha \wedge \beta) = \sum_{\substack{h_1 + h_2 = h \\ h_i \geq 0}} D_{h_1}\alpha \wedge D_{h_2}\beta \tag{3}$$

which is the h^{th} order Leibniz rule, holding for all $h \geq 0$.

The set of all D_t, defining a Schubert Calculus on $\bigwedge M$, form a group $HS_t(\bigwedge M)$ with respect to the product

$$D_t * E_t = \sum_{h\geq 0}\sum_{i+j=h}(D_i \circ E_j)t^h.$$

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SCGA(M) is based on (3) (Pieri's Formula!) and on *integration by parts* (the counterpart of Giambelli's formula):

$$D_h\alpha \wedge \beta = \sum_{i=0}^{h} D_{h-i}(\alpha \wedge \overline{D}_i\beta) \qquad (4)$$

where $\overline{D}_t$ is the formal inverse of $D_t \in HS_t(\bigwedge M)$.

SCGA 2

Here are some formulas of SCGA:

$$D_h^n(\alpha \wedge \beta) = \sum_{n_0+n_1+\ldots+n_h=n} \binom{n}{n_0, n_1, \ldots, n_h} D_h^{n_0} D_{h-1}^{n_1} \ldots D_1^{n_{h-1}} \alpha \wedge D_1^{n_1} D_2^{n_2} \ldots D_h^{n_h} \beta, \tag{5}$$

where the multinomial coefficient $\binom{n}{n_0, n_1, \ldots, n_h}$ is defined via the equality:

$$(a_0 + a_1 + \ldots + a_h)^n = \sum \binom{n}{n_0, n_1, \ldots, n_h} a_0^{n_0} a_1^{n_1} \ldots a_h^{n_h}$$

where $n_0 + n_1 + \ldots + n_h$.

In particular

$$D_1^n(\alpha \wedge \beta) = \sum_{h=0}^{n} \binom{n}{h} D_1^h \alpha \wedge D_1^{n-h} \beta$$

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Moreover, for all $(\alpha_0, \alpha_1, \ldots, \alpha_k) \in \bigwedge^{1+k} M$

$$D_1(\alpha_0 \wedge \alpha_1 \wedge \ldots \wedge \alpha_k) = \sum_{j=0}^{k} \alpha_1 \wedge \alpha_2 \wedge \ldots \wedge D_1 \alpha_j \wedge \ldots \wedge \alpha_k,$$

from which one easily gets:

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from which one easily gets:

$$D_1^n(\alpha_0 \wedge \alpha_1 \wedge \ldots \wedge \alpha_k) = \sum \binom{n}{n_0, n_1, \ldots, n_k} D_1^{n_0} \alpha_0 \wedge D_1^{n_1} \alpha_1 \wedge \ldots \wedge D_1^{n_k} \alpha_k$$

One may write many others similar formulas, easily proven by induction and basic algebra experience.

Among all we quote:

$$D_2^n(\alpha \wedge \beta \wedge \gamma \wedge \delta) =$$

$$= \sum \binom{n}{n_1, \ldots, n_{10}} D_2^{n_1} D_1^{n_2+n_3+n_4} \alpha \wedge D_2^{n_5} D_1^{n_2+n_6+n_7} \beta \wedge$$

$$\wedge D_2^{n_8} D_1^{n_3+n_6+n_9} \gamma \wedge D_2^{n_{10}} D_1^{n_4+n_7+n_9} \delta \qquad (6)$$

the sum being over all non negative integers $(n_1, n_2, \ldots, n_{10})$ such that $\sum n_i = n$.

“Classical Schubert Calculus”

It is recovered by the pair $(\bigwedge M, D_t)$, where $\bigwedge M$ is the exterior algebra of a free $\mathbb{Z}$-module M of rank, say, $1+n$, spanned by $(\epsilon^0, \epsilon^1, \ldots, \epsilon^n)$, and D_t is

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the unique extension to a map $\bigwedge M \longrightarrow \bigwedge M[t]$ *of the linear map*

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defined by $D_t(\epsilon^i) = \sum_{j\geq 0} \epsilon^{i+j} t^j$, where $\epsilon^{i+j} = 0$ if $i+j > n$, gotten by imposing the fundamental equation (2):

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$$D_t(\alpha \wedge \beta) = D_t\alpha \wedge D_t\beta.$$

The degree of $G(k, \mathbb{P}^n)$ is then given by:

$$D_1^{(1+k)(n-k)}(\epsilon^0 \wedge \epsilon^1 \wedge \ldots \wedge \epsilon^k) = d_{k,n} \cdot \epsilon^{n-k} \wedge \epsilon^{n-k+1} \wedge \ldots \wedge \epsilon^n$$

where

$$d_{k,n} = \sum_{\tau \in S_{1+k}} (-1)^{|\tau|} \binom{(1+k)(n-k)}{n-k+\tau(0), n-k+\tau(1), \ldots, n-k+\tau(k)}$$

Taking the l.c.d. and simplifying, one EASILY gets formula (1).

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Using formula (5):

$$C_{a,b} = \sum_{b_0=0}^{b} \sum_{a_0=0}^{2n-2b} \binom{2n-2-2b}{a_0, 2n-2-b-a_0} \cdot K_{a_0,b_0} \tag{7}$$

where

$$K_{a_0,b_0} = \binom{b}{b_0, n-1-a_0-2b_0, b+b_0+a_0-n+1} - \binom{b}{b_0, n-a_0-2b_0, b+b_0+a_0-n}$$

Formula (7) is a new formula in Schubert calculus.

Some Computations with Mathematica 5.1

n	a	b	$C_{a,b}$
1	**0**	**0**	**1**
2	0	1	0
	2	**0**	**1**
3	0	2	1
	2	1	1
	4	**0**	**2**

n	a	b	$C_{a,b}$
4	0	3	1
	2	2	2
	4	1	3
	6	**0**	**5**
5	0	4	3
	2	3	4
	4	2	6
	6	1	9
	8	**0**	**14**

n	a	b	$C_{a,b}$
6	0	5	6
	2	4	9
	4	3	13
	6	2	19
	8	1	28
	10	**0**	**42**

n	a	b	$C_{a,b}$
7	0	6	15
	2	5	21
	4	4	30
	6	3	43
	8	2	62
	10	1	90
	12	**0**	**132**

…more

n	a	b	$C_{a,b}$
8	0	7	36
	2	6	51
	4	5	72
	6	4	102
	8	3	145
	10	2	207
	12	1	297
	14	**0**	**429**

n	a	b	$C_{a,b}$
9	0	8	91
	2	7	127
	4	6	178
	6	5	250
	8	4	352
	10	3	497
	12	2	704
	14	1	1001
	16	**0**	**1430**

n	a	b	$C_{a,b}$
10	0	9	232
	2	8	323
	4	7	450
	6	6	628
	8	5	878
	10	4	1230
	12	3	1727
	14	2	2431
	16	1	3432
	18	**0**	**4862**

Clearly $C_a := C_{a,0} = d_{1,n}$.

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If one writes:

$$F(t) = \sum_{n \geq 0} d_{1,n+1} \cdot \frac{t^n}{n!}$$

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then Taíse Santiago proves that:

$$F(t) = e^{2t}(I_0(t) - I_1(t)),$$

where by $I_n(x)$ one means the n^{th} *modified Bessel functions* of the first kind, solution of :

$$z^2 y'' + z y' - (z^2 + n^2) y = 0$$

the *modified Bessel differential equation*.

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$$D_2^{2n}\epsilon^1 \wedge \epsilon^2 \wedge \epsilon^3 \wedge \epsilon^4 = \mathbf{C_n} \cdot \epsilon^{n+1} \wedge \epsilon^{n+2} \wedge \epsilon^{n+3} \wedge \epsilon^{n+4}$$

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where

$$\mathbf{C}_n = \sum_{\tau \in S_4} (-1)^{|\tau|} C_n^\tau \tag{8}$$

and

$$C_n^\tau = \sum \binom{2n}{n_1, \ldots, n_{10}}, \tag{9}$$

the sum over all non negative $n_1, \ldots, n_{10}$ such that:

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$$\begin{aligned} 2n_1 + n_2 + n_3 + n_4 &= n + \tau(1) \\ 2n_5 + n_2 + n_6 + n_7 &= n + \tau(2) \\ 2n_8 + n_3 + n_6 + n_9 &= n + \tau(3) \\ 2n_{10} + n_4 + n_7 + n_9 &= n + \tau(4) \end{aligned}$$

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2n_8 + n_3 + n_6 + n_9 &= n + \tau(3) \\
2n_{10} + n_4 + n_7 + n_9 &= n + \tau(4)
\end{aligned}$$

To get it one has simply applied formula (6).

Putting formulas (8) and (9) in *Mathematica 5.1* and in **R** (a program for statistical computing), one gets the following table:

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n	C_n
0	1
1	0
2	1
3	5
4	126
5	3396

n	C_n
6	114675
7	4430712
8	190720530
9	8942188632
10	449551230102

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One can generate the same table using the *Schubert Package* for *Maple*, in approximatively the same time of **R**.

Thank You

Obrigado!