Proposal of a minicourse intended to be given at

during the

25° Colóquio Brasileiro de Matemática

Título do Curso

Schubert Calculus: an Algebraic Introduction.

Autor

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SUMMARY

1. Nível;
2. Descrição do Curso;
   (a) objetivos
   (b) Conteúdos,
   (c) Distribuição de capítulos e seções
   (d) Bibliografia;
3. Pré-requisitos.

1. Nível

The course is definitely elementar, because most of the techniques involved in this presentation are borrowed from basic undergraduate linear algebra. Since Schubert Calculus is very important for enumerative algebraic geometry and it is not usually covered in doctoral courses of algebraic geometry, ph.D. students may profit of it, too.

2. Descrição do Curso

2a) **The Goal.** Let $G := G(k, n)$ be the grassmannian variety parametrizing k-dimensional linear subspaces of a n-dimensional complex vectorspace. Schubert Calculus is the set of formal rules necessaries to perform computations in the intersection (or cohomology) ring of $G(k, n)$. The course would aim to let the audience becoming familiar with Schubert Calculus *in just one week*. This would be possible by observing that the cohomology ring of $G(k, n)$ can be realized as a ring of differential operators on the k-th exterior power of a n-dimensional vector space. Within this framework, classical *Pieri*'s and *Giambelli*'s formula, the fundamental blocks of Schubert Calculus, are shown to be, respectively, nothing but than Leibniz's rule and integration by part. The lectures would be delivered either in English or Portugues.

2b) **Contents.** One begins to see how some enumerative problems can be translated in terms of Schubert calculus: the number (2) of lines intersecting 4 lines in general position in the projective 3-space is an example. Then one see the construction of the grassmannian as a holomorphic variety and explain why Schubert varieties form a cellular decomposition of it (implying that Schubert cycles are a basis for the homology). Finally one will put the emphasis on the practical problem of performing computation and, at this point, one will show that the intersection ring of the grassmannian $G(k,n)$ is a ring of linear operators on the k-th exterior power of $\mathbf{C}^n$. Many examples regarding enumerative geometry shall be worked out.

2c) – Table of Contents [1] –

[1] Each lecture is supposed to be one hour long.

2d) REFERENCES

The six main references are:

1. W. Fulton, *Intersection Theory*, Springer-Verlag, 1984.
2. Ph. Griffiths, J. Harris, *Principles of Algebraic Geometry*, Wiley-Interscience, New York, 1978.
3. R. Bott, L. Tu, *Differential Forms in Algebraic Topology*, **GTM 82**, Springer-Verlag.
4. J. Milnor, J. Stasheff, *Characteristic Classes*, Study **76**, Princeton University Press, Princeton, New Jersey, 1974.
5. W. Fulton, P. Pragacz, *Schubert Varieties and Degeneracy Loci*, Springer **LNM 1689**, 1998.
6. S. L. Kleiman, D. Laksov, *Schubert Calculus*, Amer. math. Monthly **79** (1972), 1061–1082.

The last two lectures are based on the two manuscripts

7. L. Gatto, *Schubert Calculus via Hasse-Schmidt derivations*, submitted.
8. L. Gatto, *The algebra of Schubert Calculus*, `ArXiv - math.AG/0405155.`

For general synopsis look at

8. F. Sottile, *Four Entries for the Kluwer Encyclopaedia of Mathematics*, `arXiv: math.AG/0102047`.

If you want to have some informative notions about intersection theory as well as nice examples treated in details, you may look at

10. I. Vainsencher, Classes características em Geometria Algébrica, IMPA, Rio de Janeiro, 1985.

em lingua portuguesa. The classical literature on Schubert Calculus dates back to the classical papers:

11. M. Pieri, *Formule di coincidenza per le serie algebriche* ∞^n *di coppie di punti dello spazio a* n *dimensioni*, Rend. Circ. Mat. Palermo **5** (1891), 252–268.
12. G. Z. Giambelli, *Sulle varietà rappresentate coll'annullare dei determinanti minori contenuti in un determinante simmetrico o emisimmetrico generico di forme*, Atti. R. Accad. Torino **41** (1906), 102–125.
13. H. C. H. Schubert, *Kalkül der abzählenden Geometrie*, 1879, reprinted with an introduction by S.L. Kleiman, Springer–Verlag, 1979.

14. H. C. H. Schubert, *Beziehungen zwischen den Linearen Räumen auferlegbaren charakterischen Bedingungen*, Math. Ann. **38** (1891), 598–602.
15. H. C. H. Schubert, *Anzahlbestimmungen für lineare Räume beliebiger Dimension*, Acta Math. **8** (1866), 97–118.
16. H. C. H. Schubert, *Allgemeine Anzahlfunctionen für Kegelschnitte, Flächen und Räume zweiten grades in n Dimensionen*, Math. Ann. **45**,(1895), 153–206.

Schubert calculus is related with wronskians and Hasse-Schmidt derivations. Some history below

17. F. K. Schmidt, *Die Wronskische Determinante in beliebigen differenzierbaren Funktionenkörper*, math. Z. **45** (1939), 62–74.
18. H. Hasse, F. H. Schmidt, *Noch eine Begründung der Theorie der höheren Differentialquotienten in einem algebraischen Funktionenkorper einer Unbestimmten*, J. Reine U. Angew. math. **177** (1937), 215–237.

Still classical, although more recents are:

19. G. Kempf, D. Laksov, *The determinantal formula of Schubert calculus*, Acta Math. **132** (1974), 153–162.

If one is interested also in the quantum part of the story, may consult

20. A. Bertram, *Quantum Schubert Calculus*, Adv. Math. **128**, (1997) 289–305.
21. A. Bertram, *Computing Schubert's calculus with Severi Residues: an introduction to Quantum Cohomology*, in Moduli of Vector Bundles, M. Maruyama ed., Lecture Notes in Pure and Applied Mathematics, **179**, Marcel Dekker inc., 1996.
22. W. Fulton, R. Pandahripande, *Notes on stable maps and quantum cohomology*, Algebraic Geometry, Santa Cruz 1995 (J. Kollár et al., eds.), proc. symp. Pure Mat. **62, part 2** (1997), AMS, 45–96.
23. D. Gepner, *Fusion rings and geometry*, Comm. in Math. Physics, **141**, (1991), 381–411.
24. R. Pandharipande, The small quantum cohomology ring of the Grassmannian, in "Quantum Cohomology at the Mittag-Leffler Institute", P.Aluffi ed., (1996–97), 38–44.
25. B. Siebert, G. Tian, *On Quantum Cohomology rings of Fano manifolds and a formula of Vafa and Intrilligator*, Asian J. Math **1** (1997), 679–695.
26. E. Witten, *The Verlinde Algebra and the cohomology of the Grassmannian*, in "Geometry, Topology and Physics", Conference Proceedings and

Lecture Notes in Geometric Topology, Vol. IV, pp. 357-422, International Press, Cambridge, MA, 1995.

For a readable account on the enumerative significance of quantum cohomology see (em lingua portuguesa):

27. J. Kock, I. Vainsencher, *A fórmula de Kontsevich para curvas racionais planas*, 22° Colóquio Brasileiro de Matemática, IMPA, Rio de Janeiro, 1999.

3. Pré-requisitos. The course will deal in a elementary way with important topics in Algebraic Geometry not usually treated in post-graduate programs. However, the only important pre-requisite is a (good) course of linear algebra. Basics on exterior algebras shall be developed during the lectures, if necessary).