Robust Model Predictive Control via Scenario Optimization

G.C. Calafiore*, *Senior Member, IEEE,* L. Fagiano*,**,†, *Member, IEEE*

Abstract—This paper discusses a novel probabilistic approach for the design of robust model predictive control (MPC) laws for discrete-time linear systems affected by parametric uncertainty and additive disturbances. The proposed technique is based on the iterated solution, at each step, of a finite-horizon optimal control problem (FHOCP) that takes into account a suitable number of randomly extracted scenarios of uncertainty and disturbances, followed by a specific command selection rule implemented in a receding horizon fashion. The scenario FHOCP is always convex, also when the uncertain parameters and disturbance belong to non-convex sets, and irrespective of how the model uncertainty influences the system's matrices. Moreover, the computational complexity of the proposed approach does not depend on the uncertainty/disturbance dimensions, and scales quadratically with the control horizon. The main result in this paper is related to the analysis of the closed loop system under receding-horizon implementation of the scenario FHOCP, and essentially states that the devised control law guarantees constraint satisfaction at each step with some a-priori assigned probability p, while the system's state reaches the target set either asymptotically, or in finite time with probability at least p. The proposed method may be a valid alternative when other existing techniques, either deterministic or stochastic, are not directly usable due to excessive conservatism or to numerical intractability caused by lack of convexity of the robust or chance-constrained optimization problem.

I. INTRODUCTION

In Model Predictive Control (MPC), at each sampling time t, a plant's control input $u_t \in \mathbb{R}^m$ is computed by solving a constrained finite horizon optimal control problem (FHOCP), according to a receding horizon (RH) strategy, see, e.g., [1]. MPC has received an ever-increasing attention in the last decades, mainly due to the possibility of taking into account input and state constraints explicitly in the control design. The study of robust MPC approaches, able to guarantee stability and constraint satisfaction also in the presence of uncertainty/disturbances, is still a very active research area. For the case of linear time invariant (LTI) discrete time models, an extensive literature has been developed, considering the presence of either model uncertainty or external disturbances, see, e.g., [2]-[12]. Most of the existing approaches are deterministic and aim to optimize a worst-case performance index, while enforcing constraints for all possible outcomes of the uncertainty [2]-[4] or disturbance [5]-[8]. These techniques guarantee that the designed control law is able to cope with the considered uncertainty. However, they rely on the assumption of convexity of the optimization problem, not only with respect to the control input, but also with respect to the uncertain parameters and disturbances. Moreover, the computational complexity of deterministic approaches typically grows with the complexity of the model set. In a recent and active research direction, stochastic MPC techniques have also been studied, see, e.g., [9]-[12] and the references therein. Stochastic MPC techniques exploit some known statistical description of the uncertain parameters and/or of the disturbance (e.g., the probability distribution, or the first and second moments), yet they still employ deterministic algorithms and, in order to maintain tractability of the optimization problem, they typically assume that the system matrices are either perfectly known, or they have a particular structure that preserves convexity.

We propose here a new randomized method for robust MPC design, which is able to deal with both model uncertainty and additive disturbances. Similar to stochastic MPC techniques, we exploit information on the statistics of the uncertain parameters and disturbances. However, we do not assume convexity or even connectedness of the model set or of the disturbance set. Still, the optimization problem in our approach is always convex, and the control law is able to robustly enforce constraints and trajectory convergence, with a probability higher than a user-defined value p. Furthermore, for a given value of p, the computational complexity of our approach is completely independent of the complexity of the model set. The key point enabling to achieve these features is a shift of paradigm, from a *deterministic* algorithm to a *randomized* one, i.e., an algorithm that relies on random choices in the course of its execution (see, e.g., [13]). Indeed, a key step in our main algorithm (Algorithm 4.1) is the solution of a scenario FHOCP, in which we do not consider all possible outcomes of uncertainty and disturbances, but only a finite number M of randomly chosen instances of them, named the "scenarios." A randomized approach for MPC has been studied also in [14], [15], by using a Monte Carlo technique. However, Monte Carlo approaches may be very computationally demanding and can not handle in a straightforward way the presence of state constraints. Randomization has been used also in [16], in the context of chance-constrained MPC. However, in [16] there is no guideline on how to choose M in order to have the guarantee that the probability of success is at least p (which is instead one of the features of the present approach) and, moreover, the resulting optimization problem is a mixed-integer linear program. On the contrary, the approach proposed here, named MPCS (MPC via Scenario optimization), exploits relatively recent results in Random Convex Programming (RCP, see [17]-[20]) to provide an explicit link between M and p. Moreover, we introduce a slack variable, the "constraint violation," which renders the scenario FHOCP always feasible, and that can be used to monitor the extent of the (possible) violation of the involved constraints. Further, we show how scenario optimization can be embedded in a receding horizon scheme, in order to provide a feedback controller that gives probabilistic guarantees of

*Politecnico di Torino, Dipartimento di Automatica e Informatica, Corso Duca degli Abruzzi 24 – 10129 Torino – Italy

**University of California at Santa Barbara, Department of Mechanical Engineering, Engineering II bldg. – 93106-5070 Santa Barbara – California

e–mail addresses: giuseppe.calafiore@polito.it, lorenzo.fagiano@polito.it

† Corresponding author.

robust stability and constraint satisfaction. The approach here proposed shall be particularly interesting in all those cases where the assumptions underpinning the existing deterministic or stochastic approaches for robust MPC are not met; for example, when the dependence of the system matrices on the uncertain parameters is not affine.

II. PROBLEM FORMULATION

Consider the following uncertain, discrete time LTI model:

$$x_{t+1} = A(\theta)x_t + B(\theta)u_t + B_\gamma(\theta)\gamma_t \qquad (1)$$

where $t \in \mathbb{Z}$ is the discrete time variable, $x_t \in \mathbb{R}^n$ is the system state, $u_t \in \mathbb{R}^m$ is the control input, $\gamma_t \in \Gamma \subset \mathbb{R}^{m_\gamma}$ is an unmeasured disturbance vector, $\theta \in \Theta \subseteq \mathbb{R}^g$ is the vector of uncertain parameters, and $A(\theta), B(\theta), B_\gamma(\theta)$ are matrices of suitable dimensions. Let us consider the following assumptions:

Assumption 1: (Uncertainty description) The sets Γ and $\Sigma \doteq \{A(\theta), B(\theta), B_\gamma(\theta) : \theta \in \Theta\}$ are bounded. We assume γ_t and θ to have stochastic nature, and we let $\mathbb{P}_\theta$ denote the probability measure on Θ, and $\mathbb{P}_\gamma$ the probability measure on Γ. Variables θ and γ_t are independent. Moreover, $\gamma \doteq \{\gamma_0, \gamma_1, \ldots\}$ is an independent identically distributed (i.i.d.) sequence and we let $\mathbb{P}_\gamma^\infty$ denote the probability measure on this sequence. ∎

Assumption 2: (Robust stabilizability) The pair $A(\theta), B(\theta)$ is stabilizable for any $\theta \in \Theta$. ∎

The control problem is to regulate the system state to a neighborhood of the origin, subject to (possibly uncertain) input and state constraints $x_t \in \mathbb{X}(\theta)$, $u_t \in \mathbb{U}(\theta)$, $\forall t$. The next assumption characterizes the constraint sets.

Assumption 3: (State and input constraints) For any $\theta \in \Theta$, the sets $\mathbb{X}(\theta) \subset \mathbb{R}^n$ and $\mathbb{U}(\theta) \subset \mathbb{R}^m$ are convex; they contain the origin in their interiors and they are representable by: $\mathbb{X}(\theta) = \{x \in \mathbb{R}^n : f_X(x, \theta) \preceq 0\}$, $\mathbb{U}(\theta) = \{u \in \mathbb{R}^m : f_U(u, \theta) \preceq 0\}$, where $\preceq$ denotes element-wise inequalities, each entry of the functions $f_X : \mathbb{R}^n \times \Theta \to \mathbb{R}^r$, $f_U : \mathbb{R}^m \times \Theta \to \mathbb{R}^q$ is convex in x and u, respectively, and r, q are suitable integers. ∎

The parameter θ has been included in the constraints to account for practical applications where, for example, a convex function of the states (e.g., energy, load) has to be limited below some threshold, and some parameter in the function or the threshold itself depend on uncertain physical quantities (e.g., maximal energy, breaking load). Assumptions 1 and 3 are quite mild, since Θ may be unbounded and of any form, no assumption on Σ, Γ is made except for boundedness, no restrictions on how the parameter θ influences matrices $A(\theta), B(\theta), B_\gamma(\theta)$ are imposed, as long as the system is stabilizable, and finally no assumption on the shape of the convex sets $\mathbb{X}(\theta), \mathbb{U}(\theta)$ (e.g., polytopic, ellipsoidal, ...) for given $\theta \in \Theta$ is made. Mixed constraints of the form $(x, u) \in \mathbb{X}_U(\theta)$, where $\mathbb{X}_U(\theta) \subseteq \mathbb{R}^n \times \mathbb{R}^m$ is a convex set, are not considered here for simplicity, but they can be straightforwardly included in our problem settings. Due to the presence of the generally non-zero unmeasured disturbance γ_t, regulation of the system state to the equilibrium $\bar{x} = 0$, $\bar{u} = 0$ can not be attained. Rather, we can require regulation to a neighborhood of the origin, described by a terminal set, which is robustly positively invariant under a terminal control law.

Assumption 4: (Terminal set and terminal control law) A convex set $\mathbb{X}_f$ and a linear terminal control law $u = K_f x$, $K_f \in \mathbb{R}^{m \times n}$, exist for system (1), such that $\mathbb{X}_f = \{x : f_{X_f}(x) \preceq 0\}$; $A(\theta)x + B(\theta)K_f x + B_\gamma(\theta)\gamma \in \mathbb{X}_f$, $\forall \theta \in \Theta$, $\forall \gamma \in \Gamma$, $\forall x \in \mathbb{X}_f$; finally $f_X(x, \theta) \preceq 0$, $f_U(K_f x, \theta) \preceq 0$, $\forall \theta \in \Theta$, $\forall x \in \mathbb{X}_f$, where $f_{X_f} : \mathbb{R}^n \to \mathbb{R}^l$ has convex components, and l is a suitable integer. ∎

A possible method for constructing a terminal set and a terminal control law satisfying Assumption 4 is to apply results in quadratic stability and rejection of bounded disturbances for uncertain LTI systems, see, e.g., [21], [22] and the references therein. Moreover, there is a number of contributions in the literature concerned with the computation of approximations of the (minimal) robust positively invariant terminal set $\mathbb{X}_f$, see e.g. [23], [24] and the references therein. In the rest of this note, we parameterize the control input as:

$$u_t = K_f x_t + v_t, \qquad (2)$$

where K_f is the terminal control law of Assumption 4 (which is assumed to be known and given), and v_t is a control correction to be designed. Plugging (2) into (1), we obtain the discrete-time model

$$x_{t+1} = A_{\mathrm{cl}}(\theta)x_t + B(\theta)v_t + B_\gamma(\theta)\gamma_t, \qquad (3)$$

with $A_{\mathrm{cl}}(\theta) = A(\theta) + B(\theta)K_f$, which will be the basis of our developments.

III. THE SCENARIO-BASED FINITE-HORIZON OPTIMAL CONTROL PROBLEM

Suppose that, at a given time instant t, the state x_t of system (3) is observed. We consider the problem of determining a corrective control sequence on a horizon of N instants forward in time. To this end, we build a randomized finite-horizon optimal control problem (FHOCP), as described next. Let N be the chosen horizon length, and let $v_{j|t}$, $j = 0, 1, \ldots, N-1$, be the N predicted control corrections to be applied to (3), from t to $t + N - 1$, given the knowledge of the state at time t. From (2), the corresponding predicted control input sequence is $u_{j|t} = K_f x_{j|t} + v_{j|t}$, $j = 0, 1, \ldots, N-1$. By using model (3), we thus obtain the predicted values of the states as linear functions of the current state x_t, of the predicted (to-be-determined) control sequence $\mathcal{V}_t = [v_{0|t}^\top \cdots v_{N-1|t}^\top]^\top \in \mathbb{R}^{Nm}$, and of the disturbance sequence γ:

$$x_{j|t} = A_{\mathrm{cl}}^j(\theta)x_t + \Phi_j(\theta)\mathcal{V}_t + \Upsilon_j(\theta)\gamma, \quad j = 1, \ldots, N, \quad (4)$$

where $\Phi_j(\theta)$, $\Upsilon_j(\theta)$ are suitable functions of the model matrices, $A_{\mathrm{cl}}(\theta)$, $B(\theta)$ and $B_\gamma(\theta)$. However, the predictions obtained via model (3) are *uncertain*, since they depend on θ and on γ. In our approach, we deal with this issue by considering a discrete set of predicted state and input trajectories, obtained for a number M of randomly extracted *scenarios* of θ and γ at time t. More precisely, let us collect these random parameters in $\delta = (\theta, \gamma)$, $\delta \in \Delta \doteq \Theta \times \Gamma^\infty$. As a consequence of Assumption 1, we have that δ has a probability measure that we denote with $\mathbb{P}$, which is the product measure of $\mathbb{P}_\theta$ and the measure $\mathbb{P}_\gamma^\infty$ on γ: $\mathbb{P} = \mathbb{P}_\theta \times \mathbb{P}_\gamma^\infty$. Consider then M independent extractions $\delta_t^{(1)}, \ldots, \delta_t^{(M)}$ of δ, constituting the scenarios, where each scenario has the probability distribution $\mathbb{P}$, and let $\omega_t \doteq (\delta_t^{(1)}, \ldots, \delta_t^{(M)})$ denote the "multisample" of

scenario extractions at time t. The probability distribution on ω_t is given by $\mathbb{P}^M$. Based on the random scenarios, we obtain M different state and input predictions from (4), namely, for $i = 1, \ldots, M$,

$$
\begin{aligned}
x_{0|t}^{(i)} &= x_t \\
x_{j|t}^{(i)} &= A_{\text{cl}}^j(\theta_t^{(i)})x_t + \Phi_j(\theta_t^{(i)})\mathcal{V}_t + \Upsilon_j(\theta_t^{(i)})\gamma_t^{(i)}, \\
&\qquad j = 1, \ldots, N, \\
u_{j|t}^{(i)} &= K_f x_{j|t}^{(i)} + v_{j|t}, \quad j = 0, \ldots, N-1,
\end{aligned} \tag{5}
$$

where $(\theta_t^{(i)}, \gamma_t^{(i)}) = \delta_t^{(i)}$. Let us now introduce the following cost function:

$$
J(x_t, \omega_t; \mathcal{V}_t) \doteq \max_{i=1,\ldots,M} \left(\sum_{j=0}^{N-1} d(x_{j|t}^{(i)}, \mathbb{X}_f) + \sum_{j=0}^{N-1} v_{j|t}^T \Lambda v_{j|t} \right), \tag{6}
$$

where $d(x, \mathbb{X}_f)$ is the distance between x and the terminal set $\mathbb{X}_f$, computed in some norm $\|\cdot\|$: $d(x, \mathbb{X}_f) \doteq \min_{y \in \mathbb{X}_f} \|x - y\|$, and $\Lambda = \Lambda^\top \succ 0$ is a weighting matrix chosen by the control designer. In the following, with a slight abuse of notation, we indicate the state and input constraint sets as $\mathbb{X}(\delta)$, $\mathbb{U}(\delta)$, respectively, and the related convex functions in Assumption 3 as $f_X(x, \delta)$, $f_U(u, \delta)$. Moreover, we transform the hard constraints of Assumption 3 into soft ones, by introducing a slack variable $q_t \in \mathbb{R}$, $q_t \geq 0$. Then, the scenario-based FHOCP is a random convex program defined as follows:

$$
\mathcal{P}(x_t, \omega_t): \min_{\mathcal{V}_t, z_t, q_t} z_t + \alpha q_t \tag{7a}
$$

subject to

$$
J(x_t, \omega_t; \mathcal{V}_t) \leq z_t \tag{7b}
$$

$$
f_X(x_{j|t}^{(i)}, \delta_t^{(i)}) - \mathbf{1}q_t \preceq 0; \; j = 1, \ldots, N-1, \; i = 1, \ldots, M \tag{7c}
$$

$$
f_U(u_{j|t}^{(i)}, \delta_t^{(i)}) - \mathbf{1}q_t \preceq 0; \; j = 0, \ldots, N-1, \; i = 1, \ldots, M \tag{7d}
$$

$$
f_{X_f}(x_{t+N|t}^{(i)}) - \mathbf{1}q_t \preceq 0; \; i = 1, \ldots, M. \tag{7e}
$$

$$
q_t \geq 0 \tag{7f}
$$

In (7a), the weighting scalar $\alpha > 0$ is chosen by the control designer, and $\mathbf{1}$ denotes a column vector of appropriate length, containing all ones. We denote with $\mathcal{V}_t^*(x_t, \omega_t) = \{v_{0|t}^*, \ldots, v_{N-1|t}^*\}$, $z_t^*(x_t, \omega_t)$ and $q_t^*(x_t, \omega_t)$ an optimal solution to problem (7).

Remark 3.1: (Worst-case cost and constraint violation) Due to the presence of constraint (7b), the value z_t^* is an upper bound of the worst case cost with respect to all the M extracted scenarios. We thus refer to z_t^* as the "worst-case cost." Moreover, we note that the use of the soft constraints (7c)-(7e) imply that problem (7) is always feasible. In particular, by using a sufficiently high value of α (e.g. 10^4 times higher than the typical value of z_t^*), the optimal value of q_t^* turns out to be negligible whenever the problem with hard constraints (i.e., with q_t set a priori to zero) is feasible. Contrary, when the problem with hard constraints is not feasible, the variable q_t^* provides an indication on "how much" some of the constraints are violated. For this reason, we refer to q_t^* as the "constraint violation" level. Finally, we note that there is no constraint violation in (7b), i.e. z_t^* is always greater than all the cost

functions corresponding to the sampled scenarios, and in particular it is always an upper bound of the distance between the state x_t and the terminal set $\mathbb{X}_f$ (see (6)). This feature is important for our convergence result in Section IV. ∎

Remark 3.2: (Choice of cost function and input parameterization) Prediction of the state trajectories in a closed loop fashion is quite common in the context of robust MPC, see e.g. [5], [25]. In particular, we adopt here the input parameterization (2), and we optimize over the control corrections $v_{j|t}$, $j = 0, \ldots, N-1$, i.e. over Nm decision variables. Moreover, we chose as stage cost the distance between the state and the terminal set, plus a quadratic penalty on the control correction. Indeed, these choices of control parameterization and cost function are not meant to be the sole possibility, neither the optimal, for the proposed approach. Generalization to other kinds of input parameterization (e.g. disturbance-feedback [8], [25]) and cost function (like a standard quadratic stage cost) can be done with some technical modifications in the proofs of the results reported in this note. ∎

The optimization problem $\mathcal{P}(x_t, \omega_t)$ can be rewritten in a more compact standard form. By collecting the optimization variables $(\mathcal{V}_t, z_t, q_t)$ in vector $s_t \in \mathbb{R}^{mN+2}$, the cost can be expressed as $z_t + \alpha q_t = c^\top s_t$, where $c = [0, \ldots, 0, 1, \alpha]^\top$. Moreover, it can be noted that, for any fixed value of δ_t, due to linearity of (5), the constraints (7b)-(7f) are convex in the decision variable s_t and in the state x_t. Finally, these constraints can be formally expressed compactly as $h(s_t, x_t, \delta_t^{(i)}) \leq 0$, for all $i = i, \ldots, M$, where $h : \mathbb{R}^{mN+2} \times \mathbb{R}^n \times \Delta \to \mathbb{R}$ is defined as $h(s_t, x_t, \delta_t^{(i)}) \doteq \max \left\{ \max_{j=0,\ldots,N-1} \left\{ f_X(x_{j|t}^{(i)}, \delta_t^{(i)}) - \mathbf{1}q_t, f_U(u_{j|t}^{(i)}, \delta_t^{(i)}) - \mathbf{1}q_t \right\}, f_{X_f}(x_{t+N|t}^{(i)}) - \mathbf{1}q_t, -q_t, J(x_t, \omega_t; \mathcal{V}_t) - z_t \right\}$. Notice that $h(s_t, x_t, \delta_t^{(i)})$ is convex in both s_t and x_t, since it is the point-wise maximum of convex functions. The scenario FHOCP can hence be rewritten as

$$
\mathcal{P}(x_t, \omega_t): \quad \min_{s_t} c^\top s_t \tag{8}
$$

subject to: $h(s_t, x_t, \delta_t^{(i)}) \leq 0, \; i = 1, \ldots, M.$

We denote with $s_t^*(x_t, \omega_t) = (\mathcal{V}_t^*, z_t^*, q_t^*)$ an optimal solution of $\mathcal{P}(x_t, \omega_t)$. Notice that, due to the way it has been defined, problem $\mathcal{P}(x_t, \omega_t)$ is *always* feasible. We further assume that this problem always attains a unique optimal solution.

A. Properties of the scenario FHOCP

We now consider the following problem: suppose that, given the state x_t, we solve problem $\mathcal{P}(x_t, \omega_t)$. Then, we ask what is the probability that the computed optimal control sequence $\mathcal{V}_t^*(x_t, \omega_t)$ is able to satisfy all state and input constraints over the chosen horizon, and to drive the state trajectory to the terminal set at the end of the horizon, within the computed optimal constraint violation q_t^*. Formally, this is the probability (with respect to δ) with which $h(s_t^*, x_t, \delta) \leq 0$, where we notice that h is now evaluated at the optimal scenario solution s_t^*, and the state and input trajectories that enter the definition of h are the "actual," uncertain, ones, obtained from model (3) at a random $\delta = (\theta, \gamma)$. So, we define the *reliability* R of the scenario-FHOCP as

$$
R \doteq \mathbb{P}\{\delta : h(s_t^*, x_t, \delta) \leq 0\}.
$$

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication.

Limited circulation. For review only

4

Notice that $R \in [0, 1]$ is itself a random variable, since it depends on s_t^*, which in turn depends on the random multi-extraction of the scenarios ω_t, hence $R = R(\omega_t)$. Indeed, for some extractions ω_t the reliability can be good (close to one), and for other extractions it can be bad. It is therefore critical to assess the a-priori likelihood of these two situations, that is to precisely quantify bounds on the probability of the "bad" event where $\{R < p\}$, being p some a-priori assigned level or desired reliability. To this purpose, we exploit the fact that problem $\mathcal{P}(x_t, \omega_t)$ belongs to the class of so-called Random Convex Programs (RCP) (see, e.g., [17]-[20]) and, in particular, the result in Theorem 1 of [19], concerned with feasible random convex programs, applies to our context. The following key result directly follows from Theorem 1 of [19], see also Theorem 3.3 in [20].

Theorem 3.1: Let $d = mN + 2$ be the number of decision variables in problem $\mathcal{P}(x_t, \omega_t)$, let $p \in (0, 1)$ be a given desired reliability level, let $\beta \in (0, 1)$ be a given small probability level (say, $\beta = 10^{-9}$), and let M be an integer such that

$$\Phi(p, d, M) \leq \beta, \quad (9)$$

with $\Phi(p, d, M) \doteq \sum_{j=0}^{d-1} \binom{M}{j} (1-p)^j p^{M-j}$. Then, it holds that

$$\mathbb{P}^M \{\omega_t : R(\omega_t) \geq p\} \geq 1 - \beta. \quad (10)$$
■

Remark 3.3: (Number of scenarios and "certainty equivalence") The practical importance of the result in Theorem 3.1 stems from the fact that the number M of scenarios necessary to fulfill condition (9) grows mildly with the inverse of β. More precisely, Corollary 5.1 in [20] states that condition (9) is implied by $M \geq \frac{2}{1-p} \left(\ln \beta^{-1} + d \right)$, thus M grows at most logarithmically with β^{-1} (tighter values of M for given β and p can be obtained by inverting numerically (9)). This means in turn that the parameter β may be fixed by the designer to a very low level, say $\beta = 10^{-9}$, or even $\beta = 10^{-12}$, and still the number M of scenarios necessary to guarantee (10) remains manageable. With such small values of β, we may safely say that, to all practical engineering purposes, the event $\{R(\omega_t) \geq p\}$ is the "certain" event. In other words, the possibility that $\{R(\omega_t) \geq p\}$ is not satisfied by the scenario problem is so remote that, before having any concern about it, the designer should better verify the validity of many other assumptions and approximations in the model. We will adopt such a "certainty equivalence" principle in the following, and we will use the expression "with practical certainty" as a synonym of "with probability larger than $1 - \beta$," where $\beta > 0$ is some extremely small value. This simplifies greatly the practical application of scenario techniques, and makes the whole approach more clear and understandable by both theoreticians and control practitioners. ■

The properties of the scenario FHOCP are resumed in the following proposition.

Proposition 3.1: (Finite horizon robustness) Given the state x_t of system (3) at time t, consider the scenario problem $\mathcal{P}(x_t, \omega_t)$ as an instrument to derive a finite-horizon control sequence $\mathcal{V}_t^* = \{v_{0|t}^*, \ldots, v_{N-1|t}^*\}$ to be applied to the system (3) at the subsequent instants $t, t+1, \ldots, t+N-1$. Let the number M of scenarios in problem $\mathcal{P}(x_t, \omega_t)$ be chosen so

to satisfy (9) for given reliability level $p \in (0, 1)$ and very small $\beta \in (0, 1)$. Then, with practical certainty it holds that the computed control sequence:
a) steers the state of system (3) to the terminal set $\mathbb{X}_f$ in N steps with probability at least p and constraint violation q_t^*, i.e.: $\mathbb{P}\{\delta : f_{X_f}(x_{t+N}, \delta) - \mathbf{1}q_t^* \preceq 0\} \geq p$;
b) Satisfies all state constraints with probability at least p and constraint violation q_t^*, i.e.: $\mathbb{P}\{\delta : f_X(x_{t+j}, \delta) - \mathbf{1}q_t^* \preceq 0, \forall j \in [1, N]\} \geq p$
c) Satisfies all input constraints with probability at least p and constraint violation q_t^*, i.e.: $\mathbb{P}\{\delta : f_U(u_{t+j}^*, \delta) - \mathbf{1}q_t^* \preceq 0, \forall j \in [0, N-1]\} \geq p$. ■
The proof of this result follows immediately from Theorem 3.1: eq. (10) states that, with practical certainty, the optimal solution s_t^* of the scenario problem satisfies $h(s_t^*, x_t, \delta) \leq 0$ with probability at least p, which indeed implies that points **a)**-**c)** in the corollary hold.

Remark 3.4: (Relationship with deterministic approaches) In a deterministic approach to robust MPC, a problem similar to (7) has to be solved *for all possible values* of $\delta \in \Delta$. When the problem is convex with respect to δ (which happens, for instance, when the uncertain matrices and/or the additive disturbance belong to polytopes [2], [7]), deterministically robust approaches are indeed well-established and shall be preferred to the scenario approach, especially if deterministic robustness is critical in the considered application. In all other cases, deterministic approaches are generally intractable, unless the problem is manipulated so to satisfy convexity assumptions, at the cost of higher conservativeness and reduced feasibility. In these situations, the scenario approach proposed here is a viable alternative to deterministic techniques, since it is always convex and it can be efficiently solved also with a large number of samples, while still giving probabilistic guarantees on the robustness of the solution, as shown in the example of [26]. ■

The remaining part of this note is devoted to analyzing what happens when a scenario FHOCP is solved repeatedly in time and used to control the plant in a receding-horizon fashion. In a receding-horizon approach, which is the key feature of MPC, only the first control correction in the optimal sequence $\mathcal{V}_t^*$ is applied at time t, and then the FHOCP is solved again at time $t + 1$, by exploiting the knowledge of the state x_{t+1}, etc. In the next section, we propose a technique for incorporating the scenario FHOCP into a suitable receding-horizon scheme, and we derive probabilistic guarantees of asymptotic convergence and constraint satisfaction for the resulting closed-loop system.

IV. MPC SCHEME BASED ON SCENARIO OPTIMIZATION

We here introduce a receding-horizon implementation of a control algorithm based on the scenario FHOCP, as described next. The notation is set as follows: "$*$" variables, such as $z_t^*, q_t^*, \mathcal{V}_t^* = \{v_{0|t}^*, \ldots, v_{N-1|t}^*\}$, denote the optimal solution of the scenario optimization problem $\mathcal{P}(x_t, \omega_t)$ at time t, given x_t; "$\sim$" variables, $\tilde{z}_t, \tilde{q}_t, \tilde{\mathcal{V}}_t$, denote, respectively, two scalar values and a sequence of N vectors of dimension m, as defined in the algorithm below; finally plain variables, $z_t, q_t, \mathcal{V}_t$, denote the running values of the variables z, q and of the sequence $\mathcal{V} = \{v_{0|t}, \ldots, v_{N-1|t}\}$ in the algorithm. The first entry in $\mathcal{V}_t$, namely $v_{0|t}$, is the actual control correction that is applied to the system (3) at time t. The subsequence

composed by the last $N - 1$ elements of $\mathcal{V}_t$ is denoted with $v_{1:N-1|t}$. We are now in position to describe the algorithm for MPC based on Scenario optimization (MPCS).

Algorithm 4.1: (MPCS algorithm)

(Initialization) Choose a desired reliability level $p \in (0, 1)$ and "certainty equivalence" level $\beta \in (0, 1)$ (say, $\beta = 10^{-9}$, or $\beta = 10^{-12}$). Let M be an integer satisfying (9). Choose $\varepsilon \in (0, 1]$ (see Remark 4.1 below for the meaning of ε and for guidelines on its choice). Given an initial state x_0, extract ω_0 according to $\mathbb{P}^M$, solve problem $\mathcal{P}_M(x_0, \omega_0)$ and obtain the optimal control sequence $\mathcal{V}_0^* = \{v_{0|0}^*, v_{1|0}^* \ldots, v_{N-1|0}^*\}$, and the optimal objective z_0^* and constraint violation q_0^*. Set $z_0 = z_0^*, q_0 = q_0^*, \mathcal{V}_0 = \mathcal{V}_0^*$, and apply to the system the control action $u_0 = K_f x_0 + v_{0|0}$.

1) Let $t := t + 1$, observe x_t, and set $\tilde{\mathcal{V}}_t = \{v_{1|t-1}, \ldots, v_{N-1|t-1}, 0\} = \{v_{1:N-1|t-1}, 0\}$, $\tilde{z}_t = \max(0, z_{t-1} - d(x_{t-1}, \mathbb{X}_f))$, $\tilde{q}_t = q_{t-1}$;

2) Extract the multi-sample ω_t according to $\mathbb{P}^M$, and solve problem $\mathcal{P}_M(x_t, \omega_t)$. Let $(\mathcal{V}_t^*, z_t^*, q_t^*)$ be the obtained optimal solution.

3) Evaluate the following collectively exhaustive and mutually exclusive cases:

3.a) If $z_t^* > (z_{t-1} - \varepsilon d(x_{t-1}, \mathbb{X}_f))$ and $\tilde{z}_t < d(x_t, \mathbb{X}_f)$, then set $\mathcal{V}_t = \tilde{\mathcal{V}}_t$; $z_t = 0$; $q_t = \tilde{q}_t$;

3.b) If $z_t^* > (z_{t-1} - \varepsilon d(x_{t-1}, \mathbb{X}_f))$ and $\tilde{z}_t \geq d(x_t, \mathbb{X}_f)$, then set $\mathcal{V}_t = \tilde{\mathcal{V}}_t$; $z_t = \tilde{z}_t$; $q_t = \tilde{q}_t$;

3.c) If $z_t^* \leq (z_{t-1} - \varepsilon d(x_{t-1}, \mathbb{X}_f))$, then set $\mathcal{V}_t = \mathcal{V}_t^*$; $z_t = z_t^*$; $q_t = q_t^*$;

4) Apply the control input $u_t = K_f x_t + v_{0|t}$, then go to 1). ∎

Remark 4.1: The inequality $z_t^* \leq (z_{t-1} - \varepsilon d(x_{t-1}, \mathbb{X}_f))$, checked at step 3) of the MPCS algorithm, can be interpreted as a verification of a required minimum improvement, in terms of worst-case cost, achieved by the newly computed optimal solution $(\mathcal{V}_t^*, z_t^*, q_t^*)$ of the scenario problem at time step t, with respect to the previous step. The user-defined parameter $\varepsilon \in (0, 1]$ influences such a requirement: the closer the value of ε is set to 0, the more likely it is that case $z_t^* \leq (z_{t-1} - \varepsilon d(x_{t-1}, \mathbb{X}_f))$ is met, so that the MPCS algorithm relies, at each time step, on the newly computed optimal solution. Vice-versa, the closer is the value of ε to 1, the more likely it is that the complementary condition $z_t^* > (z_{t-1} - \varepsilon d(x_{t-1}, \mathbb{X}_f))$ is detected, so that the MPCS algorithm employs the previously computed solution. ∎

The next results is concerned with the guaranteed properties, in terms of constraint satisfaction and convergence to the terminal set, of the closed loop system obtained by applying Algorithm 4.1. A numerical example is given in [26].

Theorem 4.1: (Properties of Scenario MPC) Let Assumptions 1-4 be satisfied and let $p \in (0, 1)$ be a chosen reliability level. Let $v_{0|t}$, $t = 0, 1, \ldots$ denote the sequence of control actions produced by Algorithm 4.1, and consider the closed loop system obtained by applying to (1) the control law $u_t = K_f x_t + v_{0|t}$. Then:

(a) With practical certainty, at all time steps $t = 0, 1, \ldots$, the probability that the state and input constraints are satisfied with constraint violation q_t is at least p, that is $\mathbb{P}\{\delta : f_X(x_{t+1}, \delta) - \mathbf{1}q_t \preceq 0 \cap f_U(u_t, \delta) - \mathbf{1}q_t \preceq 0\} \geq p$, $t = 0, 1, \ldots$

(b) Algorithm 4.1 either: **(i)** makes the state trajectory converge asymptotically to the terminal set, i.e. $\lim_{t \to \infty} d(x_t, \mathbb{X}_f) = 0$, or **(ii)** there exists a finite time t^* such that, with practical

certainty, the control sequence $\{v_{0|t^*}, v_{0|t^*+1}, \ldots v_{0|t^*+N-1}\}$ drives the state of the closed-loop system to the terminal set at time $t^* + N - 1$, with probability at least p and constraint violation q_{t^*}. ∎

Proof: **Preliminaries.** Notice first that, for any $t \geq 0$, if $x_t \in \mathbb{X}_f$ then the optimal solution to problem $\mathcal{P}(x_t, \omega_t)$ is $(\mathcal{V}_t^*, z_t^*, q_t^*) = (0, 0, 0)$, since the terminal control law (i.e., with $\mathcal{V}_t = 0$) is able to keep the predicted state trajectory in the terminal set while satisfying all constraints. Also, if $z_t^* = 0$ is the optimal objective of problem $\mathcal{P}(x_t, \omega_t)$, then $x_t \in \mathbb{X}_f$, since z_t^* is an upper bound of $d(x_t, \mathbb{X}_f)$ (see also Remark 3.1), therefore $z_t^* = 0 \iff x_t \in \mathbb{X}_f$. Let then $x_t \notin \mathbb{X}_f$.

Proof of statement (a). At time $t = 0$, Proposition 3.1 guarantees with practical certainty that the first control correction satisfies the constraints on u_0 and x_1 with probability no less than p and constraint violation $q_t = q_t^*$. At any generic time step $t \geq 1$, the variables $(\tilde{\mathcal{V}}_t, \tilde{z}_t, \tilde{q}_t)$ are computed. Then, two cases may occur. If $z_t^* \leq (z_{t-1} - \varepsilon d(x_{t-1}, \mathbb{X}_f))$, then case 3.c) is detected, and the first element $v_{0|t}^*$ of the optimal sequence $\mathcal{V}_t^*$ is applied to the system. Being this sequence the solution of a scenario optimization problem, with practical certainty the probability of satisfying state and input constraints is no less than p, with constraint violation $q_t = q_t^*$. If, on the other hand, $z_t^* > (z_{t-1} - \varepsilon d(x_{t-1}, \mathbb{X}_f))$, then we are either in case 3.a) or 3.b), and in both cases the element $v_{k|t-k}^*$, for some $k \in [1, N-1]$, is applied to the system. Being this value part of the solution sequence $\mathcal{V}_{t-k}^*$, with corresponding constraint violation q_{t-k}^*, again the probability of satisfying state and input constraints is no less than p, with constraint violation $q_t = \tilde{q}_t = q_{t-k}^*$. Thus, in any case, with practical certainty, at each time step the MPCS algorithm guarantees satisfaction of state and input constraints with probability no less than p and constraint violation q_t.

Proof of statement (b). Each run of Algorithm 4.1 may have one of two possible behaviors, depending on whether or not there exists a finite time $t > 0$ such that $z_t^* > (z_{t-1} - \varepsilon d(x_{t-1}, \mathbb{X}_f))$ and $\tilde{z}_t < d(x_t, \mathbb{X}_f)$, that is, whether or not the situation in step 3.a is ever satisfied. We then name $\mathcal{A}$ the situation when condition in step 3.a is met at some finite $t > 0$, and $\bar{\mathcal{A}}$ the complementary situation when this condition is not satisfied at any finite time, that is when $z_t^* \leq (z_{t-1} - \varepsilon d(\xi_{t-1}, \mathbb{X}_f))$ or $\tilde{z}_t \geq d(\xi_t, \mathbb{X}_f)$ holds for all $t > 0$.

I. Let us first consider the situation of case $\bar{\mathcal{A}}$. Consider a generic time t. At step 3) of the MPCS algorithm, if $z_t^* > (z_{t-1} - \varepsilon d(x_{t-1}, \mathbb{X}_f))$, then, since it is assumed that we are in situation $\bar{\mathcal{A}}$, it must hold that $\tilde{z}_t \geq d(x_t, \mathbb{X}_f)$, thus case 3.b) occurs, and the values $\mathcal{V}_t = \tilde{\mathcal{V}}_t$ and $z_t = \tilde{z}_t$ are set. Now, recalling that $\tilde{z}_t = \max(0, z_{t-1} - d(x_{t-1}, \mathbb{X}_f))$, two cases may occur: either $\tilde{z}_t = 0$ or $\tilde{z}_t = z_{t-1} - d(x_{t-1}, \mathbb{X}_f) > 0$. If $\tilde{z}_t = 0$, we have $0 = \tilde{z}_t \geq d(x_t, \mathbb{X}_f)$, i.e. $d(x_t, \mathbb{X}_f) = 0$, which would imply that the terminal set has been reached. Otherwise, if $\tilde{z}_t = z_{t-1} - d(x_{t-1}, \mathbb{X}_f) > 0$, then we have:

$$z_t = \tilde{z}_t \geq d(x_t, \mathbb{X}_f) \geq 0, \tag{11}$$

and $z_t - z_{t-1} = \tilde{z}_t - z_{t-1} = z_{t-1} - d(x_{t-1}, \mathbb{X}_f) - z_{t-1} = -d(x_{t-1}, \mathbb{X}_f)$. Thus,

$$z_t - z_{t-1} \leq -\varepsilon d(x_{t-1}, \mathbb{X}_f), \forall x_{t-1} \notin \mathbb{X}_f, \tag{12}$$

$$\text{and } z_t - z_{t-1} = 0 \iff x_{t-1} \in \mathbb{X}_f. \tag{13}$$

On the other hand, if at step 3) of the MPCS algorithm it happens that $z_t^* \leq (z_{t-1} - \varepsilon d(x_{t-1}, \mathbb{X}_f))$, then case 3.c) occurs, and the optimal values $\mathcal{V}_t^*$ and z_t^* are retained, i.e. $z_t = z_t^*$, $\mathcal{V}_t = \mathcal{V}_t^*$. In this case, it is straightforward to note that equations (11)-(13) still hold true. The same reasoning can be repeated for any time step, as long as the case $z_t^* \leq (z_{t-1} - \varepsilon d(x_{t-1}, \mathbb{X}_f))$ or $\tilde{z}_t \geq d(x_t, \mathbb{X}_f)$ holds true as assumed, so that we can conclude that the variable z_t enjoys the following properties:

$$
\begin{aligned}
&z_t \geq d(x_t, \mathbb{X}_f) \geq 0, \forall t \geq 0 \\
&z_t = 0 \iff x_t \in \mathbb{X}_f \\
&z_{t+1} - z_t \leq -\varepsilon d(x_t, \mathbb{X}_f), \forall x_t \notin \mathbb{X}_f, \forall t \geq 0 \\
&z_{t+1} - z_t = 0 \iff x_t \in \mathbb{X}_f.
\end{aligned}
\tag{14}
$$

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication.

Limited circulation. For review only

6

Properties (14) are sufficient to prove convergence of the state x_t to the set $\mathbb{X}_f$:

$$0 \leq \lim_{t \to \infty} d(x_t, \mathbb{X}_f) \leq \lim_{t \to \infty} z_t = 0, \Rightarrow \lim_{t \to \infty} d(x_t, \mathbb{X}_f) = 0.$$

Therefore, we obtain that in case $\bar{\mathcal{A}}$ the MPCS algorithm guarantees that $\lim_{t \to \infty} d(x_t, \mathbb{X}_f) = 0$.

II. Let us next analyze what happens in case $\mathcal{A}$. Let $\bar{t} > 0$ be the time instant at which the case $z_t^* > (z_{t-1} - \varepsilon d(x_{t-1}, \mathbb{X}_f))$ and $\tilde{z}_t < d(x_t, \mathbb{X}_f)$ is met for the first time, and let $t^* < \bar{t}$ be the last time at which case $z_t^* \leq (z_{t-1} - \varepsilon d(x_{t-1}, \mathbb{X}_f))$ was satisfied, that is the last time previous to $\bar{t}$ when an optimal command sequence was retained, together with its constraint violation q_t^*, according to case 3.c) of Algorithm 4.1; let $\ell = \bar{t} - t^* \geq 1$. According to step 3.a) of the MPCS algorithm, we set

$$\mathcal{V}_{\bar{t}} = \tilde{\mathcal{V}}_{\bar{t}}, \ z_{\bar{t}} = 0, \ q_{\bar{t}} = \tilde{q}_{\bar{t}}. \tag{15}$$

Thus, at step 4) of the algorithm, the control move $u_{\bar{t}} = K_f x_{\bar{t}} + v_{0|\bar{t}}$ is applied to the system at time $\bar{t}$, where $v_{0|\bar{t}} = v_{\ell|t^*}^*$, i.e., $v_{0|\bar{t}}$ is the optimal correction predicted for time $t^* + \ell = \bar{t}$, computed at time t^*. At time step $t = \bar{t} + 1$, the state variable $x_{\bar{t}+1}$ is observed and $(\tilde{\mathcal{V}}_{\bar{t}+1}, \tilde{z}_{\bar{t}+1}, \tilde{q}_{\bar{t}+1})$ are computed as $\tilde{z}_{\bar{t}+1} = \max(0, z_{\bar{t}} - d(x_{\bar{t}}, \mathbb{X}_f))$, $\tilde{q}_{\bar{t}+1} = q_{\bar{t}}$, $\tilde{\mathcal{V}}_{\bar{t}+1} = \{v_{1:N-1|\bar{t}}, 0\} = \{v_{\ell+1|t^*}^*, v_{\ell+2|t^*}^*, \dots, v_{N-1|t^*}^*, 0, \dots, 0\}$. Since (15) holds, it must be $\tilde{z}_{\bar{t}+1} = 0$. Then, $z_{\bar{t}+1}^*$, $q_{\bar{t}+1}^*$ and $\mathcal{V}_{\bar{t}+1}^*$ are computed at step 2), and we notice that, by definition, $z_{\bar{t}+1}^* \geq 0$. Therefore, at step 3) of the algorithm either *(i)* case 3.a) $\{z_{\bar{t}+1}^* > (z_{\bar{t}} - \varepsilon d(x_{\bar{t}}, \mathbb{X}_f))$ and $\tilde{z}_{\bar{t}+1} < d(x_{\bar{t}+1}, \mathbb{X}_f)\}$ is detected again, or *(ii)* one of cases 3.b) or 3.c) are detected, which would imply, respectively, $0 = \tilde{z}_{\bar{t}+1} \geq d(x_{\bar{t}+1}, \mathbb{X}_f)$, or $0 \leq d(x_{\bar{t}+1}, \mathbb{X}_f) \leq z_{\bar{t}+1}^* = \tilde{z}_{\bar{t}+1} = 0$. Hence (in either case) $x_{\bar{t}+1} \in \mathbb{X}_f$, so that convergence to the terminal set would be achieved. Consider then case *(i)*: the values $\mathcal{V}_{\bar{t}+1} = \tilde{\mathcal{V}}_{\bar{t}+1}$, $z_{\bar{t}+1} = 0$ and $q_{\bar{t}+1} = q_{\bar{t}}$ are set in the algorithm, and the control move $u_{\bar{t}+1} = K_f x_{\bar{t}+1} + v_{\ell+1|t^*}^*$ is applied to the system. Now, the same circumstances actually reproduce for all time steps $t = \bar{t} + k$, $k \geq 0$, so the algorithm is such that the optimal input sequence $\mathcal{V}_{t^*}^*$, computed at time t^* by solving a scenario FHOCP, is the one actually next applied to the system, and the related constraint violation q_t^* is retained for all $t \geq t^*$. Thus, in case $\mathcal{A}$, there exists a finite time t^* such that the sequence $\mathcal{V}_{t^*}^*$ is applied to the system for all subsequent instants $t = t^* + k$, $k = 0, \dots, N - 1$. Now, the sequence $\mathcal{V}_{t^*}^*$ is the result of the solution of the scenario-FHOCP $\mathcal{P}(x_{t^*}, \omega_{t^*})$, and Proposition 3.1 states that, with practical certainty, we have $R(\omega_{t^*}) \geq p$, where R is the reliability defined in Section III-A of the paper, which means that $\mathbb{P}\{\delta : h(s_{t^*}^*, x_{t^*}, \delta) \leq 0\} \geq p$. Therefore, in the situation $\mathcal{A}$, there exists a finite time t^* at which an optimal control sequence is computed by solving a scenario-FHOPC and next applied to the actual system for the subsequent N time instants: we can hence claim with practical certainty this sequence will satisfy the problem constraints and reach the terminal set within the time window from t^* to $t^* + N$, with probability at least p and constraint violation q_t^*. ∎

REFERENCES

[1] D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. Scokaert, "Constrained model predictive control: stability and optimality," *Automatica*, vol. 36, pp. 789–814, 2000.

[2] M. Kothare, V. Balakhrisna, and M. Morari, "Robust constrained model predictive control using linear matrix inequalities," *Automatica*, vol. 32, no. 10, pp. 1361–2378, 1996.

[3] Z. Wan and M. Kothare, "An efficient off-line formulation of robust model predictive control using linear matrix inequalities," *Automatica*, vol. 39, pp. 837–846, 2003.

[4] D. Muñoz de la Peña, A. Bemporad, and C. Filippi, "Robust explicit MPC based on approximate multiparametric convex programming," *IEEE Transactions on Automatic Control*, vol. 51, no. 8, pp. 1399–1403, August 2006.

[5] P. Scokaert and D. Mayne, "Min-max feedback model predictive control for constrained linear systems," *IEEE Transactions on Automatic Control*, vol. 43, no. 8, pp. 1136–1142, 1998.

[6] L. Chisci, J. A. Rossiter, and G. Zappa, "Systems with persistent disturbances: predictive control with restricted constraints," *Automatica*, vol. 37, pp. 1019–1028, 2004.

[7] W. Langson, I. Chryssochoos, S. Raković, and D. Mayne, "Robust model predictive control using tubes," *Automatica*, vol. 40, pp. 125–133, 2004.

[8] P. Goulart, E. Kerrigan, and J. Maciejowski, "Optimization over state feedback policies for robust control with constraints," *Automatica*, vol. 42, pp. 523–533, 2006.

[9] D. Bernardini and A. Bemporad, "Scenario-based model predictive control of stochastic constrained linear systems," in *Joint 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference*, Shanghai, P.R. China, 2009, pp. 6333–6338.

[10] J. Primbs and C. Sung, "Stochastic receding horizon control of constrained linear systems with state and control multiplicative noise," *IEEE Transactions on Automatic Control*, vol. 54, no. 2, pp. 221–230, 2009.

[11] M. Cannon, B. Kouvaritakis, S. Raković, and Q. Cheng, "Stochastic tubes in model predictive control with probabilistic constraints," *IEEE Transactions on Automatic Control*, vol. 56, no. 1, pp. 194–200, 2011.

[12] E. Cinquemani, M. Agarwal, D. Chatterjee, and J. Lygeros, "On convex problems in chance-constrained stochastic model predictive control," *Automatica*, vol. 47, no. 9, pp. 2082–2087, 2011.

[13] R. Tempo, G. Calafiore, and F. Dabbene, *Randomized Algorithms for Analysis and Control of Uncertain Systems.* Springer, 2005.

[14] I. Batina, "Model predictive control for stochastic systems by randomized algorithms," Ph.D. dissertation, Technische Universiteit Eindhoven, 2004.

[15] N. Kantas, J. Maciejowski, and A. Lecchini-Visintini, "Sequential Monte Carlo for model predictive control," in *Nonlinear Model Predictive Control*, ser. Lecture Notes in Control and Information Sciences, L. Magni, D. Raimondo, and F. Allgöwer, Eds. Springer Berlin / Heidelberg, 2009, vol. 384, pp. 263–273.

[16] L. Blackmore, M. Ono, A. Bektassov, and B. C. Williams, "A probabilistic particle-control approximation of chance-constrained stochastic predictive control," *IEEE Transactions on Robotics*, vol. 26, pp. 502–517, 2010.

[17] G. Calafiore and M. Campi, "Uncertain convex programs: Randomized solutions and confidence levels," *Mathematical Programming*, vol. 102, no. 1, pp. 25–46, 2005.

[18] ——, "The scenario approach to robust control design," *IEEE Transactions on Automatic Control*, vol. 51, pp. 742–753, 2006.

[19] M. Campi and S. Garatti, "The exact feasibility of randomized solutions of uncertain convex programs," *SIAM Journal on Optimization*, vol. 19, no. 3, pp. 1211–1230, 2008.

[20] G. Calafiore, "Random convex programs," *SIAM Journal on Optimization*, vol. 20, pp. 3427–3464, 2010.

[21] W. Mao, "Robust stabilization of uncertain time-varying discrete systems and comments on "an improved approach for constrained robust model predictive control," *Automatica*, vol. 39, pp. 1109–1112, 2003.

[22] B. Polyak, A. Nazin, M. Topunov, and S. Nazin, "Rejection of bounded disturbances via invariant ellipsoids technique," in *Proceedings of the 45th IEEE Conference on Decision and Control*, San Diego, CA, 2006, pp. 1429 – 1434.

[23] F. Blanchini and S. Miani, "Set-Theoretic Methods in Control," *Series: Systems & Control: Foundations & Applications*, Springer, 2008

[24] S. V. Raković and M. Barić, "Parameterized robust control invariant sets for linear systems: Theoretical advances and computational remarks," *IEEE Transactions on Automatic Control*, vol. 55, no. 7, pp. 1599–1614, 2010.

[25] F. Oldewurtel, R. Gondhalekar, C. N. Jones, and M. Morari, "Blocking parameterizations for improving the computational tractability of affine disturbance feedback MPC problems," in *Proceedings of the 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference.*, Shanghai, China, 2009, pp. 7381–7386.

[26] G. Calafiore and L. Fagiano, "Robust model predictive control via scenario optimization - numerical example," Politecnico di Torino / UC Santa Barbara, Tech. Rep., January 2011, technical report # TR_CaFa_19032012. Available: http://lorenzofagiano.altervista.org/docs/CaFa_TR_19032012.pdf.