

## 2. Examples and Review

- Linear equations
- Application examples
- Control interpretation
- Example: forces on rigid body
- Estimation Interpretation
- Example: navigation
- Block matrices
- Block diagrams

# Linear equations

some familiar equations:

$$y_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n$$

$$y_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n$$

$\vdots$

$$y_m = a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n$$

write this as  $y = Ax$ , where

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

this defines a map from  $\mathbb{R}^n$  to  $\mathbb{R}^m$ ; this map is *linear*; that is

$$A(x + y) = Ax + Ay$$

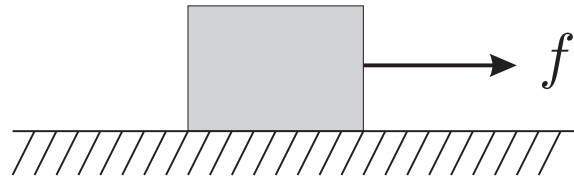
$$A(\lambda x) = \lambda Ax$$

for any  $x, y \in \mathbb{R}^n$  and any  $\lambda \in \mathbb{R}$ .

# Engineering Examples

## final position/velocity of mass from applied forces

- unit mass, with zero position/velocity at  $t = 0$ , subject to force  $f(t)$  for  $0 \leq t \leq n$
- $f(t) = x_j$  for  $t$  in the interval  $[j - 1, j)$ .  
( $x$  is the sequence of applied forces, constant in each interval)
- $y_1, y_2$  are final position and velocity (i.e. at  $t = n$ )

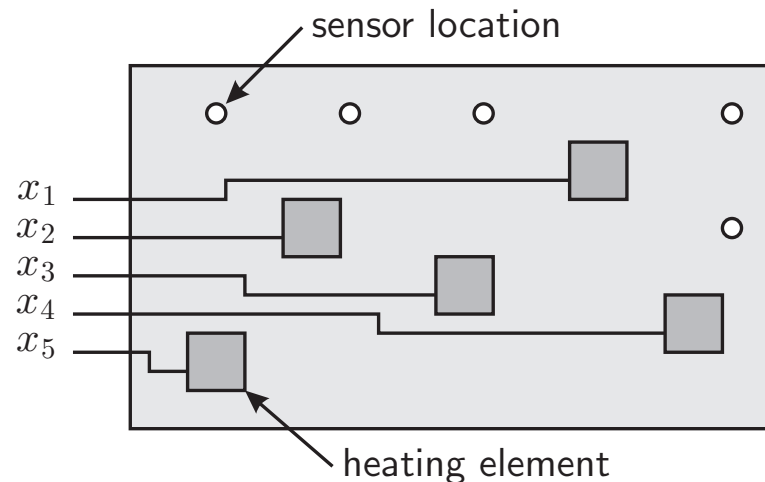


we have  $y = Ax$

- $a_{1j}$  gives influence of applied force during  $j - 1 \leq t < j$  on final position
- $a_{2j}$  gives influence of applied force during  $j - 1 \leq t < j$  on final velocity

## heating system with multiple heating elements

- $x_j$  is power of  $j$ th heating element
- $y_i$  is change in steady-state temperature at location  $i$
- thermal transport via conduction

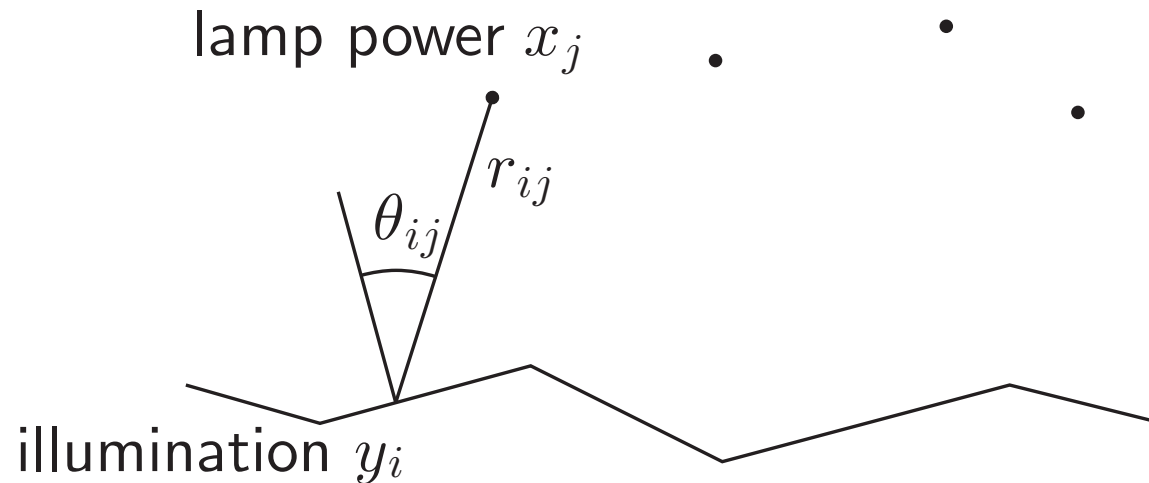


we have  $y = Ax$

- $a_{ij}$  gives influence of heater  $j$  at location  $i$  (in  $^{\circ}/W$ )
- $j$ th col. of  $A$  gives pattern of steady-state temperature rise due to  $1W$  at heater  $j$
- $i$ th row shows how heaters affect location  $i$

## illumination with multiple lamps

- $n$  lamps illuminating  $m$  (small, flat) patches, no shadows
- $x_j$  is power of  $j$ th lamp
- $y_i$  is illumination level of patch  $i$



$y = Ax$ , where

- $a_{ij} = r_{ij}^{-2} \cos \theta_{ij}$
- $j$ th column of  $A$  shows illumination pattern resulting from lamp  $j$  (at 1W)

# signal and interference power in wireless system

$n$  transmitter/receiver pairs

- transmitter  $j$  transmits to receiver  $j$  (and, inadvertently, to the other receivers)
- $p_j$  is power of  $j$ th transmitter
- $s_i$  is receiver signal power of  $i$ th receiver
- $z_i$  is receiver interference power of  $i$ th receiver
- $G_{ij}$  is path gain from transmitter  $j$  to receiver  $i$

we have  $s = Ap$  and  $z = Bp$  where

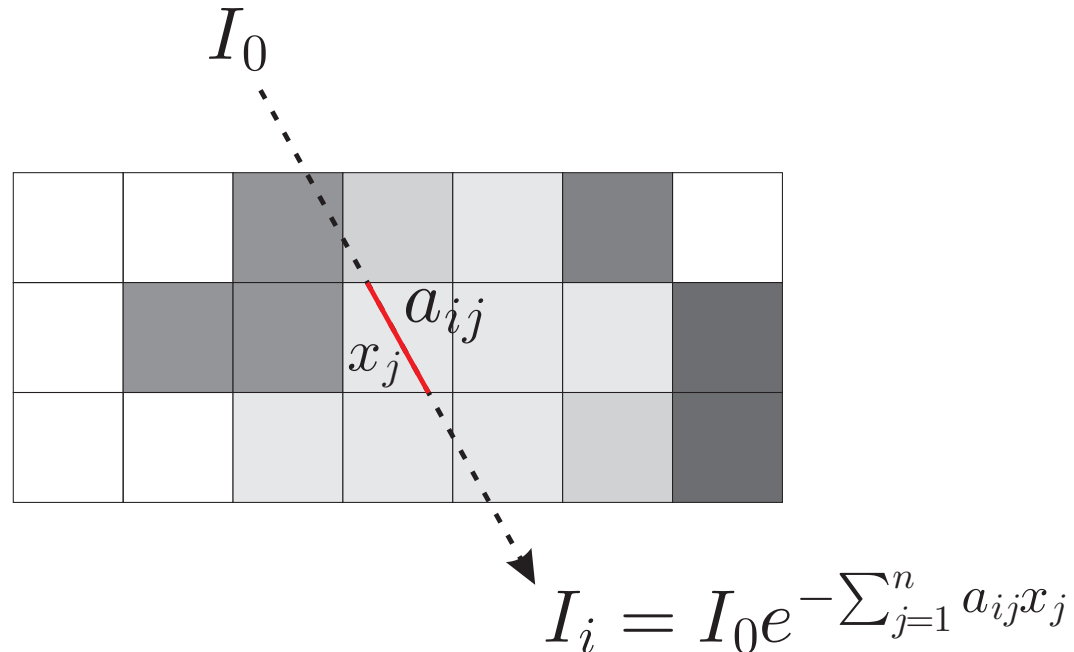
$$a_{ij} = \begin{cases} G_{ii} & \text{for } i = j \\ 0 & \text{otherwise} \end{cases} \quad b_{ij} = \begin{cases} 0 & \text{for } i = j \\ G_{ij} & \text{otherwise} \end{cases}$$

- $A$  is diagonal;  $B$  has zero diagonal
- we'd like  $A$  'large',  $B$  'small'

## cross-section image reconstruction

tomography or CAT scan, atmospheric/oceanographic remote sensing

- $m$  X-ray beams with intensity  $I_0$
- object is divided into  $n$  volume cells ('voxels')
- $x_j$  is density of cell  $j$ ,  $j = 1, \dots, n$
- $y_i = \log(I_0/I_i)$  where  $I_i$  is the measured intensity

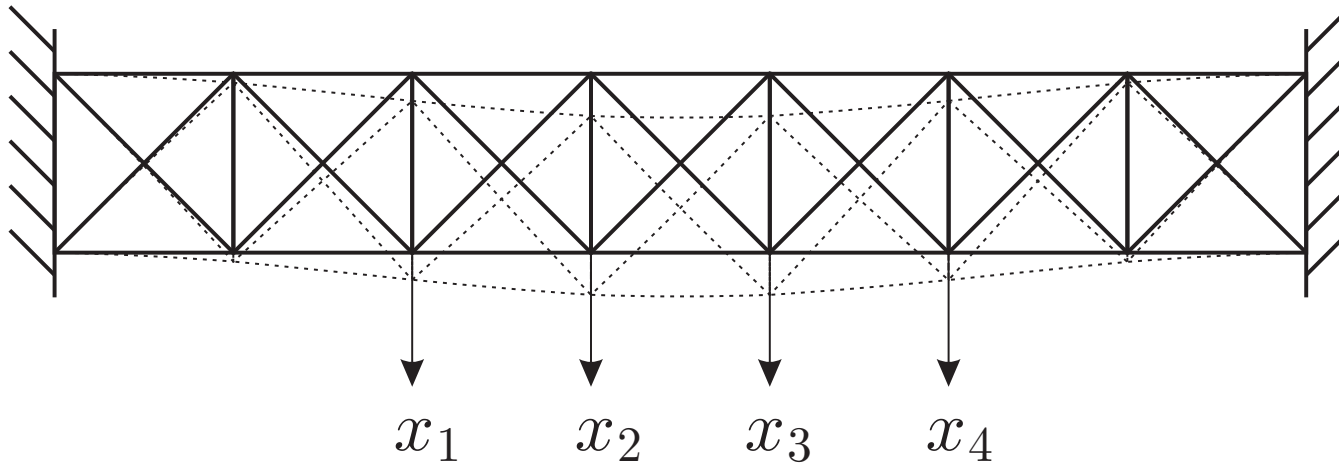


$y = Ax$  where

- $a_{ij}$  is length of path of beam  $j$  through cell  $i$

## linear elastic structure

- $x_j$  is external force applied at some node, in some direction
- $y_i$  is (small) deflection of some node, in some fixed direction



for small displacements, we have  $y \approx Ax$

- The matrix  $A$  is called the *compliance matrix*
- $a_{ij}$  gives deflection  $i$  per unit force  $j$  (in  $m/N$ )

# Cost of production

Production *inputs* (materials, parts, ...) are combined to make a number of *products*

- $x_j$  is price per unit of production input  $j$
- $a_{ij}$  is units of production input  $j$  required to manufacture one unit of product  $i$
- $y_i$  is production cost per unit of product  $i$
- we have  $y = Ax$
- $i$ th row of  $A$  is *bill of materials* for unit of product  $i$

## production inputs needed

- $q_i$  is quantity of product  $i$  to be produced
- $r_j$  is total quantity of production input  $j$  needed
- we have  $r = A^T q$

total production cost is  $r^T x = (A^T q)^T x = q^T Ax$

# Network traffic and flows

- $n$  flows with rates  $f_1, \dots, f_n$  pass from their source nodes to their destination nodes over fixed routes in a network
- $t_i$ , traffic on link  $i$ , is sum of rates of flows passing through it
- flow routes given by *flow-link incidence matrix*

$$A_{ij} = \begin{cases} 1 & \text{flow } j \text{ goes over link } i \\ 0 & \text{otherwise} \end{cases}$$

- traffic and flow rates related by  $t = Af$

## link delays and flow latency

- let  $d_1, \dots, d_m$  be link delays, and  $l_1, \dots, l_n$  be latency (total travel time) of flows
- $l = A^T d$
- $f^T l = f^T A^T d = (Af)^T d = t^T d =$  total no. of packets in network

# Control Interpretation of Linear Equations

we have the equation

$$y = Ax$$

- $x$  is a vector of inputs or design parameters we choose
- $y$  is the vector of results or outcomes
- $a_{ij}$  is the sensitivity of the  $i$ th outcome to the  $j$ th parameter

## sample problems

- find  $x$  so that  $y = y_{\text{des}}$
- find all  $x$ 's that result in  $y = y_{\text{des}}$   
(i.e., find all designs that meet the specifications)
- among all  $x$ 's that satisfy  $y = y_{\text{des}}$ , find a small one  
(i.e., find a small or efficient  $x$  that meets specifications)

## control interpretation via columns

write  $A$  in terms of its columns

$$A = [a_1 \ a_2 \ \dots \ a_n]$$

where  $a_i \in \mathbb{R}^m$  for all  $i$ . Then  $y = Ax$  means

$$\begin{aligned} y &= x_1 a_1 + x_2 a_2 + \dots + x_n a_n \\ &= \sum_{j=1}^n x_j a_j \end{aligned}$$

here each  $a_j$  is a vector

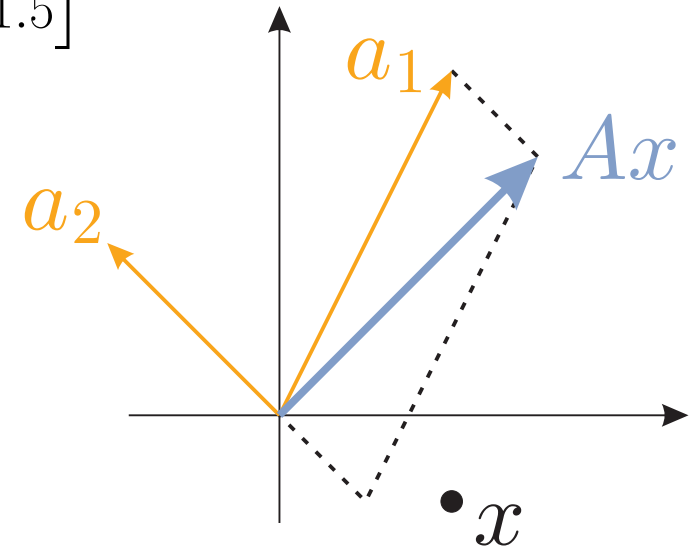
- usually we think of the matrix  $A$  as acting on  $x$  to produce  $y$
- for control, it makes sense to think of  $x$  as acting on  $A$  to produce  $y$

each column of  $A$  represents an *actuator*

## geometric interpretation of control

example:  $A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$ ,  $x = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$ ,  $y = \begin{bmatrix} 1.5 \\ 1.5 \end{bmatrix}$

$$\begin{aligned} Ax &= a_1 + (-0.5)a_2 \\ &= \begin{bmatrix} 1.5 \\ 1.5 \end{bmatrix} \end{aligned}$$



another example:

$$a_j = Ae_j$$

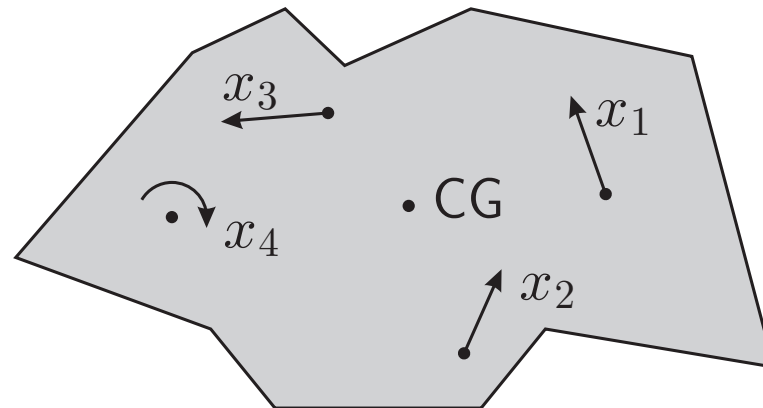
where  $e_j$  is the  $j$ th unit vector:

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, \quad e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ n \end{bmatrix}$$

- $j$ th column of  $A$  gives response to unit  $j$ th input

## application example: total force/torque on rigid body

- $x_j$  is external force/torque applied at some point/direction/axis
- $y$  is resulting total force and torque on body. six real numbers:
  - $y_1, y_2, y_3$  are  $x, y, z$  components of total force
  - $y_4, y_5, y_6$  are  $x, y, z$  components of total torque



we have  $y = Ax$

- $A$  depends on geometry (of applied forces and torques with respect to center of gravity CG)
- $j$ th column gives resulting force and torque for unit force/torque  $j$

# Estimation Interpretation of Linear Equations

we also use linear equations to describe *estimation problems*; again we have the equation

$$y = Ax$$

- $y_i$  is the  $i$ th measurement or sensor reading
- $x_j$  is the  $j$ th parameter to be estimated or determined
- $a_{ij}$  is the sensitivity of the  $i$ th sensor to the  $j$ th parameter

## sample problems

- given  $y_{\text{meas}}$ , find  $x$
- find all  $x$  that result in  $y_{\text{meas}}$   
(i.e., all  $x$  *consistent* with measurements)
- if there is no  $x$  such that  $y_{\text{meas}} = Ax$ , find  $x$  such that  $y_{\text{meas}} \approx Ax$   
(i.e., if the sensor readings are inconsistent, find  $x$  which is almost consistent)

## estimation interpretation via rows

write  $A$  in terms of its rows

$$A = \begin{bmatrix} b_1^T \\ b_2^T \\ \vdots \\ b_m^T \end{bmatrix}$$

then

$$y = \begin{bmatrix} b_1^T x \\ b_2^T x \\ \vdots \\ b_m^T x \end{bmatrix}$$

where  $b_i \in \mathbb{R}^n$ , so that

- $y_i$  is the scalar product of  $b_i$  with  $x$
- if  $b_i$  is a unit vector, then  $y_i$  is the *component* of  $x$  in the direction  $b_i$
- think of  $A$  as acting on  $x$  to produce  $y$

in particular,

each row of  $A$  represents a *sensor*

# geometric interpretation of estimation

$$b_i^T x = \text{constant}$$

is a (hyper-)plane in  $\mathbb{R}^n$  normal to  $b_i$ .

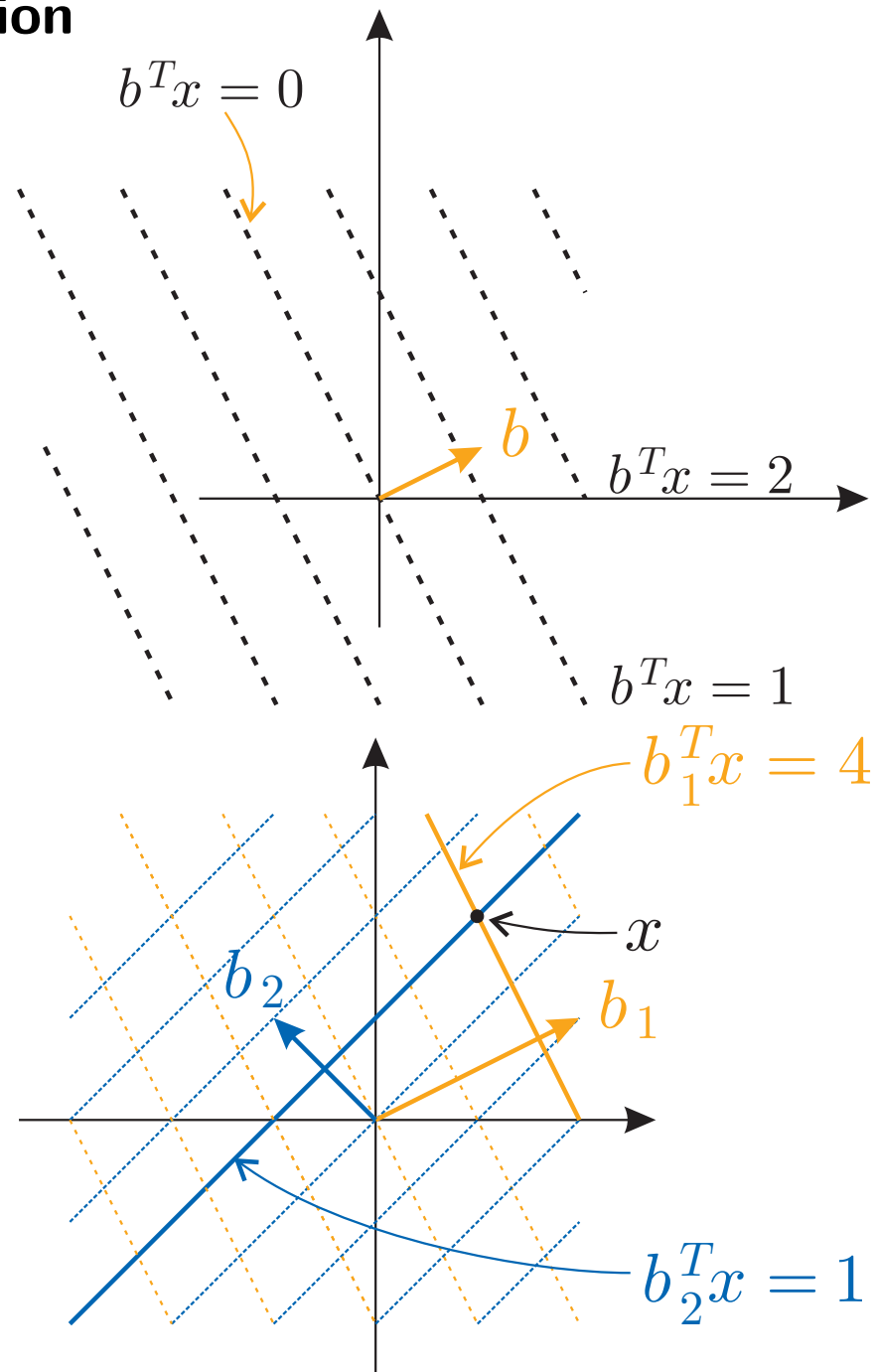
if  $Ax = y$  then  $x$  is on intersection of hyperplanes  $b_i^T x = y_i$

example:

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$$

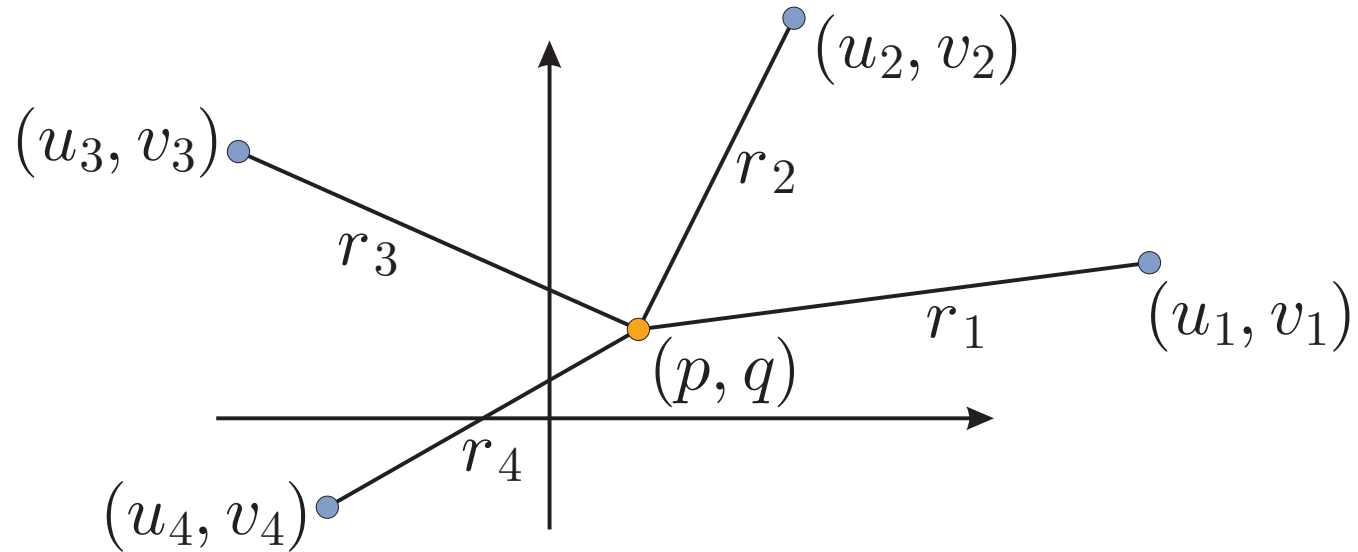
$$x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$y = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$



## example: navigation

$(p, q) \in \mathbb{R}^2$  is our location, and we measure distances  $r_i$  to  $m$  beacons at points  $(u_i, v_i)$



$$r_i = \sqrt{(p - u_i)^2 + (q - v_i)^2} = f_i(p, q)$$

## navigation example continued

Taylor expansion is

$$\begin{aligned} r_i &= f_i(0, 0) + \begin{bmatrix} \frac{\partial f_i(0,0)}{\partial p} \\ \frac{\partial f_i(0,0)}{\partial q} \end{bmatrix}^T \begin{bmatrix} p \\ q \end{bmatrix} \\ &= \sqrt{u_i^2 + v_i^2} - \frac{1}{\sqrt{u_i^2 + v_i^2}} \begin{bmatrix} u_i \\ v_i \end{bmatrix}^T \begin{bmatrix} p \\ q \end{bmatrix} \end{aligned}$$

assume  $p, q$  are small compared to  $u_i, v_i$ . then  $y \approx Ax$  where

- $A \in \mathbb{R}^{m \times 2}$ ,  $i$ th row of  $A$  is the transpose of unit vector in the direction of beacon  $i$

- $y = \begin{bmatrix} \sqrt{u_1^2 + v_1^2} - r_1 \\ \vdots \\ \sqrt{u_m^2 + v_m^2} - r_m \end{bmatrix}$  measured vector of distances

- $x = \begin{bmatrix} p \\ q \end{bmatrix}$  our location

# Block Matrices and Vectors

if  $P, Q, R, S$  are

$$P = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \quad Q = \begin{bmatrix} 7 \\ 1 \end{bmatrix} \quad R = [3 \ 3] \quad S = 7$$

then

$$A = \begin{bmatrix} P & Q \\ R & S \end{bmatrix} \quad \text{means} \quad A = \begin{bmatrix} 1 & 2 & 7 \\ 3 & 1 & 1 \\ 3 & 3 & 7 \end{bmatrix}$$

$A$  is called a *partitioned matrix* or a *block matrix*.

For this to make sense, we need the dimensions of  $P, Q, R, S$  to be *compatible*

for vectors:

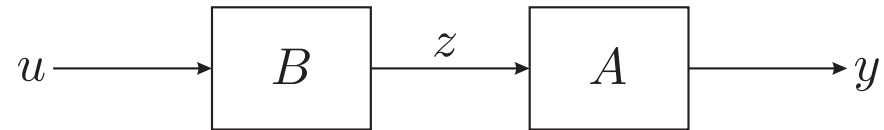
$$a = \begin{bmatrix} 5 \\ 1 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} \quad \Longrightarrow \quad \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 3 \\ 6 \\ 2 \end{bmatrix}$$

# Block Diagrams

we can also represent  $y = Ax$  by the block diagram:



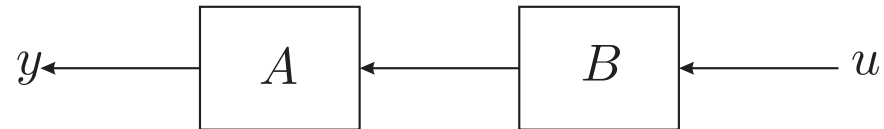
if  $y = ABu$  then



so  $z = Bu$ ,  $y = Az$  hence  $y = ABu$ .

Note the order of the blocks. In general  $AB \neq BA$ .

so we often draw block diagrams right-to-left:



# Matrix multiplication

Suppose  $C = AB$  then

- **Column interpretation:**  $i$ th column of  $C$  is  $A$  acting on  $i$ th column of  $B$

$$C = [c_1 \cdots c_p] = AB = [Ab_1 \cdots Ab_p]$$

- **Row interpretation:** similarly  $i$ th row of  $C$  is  $i$ th row of  $A$  acting (on left) on  $B$

$$C = \begin{bmatrix} \tilde{c}_1^T \\ \vdots \\ \tilde{c}_m^T \end{bmatrix} = AB = \begin{bmatrix} \tilde{a}_1^T B \\ \vdots \\ \tilde{a}_m^T B \end{bmatrix}$$

- **Inner product interpretation:** entries of  $C$  are inner products of rows of  $A$  and columns of  $B$

$$c_{ij} = \tilde{a}_i^T b_j$$

# block diagrams and block matrices

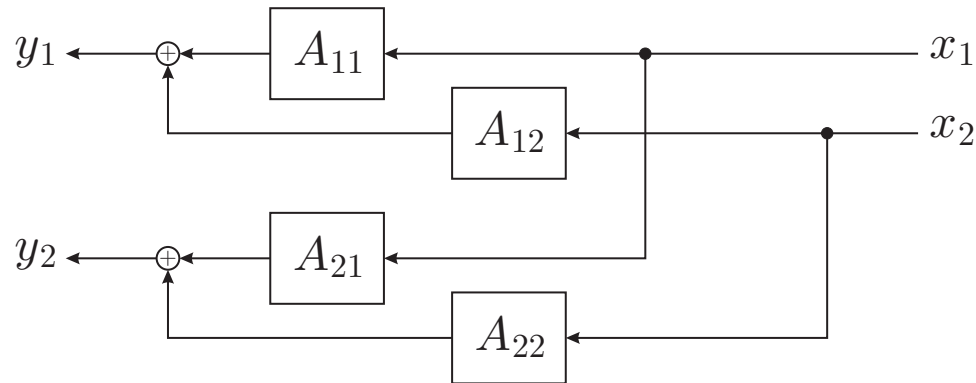
suppose

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where  $A_{11} \in \mathbb{R}^{m_1 \times n_1}$ ,  $A_{12} \in \mathbb{R}^{m_1 \times n_2}$ ,  $A_{21} \in \mathbb{R}^{m_2 \times n_1}$ ,  $A_{22} \in \mathbb{R}^{m_2 \times n_2}$ . If

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

we draw this as:



convenient shorthand:

