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G. Kaniadakis

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Maximum entropy principle and power-law tailed distributions

G. Kaniadakis[a]

Dipartimento di Fisica, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy

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Abstract. In ordinary statistical mechanics the Boltzmann-Shannon entropy is related to the Maxwell-Bolzmann distribution p_i by means of a twofold link. The first link is differential and is offered by the Jaynes Maximum Entropy Principle. Indeed, the Maxwell-Boltzmann distribution is obtained by maximizing the Boltzmann-Shannon entropy under proper constraints. The second link is algebraic and imposes that both the entropy and the distribution must be expressed in terms of the same function in direct and inverse form. Indeed, the Maxwell-Boltzmann distribution p_i is expressed in terms of the exponential function, while the Boltzmann-Shannon entropy is defined as the mean value of $-\ln(p_i)$. In generalized statistical mechanics the second link is customarily relaxed. Of course, the generalized exponential function defining the probability distribution function after inversion, produces a generalized logarithm $\Lambda(p_i)$. But, in general, the mean value of $-\Lambda(p_i)$ is not the entropy of the system. Here we reconsider the question first posed in [Phys. Rev. E **66**, 056125 (2002) and **72**, 036108 (2005)], if and how is it possible to select generalized statistical theories in which the above mentioned twofold link between entropy and the distribution function continues to hold, such as in the case of ordinary statistical mechanics. Within this scenario, apart from the standard logarithmic-exponential functions that define ordinary statistical mechanics, there emerge other new couples of direct-inverse functions, i.e. generalized logarithms $\Lambda(x)$ and generalized exponentials $\Lambda^{-1}(x)$, defining coherent and self-consistent generalized statistical theories. Interestingly, all these theories preserve the main features of ordinary statistical mechanics, and predict distribution functions presenting power-law tails. Furthermore, the obtained generalized entropies are both thermodynamically and Lesche stable.

PACS. 05.20.-y Classical statistical mechanics – 51.10.+y Kinetic and transport theory of gases – 03.30.+p Special relativity – 05.90.+m Other topics in statistical physics, thermodynamics, and nonlinear dynamical systems

1 Introduction

Apart from the exponential distribution $\exp(-x)$, several distribution functions have been considered both from the theoretical point of view as well as to analyze experimental data. Reference [1] lists more than 130 different distributions functions, more or less frequently used in statistical sciences.

In the last decades particular attention has been devoted to distribution functions exhibiting power-law tails, namely $A\,x^{-a}$ for $x \to \infty$, appearing in the phenomenology of many physical, natural and artificial systems. The power-law tailed distributions have been observed, for instance, in high energy physics (plasmas [2], cosmic rays [3,4] and particle production processes [5,6]), in condensed matter physics (anomalous diffusion [7–9], fluid motion [10], turbulence [11,12]), in natural sciences (seismology [13], meteorology [14] and heliophysics [15]), in biology (botany [16], genomics [17]), in economics (stock prices [18,19] and personal annual income [20]), in sociodemography (population of cities [21], frequencies of family names [22], telephone calls [23]), in linguistics [24,25], in philology (classical mythology [26]), and also in fields involving human activity in general (intensity of wars [27], terrorism attacks [28], citation of scientific papers [29], traffic in complex networks [30], etc.).

The very simple analytic expression for the tails i.e. $A\,x^{-a}$ in the above distributions, known as Pareto's or Zipf's law, has been regularly adopted in the literature for data analysis [31–33]. On the contrary, the question regarding the form of the analytic expression for the power-law tailed distribution that holds for $0 < x < +\infty$ remains, at the moment, an open problem.

In the last few years there have been studies of the general properties of distribution functions of arbitrary forms [34–38]. On the other hand, the general properties of an arbitrary entropic form are also well known [39–41]. Finally the connection between an arbitrary entropy and the relevant distribution given through the Jaynes maximum entropy principle [42], has been considered by several authors [36,37].

[a] e-mail: giorgio.kaniadakis@polito.it

A mechanism frequently used to explain the occurrence of statistical distributions is based on evolution equations, essentially nonlinear Fokker-Plank but also Boltzmann equations [43–54]. After fixing the form of the evolution equation the entropy of the system is automatically fixed and the probability distribution function is obtained as a stationary state of the evolution equation.

Clearly the correctness of an analytic expression for a given power-law tailed distribution, used to describe a statistical system, is strongly related to the validity of the generating mechanism. In the present paper we will consider power-law tailed distribution functions from a general prospective and independently on the particular generating mechanisms.

We recall that, according to Maximum Entropy Principle, the Boltzmann-Shannon entropy $S = -\sum p_i \ln(p_i)$, yields the exponential Maxwell-Boltzmann distribution. In generalized statistical mechanics, a special role is played by the trace-form entropies

$$S = -\langle \Lambda \rangle = -\sum_i p_i \, \Lambda(p_i), \tag{1}$$

where $\Lambda(x)$ is an arbitrary, strictly increasing function defined for $x > 0$, which for $x \to 0^+$ becomes $-\infty$, while $p = \{p_i \,, 1 \leq i \leq N\}$ is a discrete probability distribution. The function $\Lambda(x)$ can be viewed as a generalization of the ordinary logarithmic function. We remark that the entropy is the ordinary mean value of $-\Lambda(p_i)$. It is clear that the form of $\Lambda(x)$ imposes the form of the distribution function according to the maximum entropy principle.

In proposing generalized statistical theories some standard properties (positivity, continuity, concavity, symmetry, etc.) are customarily required for the entropy. Unfortunately, these properties are not sufficiently strong to impose the form of the entropy; then, the metaphor of ordinary statistical mechanics remains the unique guiding principle for further generalization. In this sense the maximum entropy principle, the cornerstone of statistical physics, is a valid and powerful tool to explore new roots in searching for generalized statistical theories.

The main goal of the present paper is to show that the maximum entropy principle, by itself, suggests a generalization of ordinary statistical mechanics. The obtained generalized statistical theories preserve the main features of ordinary statistical mechanics and predict distribution functions, showing power-law tails, some of which are already known in the literature.

This paper is organized as follows: in Section 2 we consider the maximum entropy principle within the generalized statistical theories, based on trace form entropies. In Section 3 we obtain a class of entropies depending on three real parameters. In Section 4 we consider the statistical distributions, related to this class of three-parameter entropies, exhibiting power-law tails. In Section 5 and Section 6 we discuss the thermodynamic stability and the Lesche stability of the generalized statistical theories, based on the three-parameter entropies obtained in Section 3. In Section 7 we show that the class of the three-parameter entropies contains, as special cases, some already known in the literature, entropies. In Section 8 we study a new, two-parameter subclass of entropies, belonging to the three-parameter class of entropies. In Section 9 we make some concluding remarks. Finally in the Appendix we solve a differential-functional equation which permits us to obtain the three-parameter class of entropies.

2 The maximum entropy principle

Let us consider the generalized entropy defined in equation (1). We note that the particular probability distribution given by $p = \{\delta_{ia} \,, 1 \leq i \leq N\}$ where a is a fixed integer with $1 \leq a \leq N$, describes a state of the system for which we have the maximum information. For this state we set $S = 0$ and this condition imposes for the generalized logarithm (i) $\Lambda(1) = 0$ and (ii) $0^+ \Lambda(0^+) = 0$. Furthermore, in analogy with the ordinary logarithm we normalize the generalized logarithm through (iii) $\Lambda'(1) = 1$.

We introduce the constraints functional

$$C = \mathrm{a}_0 \left[\sum_i p_i - 1 \right] + \mathbf{a} \left[\mathbf{M} - \sum_i \mathbf{g}(i) \, p_i \right], \tag{2}$$

where a_0 and $\mathbf{a} = \{\mathrm{a}_1, \mathrm{a}_2, ..., \mathrm{a}_l\}$ are the $l + 1$ Lagrange multipliers, while $\mathbf{g}(i) = \{\mathrm{g}_1(\mathrm{i}), \mathrm{g}_2(\mathrm{i}), ..., \mathrm{g}_l(\mathrm{i})\}$ are the generator functions of the l moments $\mathbf{M} = \{\mathrm{M}_1, \mathrm{M}_2, ..., \mathrm{M}_l\}$.

The variational equation

$$\frac{\delta}{\delta p_i} \left(S + C \right) = 0, \tag{3}$$

implies the maximization of the entropy S under the constraints, imposing the conservation of the norm of p

$$\sum_i p_i = 1, \tag{4}$$

and the a priori knowledge of the values of the l moments

$$\mathbf{M} = \sum_i \mathbf{g}(i) \, p_i. \tag{5}$$

Equation (3) represents the maximum entropy principle and yields the relationship

$$\frac{\partial}{\partial p_i} \, p_i \, \Lambda(p_i) = -\mathbf{a} \cdot \mathbf{g}(i) + \mathrm{a}_0, \tag{6}$$

defining unambiguously the distribution function p_i.

We explain the meaning of the Lagrange multipliers a_j by observing that $\mathrm{a}_j = \mathrm{a}_j(\mathrm{M}_1, \dots, \mathrm{M}_l)$ with $j = 0, 1, ..., l$ and $S = S_{\mathrm{M}_1, \dots, \mathrm{M}_l}$ [38]. In the first step we multiply equation (6) by p_i, then sum with respect to the index i, and finally we derive with respect to M_k with $k = 1, ..., l$, obtaining

$$\sum_i \frac{\partial p_i}{\partial \mathrm{M}_k} \frac{\partial}{\partial p_i} \, p_i \, \Lambda(p_i) + \sum_i p_i \frac{\partial}{\partial \mathrm{M}_k} \frac{\partial}{\partial p_i} \, p_i \, \Lambda(p_i) - \frac{\partial \mathrm{a}_0}{\partial \mathrm{M}_k} + \frac{\partial}{\partial \mathrm{M}_k} \sum_{j=1}^{l} \mathrm{a}_j \, \mathrm{M}_j = 0, \tag{7}$$

and then

$$\frac{\partial S}{\partial \mathrm{M}_k} - \sum_i p_i \frac{\partial}{\partial \mathrm{M}_k}\frac{\partial}{\partial p_i} p_i \Lambda(p_i) + \frac{\partial \mathrm{a}_0}{\partial \mathrm{M}_k} - \frac{\partial}{\partial \mathrm{M}_k}\sum_{j=1}^{l} \mathrm{a}_j\, \mathrm{M}_j = 0. \tag{8}$$

In the second step we derive equation (6) with respect to M_k, subsequently multiply by p_i and then sum with respect to the index i, obtaining

$$\sum_i p_i \frac{\partial}{\partial \mathrm{M}_k}\frac{\partial}{\partial p_i} p_i\, \Lambda(p_i) - \frac{\partial \mathrm{a}_0}{\partial \mathrm{M}_k} + \sum_{j=1}^{l} \frac{\partial \mathrm{a}_j}{\partial M_k} \mathrm{M}_j = 0. \tag{9}$$

Finally, by combining equations (8) and (9) we have

$$\begin{aligned}\frac{\partial S}{\partial \mathrm{M}_k} &= -\sum_{j=1}^{l}\left(\frac{\partial}{\partial \mathrm{M}_k}\mathrm{a}_j\, \mathrm{M}_j - \frac{\partial \mathrm{a}_j}{\partial \mathrm{M}_k}\mathrm{M}_j\right)\\ &= -\sum_{j=1}^{l} \mathrm{a}_j \frac{\partial \mathrm{M}_j}{\partial \mathrm{M}_k} = -\sum_{j=1}^{l} \mathrm{a}_j\, \delta_{jk} = -\mathrm{a}_k. \end{aligned}\tag{10}$$

Then, for $j = 1, ..., l$ it holds

$$\mathrm{a}_j = -\frac{\partial S}{\partial \mathrm{M}_j}, \tag{11}$$

regardless on the form of the generalized entropy.

3 Three-parameter entropies

In the following, we are interested to the particular class of probability distribution functions which are given in terms of the generalized exponential function

$$\mathcal{E}(x) = \Lambda^{-1}(x), \tag{12}$$

namely of the form

$$p_i = \alpha\, \mathcal{E}\left(-\frac{\mathbf{a}\cdot\mathbf{g}(i) - \mathrm{a}_0 + \eta}{\lambda}\right), \tag{13}$$

α, λ and η being three scaling parameters. We note that the parameters λ and η produce a scaling to the Lagrange multipliers.

Ordinary statistical mechanics corresponds to the choice $\Lambda(x) = \ln(x)$, $\mathcal{E}(x) = \exp(x)$ and $\{\alpha = 1, \lambda = 1, \eta = 1\}$, or alternatively $\{\alpha = 1/e, \lambda = 1, \eta = 0\}$. We pose now the question regarding the possible existence of further couples of functions $\Lambda(x)$, $\mathcal{E}(x)$ and set of parameters $\{\alpha, \lambda, \eta\}$ defining statistical theories, through equations (1), (6) and (13), different from ordinary statistical mechanics.

Regarding this proposition we note that equation (13), after inversion, can be written in the form

$$\lambda\, \Lambda(p_i/\alpha) + \eta = -\mathbf{a}\cdot\mathbf{g}(i) + \mathrm{a}_0, \tag{14}$$

which after comparison with equation (6) yields

$$\frac{\partial}{\partial p_i} p_i \Lambda(p_i) = \lambda\, \Lambda(p_i/\alpha) + \eta. \tag{15}$$

The latter equation is a first order differential-functional equation whose solutions determine unambiguously the form of the generalized logarithm after taking into account the conditions (i) $\Lambda(1) = 0$, (ii) $0^+\, \Lambda(0^+) = 0^+$ and (iii) $\Lambda'(1) = 1$.

After obtaining the generalized logarithm and by inversion the generalized exponential, through equations (1) and (13), the entropy and the probability distribution function are unambiguously fixed.

Regarding the meaning of equation (15) we note that the equation guarantees a twofold link between entropy and the probability distribution function: firstly we have a differential link offered by the maximum entropy principle through equation (6); secondly we have an algebraic link offered by equations (1) and (13) where the same function in direct and inverse form, i.e. $\Lambda(x)$ and $\mathcal{E}(x)$, defines the entropy and the probability distribution function according to the standard rules of ordinary statistical mechanics. Equation (15) has been proposed in reference [54], while the special case corresponding to $\eta = 0$ has been considered in reference [53]. The solutions of equation (15) for the special case $\eta = 0$, have been obtained in reference [36,53].

Preliminarily we note that equation (15) admits the classical solution corresponding to $\Lambda(x) = \ln(x)$ and $\{\alpha = 1, \lambda = 1, \eta = 1\}$, or alternatively $\{\alpha = 1/e, \lambda = 1, \eta = 0\}$. The general solution of equation (15), derived in the Appendix, is given in terms of a three-parameter function $\Lambda(p_i) = \ln_{\kappa\tau\varsigma}(p_i)$ having the form

$$\ln_{\kappa\tau\varsigma}(x) = \frac{\varsigma^{\kappa}\, x^{\tau+\kappa} - \varsigma^{-\kappa}\, x^{\tau-\kappa} - \varsigma^{\kappa} + \varsigma^{-\kappa}}{(\kappa+\tau)\varsigma^{\kappa} + (\kappa-\tau)\varsigma^{-\kappa}}. \tag{16}$$

The three parameters $\{\kappa, \tau, \varsigma\}$ are related to the parameters $\{\alpha, \lambda, \eta\}$ of equation (15) through

$$\alpha = \left(\frac{1+\tau-\kappa}{1+\tau+\kappa}\right)^{\frac{1}{2\kappa}}, \tag{17}$$

$$\lambda = \frac{\left(1+\tau-\kappa\right)^{\frac{\tau+\kappa}{2\kappa}}}{\left(1+\tau+\kappa\right)^{\frac{\tau-\kappa}{2\kappa}}}, \tag{18}$$

$$\eta = (\lambda - 1)\, \frac{\varsigma^{\kappa} - \varsigma^{-\kappa}}{(\kappa+\tau)\varsigma^{\kappa} + (\kappa-\tau)\varsigma^{-\kappa}}. \tag{19}$$

The classical solution $\Lambda(p_i) = \ln(p_i)$, $\alpha = 1/e$, $\lambda = 1$, $\eta = 0$ is contained as a limiting case ($\tau = 0$, $\kappa \to 0$) in the above three parameter class of solutions.

Let us consider the dual function of $f(x)$ defined as follows:

$$\widehat{f}(x) = -f(1/x). \tag{20}$$

It is easy to verify that the dual of a three-parameter logarithm again results to be a new three-parameter logarithm, i.e.

$$\widehat{\ln}_{\kappa\tau\varsigma}(x) = \ln_{\kappa\tau'\varsigma'}(x), \tag{21}$$

where $\tau' = -\tau$ and $\varsigma' = 1/\varsigma$. In general a three-parameter logarithm is not self-dual, but the class of the three-parameter logarithm is self-dual.

In the present context a special role is played by the scaling operation. Let us consider the class of the functions $\mathcal{F} = \{f(x),\ x > 0\}$, satisfying the condition $f(1) = 0$ and $f'(1) = 1$. Starting from the function $f(x) \in \mathcal{F}$ we define the corresponding scaled function ${}_\sigma f(x) \in \mathcal{F}$ as follows:

$$ {}_\sigma f(x) = \frac{f(\sigma x) - f(\sigma)}{\sigma\, f'(\sigma)}, \tag{22} $$

where $\sigma > 0$ is the scaling parameter.

It is remarkable that two consecutive scaling operations are equivalent to a unique scaling operation, i.e.

$$ {}_{\sigma_2}\left[\, {}_{\sigma_1} f(x)\,\right] = \ {}_{\sigma_3} f(x), \tag{23} $$

where $\sigma_3 = \sigma_2 \sigma_1$.

The scaling operation has the following group properties:

$$ {}_1 f(x) = f(x), \tag{24} $$

$$ {}_{1/\sigma}[\, {}_\sigma f(x)\,] = \ f(x), \tag{25} $$

$$ {}_{\sigma_1}\left[\, {}_{\sigma_2} f(x)\,\right] = \ {}_{\sigma_2}\left[\, {}_{\sigma_1} f(x)\,\right]. \tag{26} $$

It is straightforward to show that the three-parameter logarithm $\ln_{\kappa\tau\varsigma}(x)$ has the interesting property

$$ {}_\sigma \ln_{\kappa\tau\varsigma}(x) = \ln_{\kappa\tau\varsigma'}(x), \tag{27} $$

where $\varsigma' = \sigma\,\varsigma$. The scaled function ${}_\sigma \ln_{\kappa\tau\varsigma}(x)$ of a three-parameter logarithm again results to be a three-parameter logarithm with the same first two parameters, while the third parameter, given by $\varsigma' = \sigma\,\varsigma$, is the scaling parameter. In general a particular three-parameter logarithm is not self-scaling, but the class of the three-parameter logarithms $\ln_{\kappa\tau\varsigma}(x)$ is self-scaling.

The explicit form of the three-parameter class of entropies is given by

$$ S = -\sum_i \, p_i \, \ln_{\kappa\tau\varsigma}(p_i). \tag{28} $$

4 Power-law tailed distributions

It is immediately possible to verify that $\ln_{\kappa\tau\varsigma}(x) \in C^\infty(R^+)$ presents the following asymptotic power-law behaviour:

$$ \ln_{\kappa\tau\varsigma}(x) \underset{x\to 0^+}{\sim} -\frac{A}{x^{|\kappa|-\tau}}, \tag{29} $$

$$ \ln_{\kappa\tau\varsigma}(x) \underset{x\to +\infty}{\sim} A\, x^{|\kappa|+\tau}, \tag{30} $$

where $A = \left[(|\kappa|+\tau)\ \varsigma^{|\kappa|} + (|\kappa|-\tau)\ \varsigma^{-|\kappa|}\right]^{-1}$. For $\tau > |\kappa| - 1$ we have $\lim_{x\to 0^+}\ x\,\ln_{\kappa\tau\varsigma}(x) = 0$ so that it results

$$ \int\limits_0^1 \ln_{\kappa\tau\varsigma}(x)\, dx = -\frac{1}{(1+\tau)^2 - \kappa^2} \times \left[1 + (\tau^2 - \kappa^2)\, \frac{\varsigma^\kappa - \varsigma^{-\kappa}}{(\kappa+\tau)\ \varsigma^\kappa + (\kappa-\tau)\ \varsigma^{-\kappa}}\right]. \tag{31} $$

From equations (20) and (21) it follows

$$ \ln_{\kappa\tau\varsigma}(1/x) = -\ln_{\kappa\,\tau'\varsigma'}(x), \tag{32} $$

with $\tau' = -\tau$ and $\varsigma' = 1/\varsigma$.

After noting that when $-|\kappa| \le \tau \le |\kappa|$ it holds $d\ln_{\kappa\tau\varsigma}(x)/dx > 0$, we can define the three-parameter exponential $\exp_{\kappa\tau\varsigma}(x)$ as the inverse function of $\ln_{\kappa\tau\varsigma}(x)$. The properties of $\exp_{\kappa\tau\varsigma}(x)$ readily follow from those of $\ln_{\kappa\tau\varsigma}(x)$. For instance, the asymptotic power-law behavior of $\exp_{\kappa\tau\varsigma}(x)$ is given by

$$ \exp_{\kappa\tau\varsigma}(x) \underset{x\to\pm\infty}{\sim} (\pm A\, x)^{\,1/(\tau\pm|\kappa|)}, \tag{33} $$

while equation (32) implies

$$ \exp_{\kappa\tau\varsigma}(x)\ \exp_{\kappa\tau'\varsigma'}(-x) = 1. \tag{34} $$

The probability distribution function can been obtained after maximization of the three-parameter entropy (28), according to maximum entropy principle

$$ p_i = \alpha\ \exp_{\kappa\tau\varsigma}\left(-\,\frac{\mathbf{a}\cdot\mathbf{g}(i) - \mathrm{a}_0 + \eta}{\lambda}\right). \tag{35} $$

We remark that equations (16), (28) and (35) define a very wide class of statistical theories, candidates to generalize the ordinary statistical mechanics after passing a series of validity tests, some of which will be considered in the next sections.

5 Thermodynamic stability

We denote with $q = \{q_i\}$ the optimal distribution which according to maximum entropy principle, is defined by the algebraic equation (6), i.e.

$$ \frac{\partial}{\partial q_i}\, q_i\, \ln_{\kappa\tau\varsigma}(q_i) = -\mathbf{a}\cdot\mathbf{g}(i) + \mathrm{a}_0. \tag{36} $$

After introducing the entropy density for a general probability distribution function p_i according to

$$ \sigma(p_i) = -p_i\, \ln_{\kappa\tau\varsigma}(p_i), \tag{37} $$

the constrained entropy $\Phi = S + C$ can be written as

$$ \Phi(p) = \sum_i \left(\sigma(p_i) - \frac{\partial\sigma(q_i)}{\partial q_i}\, p_i\right) + \mathrm{a}_o - \mathbf{a}\cdot\mathbf{M}, \tag{38} $$

while its maximum value is given by

$$ \Phi(q) = \sum_i \left(\sigma(q_i) - \frac{\partial\sigma(q_i)}{\partial q_i}\, q_i\right) + \mathrm{a}_o - \mathbf{a}\cdot\mathbf{M}. \tag{39} $$

In order to calculate the difference of the above two functionals

$$ \Phi(p) - \Phi(q) = \sum_i \left[\sigma(p_i) - \sigma(q_i) - \frac{\partial\sigma(q_i)}{\partial q_i}(p_i - q_i)\right], \tag{40} $$

when $p_i \approx q_i$, we perform the Taylor expansion

$$\sigma(p_i) \approx \sigma(q_i) + \frac{\partial \sigma(q_i)}{\partial q_i}(p_i - q_i) + \frac{1}{2}\frac{\partial^2 \sigma(q_i)}{\partial q_i^2}(p_i - q_i)^2, \quad (41)$$

and obtain

$$\Phi(p) - \Phi(q) = \sum_i \frac{1}{2}\frac{\partial^2 \sigma(q_i)}{\partial q_i^2}(p_i - q_i)^2. \quad (42)$$

At this point we recall that $\Phi(q_i)$ represents the maximum value of $\Phi(p_i)$ so that it holds

$$\Phi(p) - \Phi(q) \leq 0. \quad (43)$$

The latter inequality and equation (42) imply

$$\frac{\partial^2 \sigma(x)}{\partial x^2} \leq 0, \quad (44)$$

for any value of x. After taking into account the definition of the density entropy, we get the following property for the generalized logarithm

$$\frac{\partial^2}{\partial x^2}\; x \ln_{\kappa\tau\varsigma}(x) \geq 0. \quad (45)$$

The latter relationship can be verified easily by starting with equation (15) and after taking into account that for $-|\kappa| \leq \tau \leq |\kappa|$ it results $d\ln_{\kappa\tau\varsigma}(x)/dx > 0$. Therefore we can conclude that the system described by the entropy (28) is thermodynamically stable for $-|\kappa| \leq \tau \leq |\kappa|$.

6 Lesche stability

The Lesche stability condition [36,40,41], imposes that any physically meaningful entropy, depending on a probability distribution function, should exhibit a small relative error

$$R = \left| \frac{S(p) - S(q)}{S_{max}} \right|, \quad (46)$$

with regard to small changes in the probability distributions

$$D = ||p - q||. \quad (47)$$

Mathematically this implies that for any $\varepsilon > 0$ there exists $\delta > 0$ such that $R \leq \varepsilon$ holds for all the distribution functions satisfying $D \leq \delta$. It is known that the Lesche stability condition holds for the Boltzmann-Shannon entropy.

In the following, by adopting the procedure of reference [36], we will show that the Lesche stability condition holds for the entire class of three-parameter entropies (28).

By using the procedure of reference [36], we introduce the auxiliary function

$$A(p, s) \equiv \sum_{i=1}^{N} \left[p_i - \alpha \exp_{\kappa\tau\varsigma}\left(-\frac{s}{\lambda}\right) \right]_+ , \quad (48)$$

where we have used the definitions $[x]_+ \equiv \max(x, 0)$. The definition of $A(p, s)$ implies that for any s the following relationship holds

$$\Big| A(p, s) - A(q, s) \Big| \leq \sum_{i=1}^{N} |p_i - q_i| \equiv \|p - q\|_1, \quad (49)$$

while for $s \geq -\lambda \ln_{\kappa\tau\varsigma}(1/N)$ it results

$$\Big| A(p, s) - A(q, s) \Big| < N\,\alpha \exp_{\kappa\tau\varsigma}\left(-\frac{s}{\lambda}\right). \quad (50)$$

After taking into account the definition of $A(p, s)$ and equation (15), it is easy to verify that the entropy (28) can be written in form

$$S(p) = \int_{-1}^{+\infty} [1 - A(p, s)]\, ds \;\; - 1 - \eta. \quad (51)$$

From the latter relationship it follows that the absolute difference of the entropies of the two different distributions $p = \{p_i\}$ and $q = \{q_i\}$ verifies the inequality

$$\begin{aligned} \Big| S(p) - S(q) \Big| &= \left| \int_{-1}^{+\infty} [A(p, s) - A(q, s)]\; ds \right| \\ &\leq \int_{-1}^{\ell} \Big| A(p, s) - A(q, s) \Big|\, ds \\ &+ \int_{\ell}^{+\infty} \Big| A(p, s) - A(q, s) \Big|\, ds. \end{aligned} \quad (52)$$

Choosing $-\lambda \ln_{\kappa\tau\varsigma}(1/N) \leq \ell < +\infty$, by using equation (49) in the first integral and equation (50) in the second integral of equation (52), we obtain

$$\begin{aligned} \Big| S(p) - S(q) \Big| &\leq (\ell + 1)\; \|p - q\|_1 \\ &+ N\,\alpha \int_{\ell}^{+\infty} \exp_{\kappa\tau\varsigma}\left(-\frac{s}{\lambda}\right)\, ds. \end{aligned} \quad (53)$$

In particular the latter relationship holds for $\ell = \bar{\ell}$ being

$$\bar{\ell} = -\lambda \ln_{\kappa\tau\varsigma}\left(\frac{\|p - q\|_1}{\alpha\, N}\right), \quad (54)$$

the values of ℓ that minimizes the r.h.s. of the expression, as long as $\bar{\ell} \geq -\lambda \ln_{\kappa\tau\varsigma}(1/N)$, which is true when

$$\delta = \|p - q\|_1 \leq \alpha, \quad (55)$$

i.e. for sufficiently close distributions, according to the metric. After some simple algebra one obtains the relative difference of entropies

$$\left| \frac{S(p) - S(q)}{S_{\max}} \right| \leq \epsilon(\,\delta\,,\, N), \quad (56)$$

and

$$\epsilon(\,\delta\,,\,N)=\frac{\delta}{\ln_{\,\kappa\tau\varsigma}(1/N)}\,\left[\ln_{\,\kappa\tau\varsigma}\,(\delta/N)-1+\frac{\eta}{\lambda}\right], \qquad (57)$$

where $S_{\max}\,\equiv\,-\ln_{\,\kappa\tau\varsigma}(1/N)$. After taking into account that $\lim_{x\to 0^+} x\,\ln_{\,\kappa\tau\varsigma}(x)=0$, we find

$$\lim_{\delta\to 0^+}\,\epsilon(\,\delta,N)=0^+, \qquad (58)$$

such that if the two distributions are sufficiently close, $\delta\to 0^+$, the corresponding difference ϵ of entropies (28) can be made small at will and then we can conclude that the entropy (28) is Lesche stable.

In the thermodynamic limit $N\to\infty$ we have

$$\lim_{N\to\infty}\,\epsilon(\,\delta,N)=\epsilon(\delta)=\delta^{\,1-|\kappa|+\tau}, \qquad (59)$$

and results that $\lim_{\delta\to 0^+}\,\epsilon(\,\delta)=0^+$. Thus the entropy (28) is Lesche stable also in the thermodynamic limit.

7 Special cases

In the present section we consider a few special cases of the three parameter generalized logarithm (16), some of which are already known in the literature. The ordinary logarithm, which is self-dual and self-scaling, can be obtained as the limiting case when $\kappa\to 0$ and $\tau=0$, independently on the value of the parameter ς, such that we can write $\ln(x)=\ln_{\,0\,0\,\varsigma}(x)$.

i) *Self-dual logarithm*: the choice corresponding to $\tau=0$, $\varsigma=1$ and $-1<\kappa<1$, yields the generalized logarithm $\ln_\kappa(x)=\ln_{\,0\,0\,1}(x)$, which together with its inverse function assume the form

$$\ln_\kappa(x)=\frac{x^\kappa-x^{-\kappa}}{2\,\kappa}, \qquad (60)$$

$$\exp_\kappa(x)=\left(\sqrt{1+\kappa^2x^{\,2}}+\kappa x\right)^{1/\kappa}. \qquad (61)$$

The self-duality of the above functions is expressed through the relationships

$$\ln_\kappa(1/x)=-\ln_\kappa(x), \qquad (62)$$

$$\exp_\kappa(x)\exp_\kappa(-x)=1. \qquad (63)$$

It is remarkable that the self-duality property, if imposed to the three-parameter logarithm, i.e. $\ln_{\kappa\tau\varsigma}(1/x)=-\ln_{\kappa\tau\varsigma}(1/x)$, automatically fixes the parameters $\tau=0$ and $\varsigma=1$, so that $\ln_\kappa(x)$ remains the only one self-dual generalized logarithm.

The following integral representation for $\ln_\kappa(x)$ holds

$$\ln_\kappa(x)=\frac{1}{2}\int_{1/x}^{x}\frac{dt}{t^{1+\kappa}}, \qquad (64)$$

which in the limit of $\kappa\to 0$ reduces to a well known property of the ordinary logarithm.

The properties of the functions $\ln_\kappa(x)$ and $\exp_\kappa(x)$ have been study extensively in the literature due to their relevance in relativistic statistical mechanics. The deformation mechanism introduced by the parameter κ is induced directly by the Lorentz transformations of Einstein's special relativity. Indeed $1/\kappa$ is the reciprocal of light speed while $1/\kappa^2$ is the particle rest energy, in dimensionless forms. On the other hand the self-duality relation given by equation (63) leads to the relativistic dispersion relation, obtaining in this way a direct link with the microscopic dynamics of the system. An important property of $\exp_\kappa(x)$ is given by

$$\exp_\kappa(q_{\scriptscriptstyle A}\oplus q_{\scriptscriptstyle B})=\exp_\kappa(q_{\scriptscriptstyle A})\exp_\kappa(q_{\scriptscriptstyle B}), \qquad (65)$$

$q_{\scriptscriptstyle A}\oplus q_{\scriptscriptstyle B}$ being the additivity law of the dimensionless relativistic momenta, defined through

$$q_{\scriptscriptstyle A}\oplus q_{\scriptscriptstyle B}=q_{\scriptscriptstyle A}\sqrt{1+\kappa^2q_{\scriptscriptstyle B}^2}+q_{\scriptscriptstyle B}\sqrt{1+\kappa^2q_{\scriptscriptstyle A}^2}. \qquad (66)$$

A further property of $\exp_\kappa(x)$ is given by

$$\frac{d}{d_\kappa\,q}\,\exp_\kappa(q)=\exp_\kappa(q), \qquad (67)$$

where the κ-derivative

$$\frac{d}{d_\kappa\,q}=\sqrt{1+\kappa^2q^2}\,\frac{d}{d\,q}, \qquad (68)$$

emerge within the Lorentz-invariant differential calculus.

In order to better explain how the functions $\exp_\kappa(x)$ and $\ln_\kappa(x)$ can define a statistical theory we consider briefly the paradigm of statistical mechanics. We indicate with W_i the microscopic energy, with β the reciprocal of the temperature and with μ the chemical potential, so that, the Lagrange multipliers are given by $a_0=\beta\mu$ and $a_1=\beta$. The system entropy is given by

$$S=-\sum_i p_i\,\ln_{\,\kappa}(p_i), \qquad (69)$$

while the maximum entropy principle yields the probability distribution function

$$p_i=\alpha\,\exp_{\,\kappa}\left(-\,\beta\,\frac{W_i-\mu}{\lambda}\right), \qquad (70)$$

μ being the normalization constant, while the expressions of the parameters λ and α are given by equations (17) and (18) respectively, after posing $\tau=0$.

Statistical mechanics based on equations (69) and (70) has been introduced in [51,52], while in [53,54] its relativistic origin has been shown.

In the last few years various authors have considered the foundations of this statistical theory, e.g., H-theorem and molecular chaos hypothesis [55,56], thermodynamic stability [57,58], Lesche stability [59–62], Legendre structure of ensued thermodynamics [63], Gibbs theorem [64], geometrical aspects of the theory [65] etc. On the other hand specific applications to physical systems have been considered, e.g., cosmic rays [53], relativistic [66] and classical [67] plasmas in the presence of external electromagnetic fields, relaxation in relativistic plasmas under wave-particle interactions [68,69], kinetics of interacting atoms and photons [70], particle systems in external conservative force fields [71], astrophysical systems [72,73], quark-gluon plasma formation [74], relativistic quantum hadrodynamics [75], etc. Other applications concern nonlinear diffusion [76], dynamical systems at the edge of chaos [77–79], fractal systems [80], random matrix theory [81,82], error theory [83], game theory [84], information theory [85], etc. Also applications to economic systems have been considered, e.g., to study personal income distribution [86–88] and to model deterministic heterogeneity in tastes and product differentiation [89,90] etc.

ii) *Self-scaling logarithm*: the case corresponding to $\tau = -\kappa = \alpha/2$, is undoubtedly the more discussed in the literature and produces the following generalized self-scaling logarithm and exponential:

$$\ln_\alpha(x) = \frac{1-x^{-\alpha}}{\alpha}, \tag{71}$$

$$\exp_\alpha(x) = (1-\alpha\, x)^{-1/\alpha}. \tag{72}$$

The self-scaling property is given by

$$_\sigma \ln_\alpha(x) = \ln_\alpha(x). \tag{73}$$

Both functions (71) and (72) have been considered first by Euler in 1779 and explicitly has remarked that the function $(x^\omega - 1)/\omega$ reduces to the natural logarithm in the $\omega \to 0$ limit [91]. Successively, the function (72) was adopted in 1908 by Gosset to construct the Student distribution [92], which in 1968 was introduced into plasma physics by Vasyliunas [3]. In mathematical statistics the function (72) has been used to construct the Burr-distribution [93]. On the other hand the function (71) was adopted in 1967 by Harvda e Charvat [94] to propose a generalized entropy in information theory. Finally both the functions (71) and (72) were adopted by Tsallis in 1988, using the new parameter $q = \alpha + 1$ to define the q-logarithm $\ln_q(x)$ and q-exponential $\exp_q(x)$,

$$\ln_q(x) = \frac{x^{1-q}-1}{1-q}, \tag{74}$$

$$\exp_q(x) = (1+(1-q)\,x)^{1/(1-q)}, \tag{75}$$

in order to develop nonextensive statistical mechanics [95,96].

iii) The case corresponding to the choice $\kappa = (a - 1/a)/2$, $\tau = (a+1/a)/2-1$ and $\varsigma = 1$ defines a generalized logarithm

$$\ln_a(x) = \frac{x^{a-1} - x^{1/a-1}}{a-1/a}, \tag{76}$$

leading, by means of equation (28), to an entropy introduced in the literature in 1997 by Abe [97] presenting the symmetry $a \leftrightarrow 1/a$.

iv) Only in very few cases, the generalized logarithm given by Equation (16), can be inverted to obtain analytically the corresponding generalized exponential. For instance the choice $\kappa = 3\gamma/2$, $\tau = \gamma/2$, $\varsigma = 1$, yields an invertible generalized logarithm

$$\ln_\gamma(x) = \frac{x^{2\gamma} - x^{-\gamma}}{3\gamma}, \tag{77}$$

corresponding to the generalized exponential

$$\exp_\gamma(x) = \left[\left(\frac{1+y}{2}\right)^{1/3} + \left(\frac{1-y}{2}\right)^{1/3} \right]^{1/\gamma}, \tag{78}$$

with $y = \sqrt{1-4\gamma^3 x^3}$, [36].

v) The three-parameter class of generalized logarithm (16) contains a two-parameter subclass, already known in the literature, defined by posing $\varsigma = 1$

$$\ln_{\kappa\tau 1}(x) = x^\tau \, \ln_\kappa(x). \tag{79}$$

The latter subclass of two-parameter-logarithms is not self-scaling. From equation (27) with $\sigma = 1/\varsigma$, one obtains the following relationships:

$$\ln_{\kappa\tau\varsigma}(x) = \;_\varsigma \ln_{\kappa\tau 1}(x), \tag{80}$$

$$\ln_{\kappa\tau 1}(x) = \;_{1/\varsigma} \ln_{\kappa\tau\varsigma}(x). \tag{81}$$

After introducing the new parameters α and β through $\kappa = (\alpha-\beta)/2$ and $\tau = (\alpha+\beta)/2$, the two-parameter logarithm $\ln_{\kappa\tau 1}(x) = \ln_{\alpha\beta}(x)$ can be written in the form

$$\ln_{\alpha\beta}(x) = \frac{x^\alpha - x^\beta}{\alpha - \beta}, \tag{82}$$

which is more familiar in the literature and has been considered firstly by Euler [91]. In general this two-parameter logarithm cannot be inverted analytically, thus it is impossible to define explicitly the corresponding generalized exponential. The function (82) has been used to construct the two-parameter entropies introduced in the information theory in 1975 by Mittal [98], and by Sharma and Taneja [99]. The generalized logarithm (82) was reconsidered in statistical mechanics in 1998 by Borges and Roditi [100] and recently in references [35,36]. Clearly the κ-logarithm does not emerge easily from (82) and probably this is the reason why it has not been discussed in the literature before 2001.

vi) The three-parameter class of logarithm (16) contains a second, new two-parameter subclass of logarithms, namely the scaled κ-logarithms, obtained by posing $\tau = 0$. The corresponding class of generalized entropies and distribution functions will be discussed in the next section.

The one-parameter and two-parameter generalized logarithms, already known in the literature and discussed in the present section, emerging as special cases of the three-parameter logarithm (16), are reported in Table 1.

8 Scaled κ-statistical mechanics

We consider here the two-parameter generalized logarithms $\ln_{\kappa\varsigma}(x) = \ln_{\kappa\, 0\, \varsigma}(x)$, obtained by posing $\tau = 0$ in

Table 1. Generalized logarithms already known in the literature, obtained here as special cases of the three-parameter logarithm. In the third column are reported the values of the parameters κ, τ, and ς to insert in the general expression of the three-parameter logarithm, to obtain as special cases the zero-, one-, and two-parameter logarithms.

	Generalized logarithm expression	Parameter values	Introduced in Ref.
three-parameter logarithm (most general case)	$\ln_{\kappa\tau\varsigma}(x) = \frac{\varsigma^\kappa x^{\tau+\kappa} - \varsigma^{-\kappa} x^{\tau-\kappa} - \varsigma^\kappa + \varsigma^{-\kappa}}{(\kappa+\tau)\varsigma^\kappa + (\kappa-\tau)\varsigma^{-\kappa}}$		[54]
1. zero-parameter logarithm	$\ln(x)$	$\tau = 0, \quad \kappa \to 0$	Napier
2. one-parameter logarithm	$\ln_\kappa(x) = \frac{x^\kappa - x^{-\kappa}}{2\kappa}$	$\tau = 0, \varsigma = 1$	[51]
3. one-parameter logarithm	$\ln_q(x) = \frac{x^{1-q} - 1}{1-q}$	$\kappa = \tau = (1-q)/2$	[96]
4. one-parameter logarithm	$\ln_a(x) = \frac{x^{a-1} - x^{1/a-1}}{a-1/a}$	$\varsigma = 1, \quad \kappa = (1-1/a)/2$ $\tau = (1+1/a)/2 - 1$	[97]
5. two-parameter logarithm	$\ln_{\alpha\beta}(x) = \frac{x^\alpha - x^\beta}{\alpha - \beta}$	$\varsigma = 1, \quad \kappa = (\alpha-\beta)/2$ $\tau = (\alpha+\beta)/2$	[98,99]
6. two-parameter logarithm	$\ln_{\kappa\varsigma}(x) = \frac{\varsigma^\kappa x^\kappa - \varsigma^{-\kappa} x^{-\kappa} - \varsigma^\kappa + \varsigma^{-\kappa}}{\kappa(\varsigma^\kappa + \varsigma^{-\kappa})}$	$\tau = 0$	[54]

equation (16). It is easy to verify that $\ln_{\kappa\varsigma}(x)$ is the scaled function of $\ln_\kappa(x)$. The following relationships hold

$$\ln_{\kappa\varsigma}(x) = {}_\varsigma \ln_\kappa(x), \tag{83}$$

$$\ln_\kappa(x) = {}_{1/\varsigma}\ln_{\kappa\,\varsigma}(x), \tag{84}$$

$$\ln_\kappa(x) = \ln_{\kappa\,1}(x), \tag{85}$$

and results $\ln_{0\,\varsigma}(x) = \ln(x)$. The scaled κ-logarithm assumes the form

$$\ln_{\kappa\varsigma}(x) = \frac{\varsigma^\kappa x^\kappa - \varsigma^{-\kappa} x^{-\kappa} - \varsigma^\kappa + \varsigma^{-\kappa}}{\kappa\varsigma^\kappa + \kappa\varsigma^{-\kappa}}, \tag{86}$$

and can also be written as

$$\ln_{\kappa\varsigma}(x) = \frac{1}{a}\ln_\kappa(\varsigma x) - \frac{b}{a}, \tag{87}$$

where $a = \sqrt{1 + \kappa^2[\ln_\kappa(\varsigma)]^2}$ and $b = \ln_\kappa(\varsigma)$. An important property of the scaled κ-logarithm is that its inverse function, the scaled κ-exponential, exists and assumes a very simple form for any value of κ and ς:

$$\exp_{\kappa\varsigma}(x) = \frac{1}{\varsigma}\exp_\kappa(ax + b)\,. \tag{88}$$

The following Taylor expansions hold:

$$\ln_{\kappa\varsigma}(1 + x) \underset{x\to 0}{\sim} x - \left(1 - \frac{b}{a}\kappa^2\right)\frac{x^2}{2}, \tag{89}$$

$$\exp_{\kappa\varsigma}(x) \underset{x\to 0}{\sim} 1 + x + \left(1 - \frac{b}{a}\kappa^2\right)\frac{x^2}{2}, \tag{90}$$

while the power-law asymptotic behaviours of $\ln_{\kappa\varsigma}(x)$ and $\exp_{\kappa\varsigma}(x)$ are given by

$$\ln_{\kappa\varsigma}(x) \underset{x\to 0^+}{\sim} -\frac{x^{-|\kappa|}}{2a|\kappa|}, \tag{91}$$

$$\ln_{\kappa\varsigma}(x) \underset{x\to +\infty}{\sim} \frac{x^{|\kappa|}}{2a|\kappa|}, \tag{92}$$

$$\exp_{\kappa\varsigma}(x) \underset{x\to\pm\infty}{\sim} |2a\kappa|^{\pm 1/|\kappa|}\,. \tag{93}$$

The new class of two-parameter entropies is given by

$$S = -\sum_i p_i \ \ln_{\kappa\varsigma}(p_i). \tag{94}$$

Consequently the maximum entropy principle produces a distribution function having the form

$$p_i = \alpha\, \exp_{\kappa\varsigma}\left(-\frac{\mathbf{a}\cdot\mathbf{g}(i) - \mathrm{a}_0 + \eta}{\lambda}\right), \tag{95}$$

with

$$\alpha = \left(\frac{1-\kappa}{1+\kappa}\right)^{1/2\kappa}, \tag{96}$$

$$\lambda = \sqrt{1 - \kappa^2}, \tag{97}$$

$$\eta = \frac{\sqrt{1-\kappa^2} - 1}{\kappa}\,\frac{\varsigma^\kappa - \varsigma^{-\kappa}}{\varsigma^\kappa + \varsigma^{-\kappa}}. \tag{98}$$

The κ-logarithm results to be the semi-sum of two scaled κ-logarithms with scaling parameters ς and $1/\varsigma$ respectively, i.e.

$$\ln_\kappa(x) = \frac{1}{2}\left[\ln_{\kappa\,\varsigma}(x) + \ln_{\kappa\,1/\varsigma}(x)\right]. \tag{99}$$

A further connection between the κ-logarithm and the scaled κ-logarithm is given by

$$\ln_{\kappa\varsigma}(x) = \left[1 - \frac{b}{a}\right]\ln_\kappa(x) + \frac{b}{a}\ln_{\kappa\,\varsigma'}(x), \tag{100}$$

with $\varsigma' = 1/\alpha$. The following two expressions of $\ln_{\kappa\,\varsigma'}(x)$ hold

$$\ln_{\kappa\,\varsigma'}(x) = \lambda\ln_\kappa(x/\alpha) - 1\ , \tag{101}$$

$$\ln_{\kappa\,\varsigma'}(x) = \frac{d}{dx}\left[x\,\ln_\kappa(x)\right] - 1. \tag{102}$$

It is remarkable that $\ln_{\kappa\varsigma}(x)$ in the limits of $\varsigma \to 0$ and $\varsigma \to \infty$ reduces to the self-scaling $q-$logarithm given by $\ln_q(x) = (1 - x^{1-q})/(q-1)$. After posing $q = 1 + |\kappa|$ we obtain

$$\ln_q(x) = \ln_{\,\kappa\,0}(x), \tag{103}$$

$$\ln_{2-q}(x) = \ln_{\,\kappa\,\infty}(x). \tag{104}$$

On the other hand equations (99) and (100) in the $\varsigma \to 0$ limit, reduce to

$$\ln_\kappa(x) = \frac{1}{2}\left[\,\ln_q(x) + \ln_{\,2-q}(x)\,\right], \tag{105}$$

$$\ln_q(x) = \left(1 + \frac{1}{|\kappa|}\right)\ln_\kappa(x) - \frac{1}{|\kappa|}\,\ln_{\,\kappa\,\varsigma'}(x), \tag{106}$$

with $q = 1 + |\kappa|$ and $\varsigma' = 1/\alpha$.

9 Conclusions

As conclusions, we recall briefly the main results obtained here. The requirement that the generalized trace-form entropy

$$S(p\,) = -\,\langle\,\Lambda\,\rangle = -\sum_i\,p_{_i}\,\Lambda(p_{_i}), \tag{107}$$

after maximization, according to maximum entropy principle, yields a probability distribution function given by

$$p_i = \alpha\,\Lambda^{-1}\left(-\frac{\mathbf{a}\cdot\mathbf{g}(i) - \mathrm{a}_{_0} + \eta}{\lambda}\right), \tag{108}$$

is sufficient to fix the form of $\Lambda(x)$ and therefore the form of the entropy and of the probability distribution function.

The obtained class of generalized logarithms, $\Lambda(x) = \ln_{\,\kappa\tau\varsigma}(x)$, depends on three free parameters according to

$$\ln_{\,\kappa\tau\varsigma}(x) = \frac{\varsigma^\kappa\,x^{\tau+\kappa} - \varsigma^{-\kappa}\,x^{\tau-\kappa} - \varsigma^\kappa + \varsigma^{-\kappa}}{(\kappa+\tau)\varsigma^\kappa + (\kappa-\tau)\varsigma^{-\kappa}}. \tag{109}$$

The relevant three-parameter entropy is both thermodynamically stable and Lesche stable and contains as special cases all the one-parameter and two-parameter trace form entropies appeared in the literature.

The ensuing three-parameter probability distribution functions represent the minimal deformation of the Maxwell-Boltzmann exponential distribution compatible with the maximum entropy principle and exhibit power-law tails.

Appendix A

Here we solve the equation

$$\frac{\partial}{\partial x}\,x\Lambda(x) = \lambda\;\Lambda(x/\alpha) + \eta, \tag{A.1}$$

with the conditions $\Lambda(1) = 0$, $\Lambda'(1) = 1$, $0^+\,\Lambda(0^+) = 0$ and $\Lambda(0^+) < 0$.

After expressing the function $\Lambda(x)$ in terms of the auxiliary function $L(x)$ according to

$$\Lambda(x) = \frac{L(\varsigma x) - L(\varsigma)}{\varsigma\,L'(\varsigma)}. \tag{A.2}$$

and after setting

$$\eta = (\lambda - 1)\,\frac{L(\varsigma)}{\varsigma\,L'(\varsigma)}, \tag{A.3}$$

we obtain the following equation for the function $L(x)$

$$\frac{\partial}{\partial x}\,x\,L(x) = \lambda\;L\!\left(\frac{x}{\alpha}\right). \tag{A.4}$$

The conditions associated to the function $L(x)$ easily follow from those related to the function $\Lambda(x)$, i.e. $L(1) = 0$, $L'(1) = 1$, $0^+\,L(0^+) = 0$ and $L(0^+) < 0$.

The general solution of equation (A.4) has been obtained in reference [36]. In the following we recall briefly the main steps of the solution procedure.

First we perform the change of variable $L(x) = (1/x)\,h(\lambda\,\alpha\,\ln x)$ and $x = \exp\left(t/(\lambda\,\alpha)\right)$ and reduce equation (A.4) to a first order homogeneous differential-difference equation, belonging to the class of delay equations

$$\frac{d\,h(t)}{d\,t} - h(t - t_{_0}) = 0, \tag{A.5}$$

where $t_0 = \lambda\,\alpha\,\ln\alpha$. The general solution of this equation is given by

$$h(t) = \sum_{i=1}^{n}\,\sum_{j=0}^{m_i - 1}\,a_{_{ij}}(s_{_1},\,\ldots,\,s_{_n})\,t^j\,e^{s_i\,t}, \tag{A.6}$$

where n is the number of real solutions $s_{_i}$ (of multiplicity $m_{_i}$) of the characteristic equation $s_{_i} - e^{-t_0\,s_i} = 0$. In general the integration constants $a_{_{ij}}$ depend on the parameter s_i. The latter algebraic equation does not admit real solutions for $t_{_0} < -1/e$. For $t_{_0} \geq -1/e$ the number and multiplicity of the real solutions of the characteristic equation depend on the value of t_0 according to

(a) for $-1/e < t_{_0} < 0$, $n = 2$ and $m_{_i} = 1$;
(b) for $t_{_0} = -1/e$; $n = 1$ and $m = 2$;
(c) for $t_{_0} \geq 0$, $n = 1$ and $m = 1$.

The solution of (A.4) for the case (a) can be written in the form

$$L(x) = A_{_1}(\kappa_{_1},\,\kappa_{_2})\,x^{\kappa_1} + A_{_2}(\kappa_{_1},\,\kappa_{_2})\,x^{\kappa_2}, \tag{A.7}$$

where $\kappa_{_i} = \lambda\,\alpha\,s_{_i} - 1$ and $A_{_i}(\kappa_{_1},\,\kappa_{_2})$ the integration constants. The boundary conditions $L(1) = 0$ and $L'(1) = 1$ imply $A_{_1} = -A_{_2}$ and $\kappa_{_1}\,A_{_1} + \kappa_{_2}\,A_{_2} = (\kappa_{_1} - \kappa_{_2})\,A_{_1} = 1$ respectively. After introducing the new parameters $\kappa = (\kappa_{_1} - \kappa_{_2})/2$ and $\tau = (\kappa_{_1} + \kappa_{_2})/2$, we can write the solution as follows

$$L(x) = x^\tau\,\frac{x^\kappa - x^{-\kappa}}{2\,\kappa}. \tag{A.8}$$

The condition $L(0^+) < 0$ implies $\tau \leq |\kappa|$. From the characteristic equations we obtain the system $1 \pm \tau + \kappa = \lambda\,\alpha^{-\tau\mp\kappa}$, which can be solved to determine the two constants α and λ. The constant η follows from equations (A.3) and (A.8). We obtain

$$\alpha = \left(\frac{1+\tau-\kappa}{1+\tau+\kappa}\right)^{\frac{1}{2\kappa}}, \tag{A.9}$$

$$\lambda = \frac{\left(1+\tau-\kappa\right)^{\frac{\tau+\kappa}{2\kappa}}}{\left(1+\tau+\kappa\right)^{\frac{\tau-\kappa}{2\kappa}}}, \tag{A.10}$$

$$\eta = (\lambda - 1)\,\frac{\varsigma^{\kappa} - \varsigma^{-\kappa}}{(\kappa+\tau)\varsigma^{\kappa} + (\kappa-\tau)\varsigma^{-\kappa}}, \tag{A.11}$$

with $\tau > |\kappa| - 1$. This latter condition guarantees that $0^+ L(0^+) = 0$.

Case (b) corresponds to the limit of case (a) $\kappa_1 \to \kappa_2$ producing $L(x) = x^{\tau}\ln x$. Case (c) gives the trivial solution $L(x) = a\,x^b$ with $b = \lambda\,\alpha\,s - 1$ which can not be used to define a generalized logarithm.

The most general expression of $L(x)$ is given by equation (A.8). Finally, after taking into account equation (A.2) we can write the explicit form of the generalized logarithm $\Lambda(x)$ depending on the three parameters $\{\kappa, \tau, \varsigma\}$:

$$\Lambda(x) = \frac{\varsigma^{\kappa}\,x^{\tau+\kappa} - \varsigma^{-\kappa}\,x^{\tau-\kappa} - \varsigma^{\kappa} + \varsigma^{-\kappa}}{(\kappa+\tau)\varsigma^{\kappa} + (\kappa-\tau)\varsigma^{-\kappa}}. \tag{A.12}$$

The conditions $\Lambda(0^+) < 0$ and $0^+\Lambda(0^+) = 0$ imply that $\tau \leq |\kappa|$ and $\tau > |\kappa| - 1$ respectively.

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