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# Checking the consistency of the solution in ordinal semi-democratic decision-making problems ☆

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ABSTRACT

An interesting decision-making problem is that aggregating multi-agent preference orderings into a consensus ordering, in the case the agents' importance is expressed in the form of a rank-ordering. Due to the specificity of the problem, the scientific literature encompasses a relatively small number of aggregation techniques. For the aggregation to be effective, it is important that the consensus ordering well reflects the input data, *i.e.*, the agents' preference orderings and importance rank-ordering.

The aim of this paper is introducing a new quantitative tool – represented by the so-called $p$ indicators – which allows to check the degree of consistency between consensus ordering and input data, from several perspectives. This tool is independent from the aggregation technique in use and applicable to a wide variety of practical contexts, *e.g.*, problems in which preference orderings include omissions and/or incomparabilities between some alternatives. Also, the $p$ indicators are simple, intuitive and practical for comparing the results obtained from different techniques. The description is supported by various application examples.



## 1. Introduction

A relatively little discussed decision-making problem is that of aggregating multi-agent preference orderings into a consensus ordering, in the specific case in which the agents' importance is expressed in the form of a rank-ordering. The problem of interest is characterized by the following elements:

- A set of alternatives to be prioritized ($a$, $b$, $c$, $d$, $e$, *etc.*).
- A set of $m$ decision-making agents[1] ($D_1, D_2, \ldots, D_m$) expressing their opinion on the alternatives, through preference orderings (*e.g.*, $a > [(b \sim c) \| d] > e > \ldots$, where symbols "$>$", "$\sim$" and "$\|$" respectively mean "strictly preferred to", "indifferent to" and "incomparable to").
- An importance hierarchy of the agents, which is expressed through a *linear* rank-ordering (*e.g.*, $D_1 > D_2 > (D_3 \sim D_4) > \ldots$) and not through a set of weights, as in most of the decision-making problems [19,25,26];
- A *consensus*[2] *ordering* of the alternatives, which represents the solution of the problem.

Franceschini et al. [13] classified this specific problem as *ordinal semi-democratic*; the adjective *semi-democratic* indicates that agents do not necessarily have the same importance, while *ordinal* indicates that their rank is defined by a linear ordering [20]. This problem is potentially adaptable to a large number of practical contexts, in which the agents' importance prioritization is dubious and controversial: in these situations, the formulation of a rank-ordering is certainly simpler and more intuitive than that of a set of weights defined on a *ratio* scale [5,14,17]. Possible examples are

- Management decision problems in which agents are the members of the management board of a company/organization and their importance reflects the relevant hierarchical level (*e.g.*, CEO, general manager(s), operations manager(s), office manager(s), *etc.*).
- Marketing decision problems in which agents are respondents to questionnaires/interviews and their importance reflects the relevant level of education (*e.g.*, Ph.D., M.Sc., B.Sc., high school, *etc.*);
- Competitions for academic positions in which agents are the members of committees and their importance reflects the relevant academic position (*e.g.*, full professor, associate professor, assistant professor, *etc.*).

The problem of aggregating preference orderings, when there is no agents' importance hierarchy (*fully democratic* case) or it is expressed through a set of weights, is quite old and has been studied in various fields, stimulating the development of a variety

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[1] By a "decision-making agent", we will consider any of a wide variety of different types of entities; examples could be human beings, individual criteria in a multicriteria decision-making process, intelligent entities in the field of artificial intelligence, etc.

[2] The adjective "consensus" means that this ordering should reflect the whole preference orderings as much as possible, even in the inevitable presence of divergences.

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of solution techniques [10,11,15]. For example, in the field of *social choice* and *voting theory*, we recall the pioneering method by Condorcet [6] and that by Borda [2], while, in the field of *multi-criteria decision making*, the Electre [9], Promethee [4,8], AHP [22] or TOPSIS [1] methods.

On the other hand, the *ordinal semi-democratic* problem has so far received relatively little attention, probably due to its specificity; we recall the contribution of Yager [24], proposing a practical aggregation technique (hereafter denominated as "Yager's algorithm", abbreviated as YA), and the contributions of Wang [27] and Franceschini et al. [13], presenting two ameliorative variants of the YA.

These aggregation techniques, and maybe those that will be proposed in the future, have their *pro* and *contra*. For this reason, an interesting question is: for a generic ordinal semi-democratic decision-making problem, how could we identify the best aggregation technique? We are aware that it is probably impossible to answer this question rigorously, since the "true" solution for a generic problem is not known *a priori* [9,7,29]. Nevertheless, the performance of different aggregation techniques maybe assessed, at least roughly, according to various aspects, such as

- The ability to produce a solution, which is consistent with the input data;
- The adaptability to a variety of input data, *e.g.*, preference orderings including omissions and/or incomparabilities between alternatives;
- The efficiency in using the input data for constructing the consensus ordering; *e.g.*, an algorithm that focuses on the lower/upper part of the preference orderings only or an algorithm that ignores the preference orderings of certain agents cannot be considered as very efficient.
- Computational complexity.

Among the aspects above, that concerning the consistency of the solution is particularly important. The argument of consistency has been used by Wang [27] and Franceschini et al. [13] to prove, at least at a conceptual level, the superiority of their variants with respect to the YA. In this context, *consistency* is defined as the ability of a solution to reflect the agents' preference orderings, while reflecting their importance hierarchy, *i.e.*, giving priority to the more important agents.

In the scientific literature, various tools for consistency checking have been proposed. A common feature is that they use some measures of correlation/similarity to compare the consensus ordering with the agents' preference orderings [21,28]. For example, popular statistics are the Kendall's tau, the Spearman's rho, Spearman's footrule, and Cayley's distance; see Ref. [18] for an overview. However, the application range of these tools may be limited by several aspects, such as

- The degree of "completeness" of the preference orderings; for example, many techniques are not easily applicable when some alternatives are tied, omitted or incomparable between each other [3].
- The form in which the importance hierarchy of the agents is expressed.

The aim of this paper is to provide a simple and practical tool to check the degree of consistency between the consensus ordering and the input data, for specific ordinal semi-democratic decision-making problems. The proposed tool enables two types of consistency evaluations

- At a *local* level, by comparing the consensus ordering with the preference ordering of each *j*-th agent.
- At a *global* level, by comparing the consensus ordering with the whole set of preference orderings, taking into account the agents' importance rank-ordering, under the assumption that the most important agents should have a predominant influence on the construction of the consensus ordering.

The consistency verification is performed through the so-called *p* indicators, as we will show later in the paper. The remainder of the paper is organized in two sections. Section 2 introduces the *p* indicators, focusing on their construction and practical use. The description is supported by several examples. Section 3 summarizes the original contributions of this research, focusing on its implications, limitations and possible future developments.

## 2. The *p* indicators

Before getting into the discussion of *p* indicators, we anticipate that they are virtually applicable to every aggregation technique, since they are obtained by comparing the paired-comparison relationships derived from one agent's preference orderings with those derived from the consensus ordering. The decision of using paired-comparison relationships is motivated by several reasons

1. They allow to express the preference between two alternatives in a natural and intuitive way.
2. They can be derived from (preference and consensus) orderings, even if some alternatives are tied, omitted or incomparable between each others. For the purpose of example, Fig. 1 illustrates the transformation of a fictitious *partial* ordering, with one omitted alternative (*d*) and two incomparable alternatives (*a* and *e*), into paired-comparison relationships [20].
3. They could also be derived from agents' judgements expressed in other forms (*e.g.*, measurements/evaluations on ordinal/interval/ratio scales), as long as they admit relationships of order among the alternatives.

The remainder of this section is divided into two sub-sections: Section 2.1 provides a general description of the *p* indicators, with an application example, and Section 2.2 illustrates the use of *p* indicators, for comparing the results provided by two different aggregation techniques, when applied to the same problem.

### 2.1. General description

Table 1 presents a summary scheme of the proposed indicators. The $p_j$ indicators, associated with each *j*-th of the total *m* agents, allow to assess the consistency at a *local* level, while indicators $p_A$ and $p_B$ – in turn aggregated into $p_O$ – allow to assess the consistency at a *global* level; the combination of all these indicators enables a structured evaluation of the degree of consistency between consensus ordering and input data, in a generic ordinal semi-democratic decision-making problem. These two types of indicators are defined and described in Sections 2.1.1 and 2.1.2, respectively.

#### 2.1.1. $p_j$ indicators

A preliminary operation for determining the $p_j$ indicators is constructing a table, which contains the paired-comparison relationships obtained from the agents' preference orderings and the consensus ordering. For the purpose of example, let us consider the fictitious decision-making problem in Table 2, in which $m=4$ agents formulate their preference orderings concerning $n=5$ alternatives (*a*–*e*); the importance hierarchy of agents is expressed in the form of the rank-ordering $D_1 > D_2 > (D_3 \sim D_4)$. Incidentally, the preference ordering by $D_4$ is the same (partial) ordering

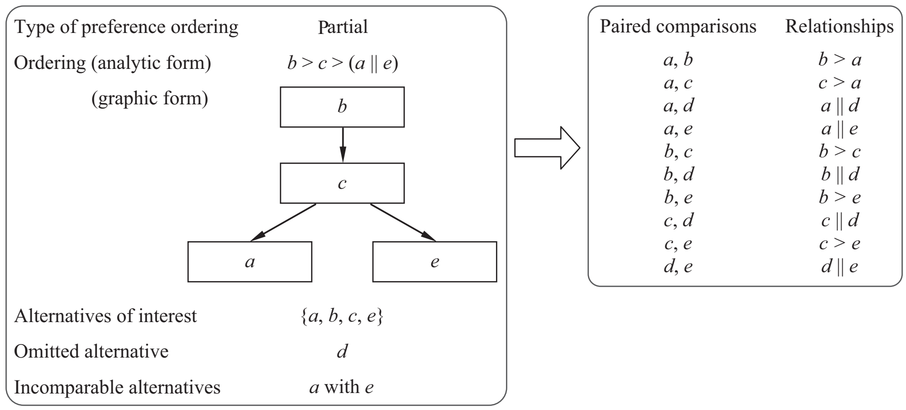


**Fig. 1.** Transformation of a fictitious (partial) preference ordering into paired-comparison relationships. Symbols " > ", "∼" and "||" respectively mean "strictly preferred to", "indifferent to" and "incomparable to".

represented in Fig. 1. Through some aggregation technique (no matter what), we assume that agents' preference orderings are aggregated into the consensus ordering: $d > a > b > c > e$. At first glance, this consensus ordering seems rather consistent with the agents' preference orderings, since the alternatives $a$ and $d$ generally figure in the top positions, $b$ in the intermediate positions, and $c$ and $e$ in the bottom positions of the preference orderings. Nevertheless, it is unclear whether the agents' importance hierarchy has been taken into account adequately. We will try to clarify this issue by defining suitable quantitative indicators.

Agents' preference orderings are transformed into the four sets of paired-comparison relationships reported in Table 3(a). Likewise agents' preference orderings, the consensus ordering is transformed into the set of paired-comparison relationships reported in the last column of Table 3(a).

Each $j$-th agent is associated with an indicator ($c_j$) corresponding to the number of paired comparisons, which are usable for evaluating the compatibility between the $j$-th preference ordering and the consensus ordering. Conventionally, we classify as *usable* a paired comparison not producing any relationship of incomparability ("||"), neither in the agent's preference orderings, nor in the consensus ordering, but only relationships of strict preference (" > ") or indifference ("∼"). Obviously, $c_j \le C_2^n \quad \forall j$-th agent, being $C_2^n = n \cdot (n-1)/2$ the total number of paired–comparisons for $n$ generic alternatives (*e.g.*, 10 in this specific example, since $n=5$).

Subsequently, we construct a "consistency table" (in Table 3(b)), which turns the paired-comparison relationships of each agent into scores, according to the scoring system in Table 4. The conventional choice of assigning 0.5 points in the case of *weak consistency* is justified by the fact that this is the intermediate case between that one of *full consistency* (with score 1) and that of *full inconsistency* (with score 0). The consistency table also reports the sum of the scores ($x_j$) relating to each $j$-th agent.

Next, for each $j$-th agent, the portion of consistent paired–comparisons can be calculated as

$$p_j = \frac{x_j}{c_j} \tag{1}$$

being $x_j$ the total score related to the $j$-th agent; $c_j$ the number of usable paired comparisons related to the $j$-th agent.

It can be noticed that the non-usable paired comparisons do not influence the evaluation of consistency, since they have no contribution neither in the $x_j$ nor in the $c_j$ terms.

Similarly to other existing measures – *e.g.*, the Kendall's tau ($\tau_j$) and the Spearman's rho ($\rho_j$)–$p_j$ can be interpreted as an indicator of correlation between pairs of orderings [16,23]. Curiously, in the case of orderings without tied, omitted or incomparable alternatives, $\tau_j$ and $p_j$ are linearly related, as shown below

$$\tau_j = \frac{\text{no. of concordant pairs} - \text{no. of discordant pairs}}{\text{total no. of pair combinations}} = \frac{x_j - (C_2^n - x_j)}{C_2^n} = 2 \cdot p_j - 1 \tag{2}$$

In this sense, $p_j$ can be considered as a variant of $\tau_j$. Two advantages of $p_j$ are: (i) the simplicity in the calculation, even for orderings with tied, omitted or incomparable alternatives, and (ii) the immediate meaning.

The relatively high values (*i.e.*, $\ge 80\%$) of the $p_j$ indicators related to $D_1$, $D_2$ and $D_3$ indicate that, in this case, the aggregation produces a consensus ordering reflecting the agents' preference orderings quite well. The preference ordering by $D_4$ is not very consistent with the consensus ordering, as depicted by the relatively low value of $p_4 = 60\%$; however this inconsistency is justified by the fact that (i) $D_4$ is the least important agent (along with $D_3$) and (ii) the corresponding preference ordering provides a limited number of usable paired comparisons (*i.e.*, $c_4 = 5$) for the consistency assessment.

Moreover, the consensus ordering seems to reflect the agents' importance hierarchy relatively well, since $p_j$ values tend to decrease as the agents' importance decreases; this means that the more important the agents, the higher the consistency between consensus ordering and preference orderings. Section 2.1.2 will go into this issue.

#### 2.1.2. $p_A$, $p_B$ and $p_O$

A rough indication of the overall compatibility between the consensus ordering and the whole set of preference orderings is given by $p_A$, defined as the ratio of $\Sigma x_j$ (*i.e.*, the sum of the total scores $x_j$) and $\Sigma c_j$ (*i.e.*, the total number of usable paired comparisons)

$$p_A = \frac{\sum_{j=1}^{m} x_j}{\sum_{j=1}^{m} c_j} = \sum_{j=1}^{m} \left( \frac{x_j}{c_j} \cdot \frac{c_j}{\sum_{j=1}^{m} c_j} \right) = \sum_{j=1}^{m} \left( p_j \cdot \frac{c_j}{\sum_{j=1}^{m} c_j} \right) \tag{3}$$

The fourth term in Eq. 3 shows that $p_A$ can also be interpreted as a weighted average of the $p_j$ values, with respect to the corresponding $c_j$ values. By applying Eq. 3 to the data in Table 3(b) we obtain $p_A = 83.8\%$. We remark that $p_A$ can be used to complement the fragmented information from the $p_j$ indicators, but it does not take account of the agents' importance hierarchy.

The degree of consistency with the agents' importance hierarchy can be evaluated through $p_B$, *i.e.*, a second global indicator, based on the assumption that the preference orderings of the more important agents should influence the consensus ordering more

**Table 1**
Summary scheme of the *p* indicators.

| Type of evaluation | Indicator(s) | Short description |
|---|---|---|
| Local consistency | $p_j$ | Indicators depicting the degree of consistency between the consensus ordering and the preference ordering by each *j*-th of the total *m* agents ($j=1, 2, \ldots, m$). |
| Global consistency | $p_A$ | Indicator representing the degree of consistency between the consensus ordering and the whole set of preference orderings, regardless of the agents' importance rank-ordering. |
| | $p_B$ | Indicator representing the consistency between the $p_j$ indicators and the agents' importance rank-ordering, under the assumption that the preference orderings by the most important agents should have a predominant influence on the construction of the consensus ordering. |
| | $p_O$ | Overall synthesis indicator, which aggregates $p_A$ and $p_B$. |

**Table 2**
Input and output data of a fictitious decision-making problem concerning the aggregation of multi-agent preference orderings of five alternatives (*a*–*e*) into a consensus ordering.

| (a) Input | | | | | | (b) Output |
|---|---|---|---|---|---|---|
| Agent | Type of ordering | Preference ordering | Omitted alternatives | Importance rank-ordering | → | Consensus ordering |
| $D_1$ | Linear | $(a\sim d)>b>c>e$ | None | $D_1>D_2>(D_3\sim D_4)$ | | $d>a>b>c>e$ |
| $D_2$ | Partial | $d>(a\|\|b)>e>c$ | None | | | |
| $D_3$ | Linear | $a>(b\sim d)>(c\sim e)$ | None | | | |
| $D_4$ | Partial | $b>c>(a\|\|e)$ | D | | | |

than the other ones; therefore they should have higher $p_j$ values. Following this reasoning, we compare the importance rank-ordering of the agents (*i.e.*, $D_1>D_2>(D_3\sim D_4)$) with the ordering based on the relevant $p_j$ values. Since $p_1>p_2>p_3>p_4$, the related agents' importance rank-ordering is $D_1>D_2>D_3>D_4$.

The comparison between the two orderings is carried out by repeating the exercise seen in Section 2.1.1 when constructing the $p_j$ indicators, namely: (i) decomposing the two orderings into paired comparison relationships, (ii) comparing them and assigning some scores according to the conventions in Table 5[3], (iii) determining the sum of the $x_B$ scores, and (iv) constructing $p_B$, defined as

$$p_B=\frac{x_B}{c_B} \quad (4)$$

being $c_B=C_2^m$ the number of usable paired comparisons (*i.e.*, 6 in this case). We observe that both the orderings are linear and do not include any incomparable alternatives. Table 6 shows the paired comparison data relating to the two orderings and the so-called *consistency string*[4], which contains the corresponding scores. For the problem exemplified, the value of $p_B=100\%$ indicates excellent compatibility between the consensus ordering and the $p_j$ ordering.

Pushing to the extreme the synthesis, the two complementary indicators $p_A$ and $p_B$ can be synthesized into an overall indicator

$$p_O = min(p_A, p_B) \quad (5)$$

The synthesis by the *min*() operator allows condensing the results of the consistency analysis into a single number. It is implicitly assumed that the aspects described by $p_A$ and $p_B$ are equally important.

### 2.2. Comparisons between different solutions

This section illustrates the use of the *p* indicators for comparing the solutions obtained from different techniques, which are applied to the same ordinal semi-democratic decision-making problem. Let us consider a fictitious problem, in which five different alternatives (*a*–*e*) are examined. These alternatives are evaluated by three agents ($D_1$ to $D_3$), ranked in terms of importance, according to the linear ordering $D_1>D_2>D_3$. Table 7(a) summarizes the input data of the problem.

The agents' preference orderings are aggregated into a single consensus ordering through two different aggregation techniques: (i) the YA and (ii) Enhanced Yager's Algorithm, hereafter abbreviated as EYA. These techniques generate the two consensus orderings reported in Table 7(b). For more information on the YA and the EYA, we refer the reader to Yager [24] and Franceschini et al. [13], respectively.

In spite of being originated from the same input data, the two consensus orderings are deeply different from each other, except for the fact that alternative *b* is in the second position and *e* is in the last position for both of them. Which consensus ordering better reflects the input data?

At a qualitative level, the second one is probably better than the first one, as it is not so dissimilar from the preference orderings of the two most important agents (*i.e.*, $D_1$ and $D_2$). At a quantitative level, the *local consistency* can be assessed through the $p_j$ indicators. To facilitate the construction of these indicators, preference orderings are turned into paired-comparison data (see the first four columns in Table 8). In this specific case, the preference orderings are all *linear* (and not *partial*), therefore they will not include any omissions or incomparabilities between the alternatives. Likewise the preference

[3] Two are the differences between the scoring system in Table 5 and that in Table 4. The first is that the new scoring system does not include the possibility of incomparability, since the $p_j$ ordering and the agents' importance rank-ordering are both *linear* and do not include any incomparability relationship [20]. The second is that the new scoring system is more "indulgent" than that in Table 4, since it assigns a unitary score even the case of weak consistency (*i.e.*, consistency with respect to the relationship "$>$ or $\sim$"). This choice is justified by the fact that it is very unlikely to expect a strict correspondence between relationships of indifference, in both the importance rank-ordering and the $p_j$ ordering; in fact, two $p_j$ values are considered indifferent only if they coincide exactly. Moreover, the adoption of the scoring system in Table 4 could lead to paradoxical situations, such as the following one: three agents formulate three preference orderings, which are completely consistent with the consensus ordering, *i.e.*, $p_1=p_2=p_3=100\%$ (this would be the extreme case where all the preference orderings and the consensus ordering coincide), and the agents' importance rank-ordering is $D_1>D_2>D_3$. Applying the scoring system in Table 4, we would obtain $p_B$=50%, which is an unduly penalizing result for this situation of ideal consistency; on the other hand, when applying the scoring system in Table 5, we would obtain $p_B$=100%, eliminating the paradox.

[4] The expression "consistency string" is used to avoid confusion with the "consistency table", defined in Section 2.1.1.

**Table 3**
(a) Table of paired-comparison data relating to the agents' preference orderings and the consensus ordering in Table 2. (b) Consistency table relating to the previous orderings; scores are assigned according to the scoring system in Table 4.

(a)

| Paired comparison | From agents' preference orderings | | | | From consensus ordering |
|---|---|---|---|---|---|
| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | |
| $a$, $b$ | $a>b$ | $a\|b$ | $a>b$ | $b>a$ | $a>b$ |
| $a$, $c$ | $a>c$ | $a>c$ | $a>c$ | $c>a$ | $a>c$ |
| $a$, $d$ | $a\sim d$ | $d>a$ | $a>d$ | $a\|d$ | $d>a$ |
| $a$, $e$ | $a>e$ | $a>e$ | $a>e$ | $a\|e$ | $a>e$ |
| $b$, $c$ | $b>c$ | $b>c$ | $b>c$ | $b>c$ | $b>c$ |
| $b$, $d$ | $d>b$ | $d>b$ | $b\sim d$ | $b\|d$ | $d>b$ |
| $b$, $e$ | $b>e$ | $b>e$ | $b>e$ | $b>e$ | $b>e$ |
| $c$, $d$ | $d>c$ | $d>c$ | $d>c$ | $c\|d$ | $d>c$ |
| $c$, $e$ | $c>e$ | $e>c$ | $c\sim e$ | $c>e$ | $c>e$ |
| $d$, $e$ | $d>e$ | $d>e$ | $d>e$ | $d\|e$ | $d>e$ |

(b)

| Paired comparison | Scores | | | | |
|---|---|---|---|---|---|
| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | |
| $a$, $b$ | 1 | N/A | 1 | 0 | |
| $a$, $c$ | 1 | 1 | 1 | 0 | |
| $a$, $d$ | 0.5 | 1 | 0 | N/A | |
| $a$, $e$ | 1 | 1 | 1 | N/A | |
| $b$, $c$ | 1 | 1 | 1 | 1 | |
| $b$, $d$ | 1 | 1 | 0.5 | N/A | |
| $b$, $e$ | 1 | 1 | 1 | 1 | |
| $c$, $d$ | 1 | 1 | 1 | N/A | |
| $c$, $e$ | 1 | 0 | 0.5 | 1 | |
| $d$, $e$ | 1 | 1 | 1 | N/A | |
| $c_j$ | 10 | 9 | 10 | 5 | |
| $x_j$ | 9.5 | 8 | 8 | 3 | |
| $p_j$ | 95% | 88.9% | 80% | 60% | $p_A=$ 83.8% |

$c_j$ is the number of "usable" paired-comparisons; $x_j$ is total score of the $j$-th agent; $p_j$ is the ratio $x_j/c_j$.

**Table 4**
Scoring system used in the construction of the "consistency table".

| Case | Score |
|---|---|
| 1. *Full consistency*, *i.e.*, identical relationship of strict preference ("$>$") or indifference ("$\sim$"). | 1 |
| 2. *Weak consistency*, *i.e.*, consistency with respect to the weak preference relationships only ("$>$ or $\sim$", *i.e.*, strict preference or indifference); *e.g.*, when comparing the relationship $a>b$ with $a\sim b$. | 0.5 |
| 3. *Full inconsistency* (with respect to both strict and weak preference relationships); *e.g.*, when comparing the relationship $a>b$ with $b>a$. | 0 |
| 4. *Incomparability* between the two alternatives, in the agents' preference orderings and/or in the fused ordering. | N/A |

**Table 5**
Conventional scores used for constructing the "consistency string", when comparing the paired-comparison relationships from the $p_j$ ordering with those from the agents' importance hierarchy.

| Case | Score |
|---|---|
| 1. *Full* or *weak consistency*, *i.e.*, consistency with respect to strict preference or indifference relationship ("$>$ or $\sim$"); *e.g.*, when comparing the relationship $D_1>D_2$ with itself or with $D_1\sim D_2$. | 1 |
| 2. *Full inconsistency* (with respect to strict preference or indifference relationships); *e.g.*, when comparing the relationship $D_1>D_2$ with $D_2>D_1$. | 0 |

orderings, the two consensus orderings are turned into paired-comparison data (see the last two columns of Table 8).

In Sections 2.2.1 and 2.2.2, we construct the $p$ indicators relating to the solution from the YA and the EYA respectively; Section 2.2.3 provides a comparison between $p_j$, which is the "elementary block" of the $p$ indicators, and other popular correlation measures; in Section 2.2.4, we represent the $p$ indicators by means of the so-called $p$-diagram.

### 2.2.1. Case of the YA

Table 9 contains the consistency table concerning the consensus ordering of the YA (compare it with the data in Table 8). Due to the absence of omitted and incomparable alternatives in the preference orderings and consensus ordering, the total number of usable paired comparisons will be $c_j=C_2^{n=5}=10 \quad \forall$ $j$-th agent.

The $p_j$ values related to the three agents are reported at the bottom of Table 9. The fact that they are relatively low (*i.e.*, $\leq 75\%$) denotes a relatively poor consistency between consensus ordering

**Table 6**
(a) Paired-comparison data relating to the $p_j$ ordering (*i.e.*, $D_1>D_2>D_3>D_4$) and the agents' importance rank-ordering (*i.e.*, $D_1>D_2>(D_3\sim D_4)$). (b) Consistency string relating to the same orderings; scores are assigned according to the scoring system in Table 5.

(a)

| Paired comparison | From agents' importance rank-ordering | From the $p_j$ ordering |
|---|---|---|
| $D_1$, $D_2$ | $D_1>D_2$ | $D_1>D_2$ |
| $D_1$, $D_3$ | $D_1>D_3$ | $D_1>D_3$ |
| $D_1$, $D_4$ | $D_1>D_4$ | $D_1>D_4$ |
| $D_2$, $D_3$ | $D_2>D_3$ | $D_2>D_3$ |
| $D_2$, $D_4$ | $D_2>D_4$ | $D_2>D_4$ |
| $D_3$, $D_4$ | $D_3\sim D_4$ | $D_3>D_4$ |

(b)

| Paired comparison | Scores |
|---|---|
| $D_1$, $D_2$ | 1 |
| $D_1$, $D_3$ | 1 |
| $D_1$, $D_4$ | 1 |
| $D_2$, $D_3$ | 1 |
| $D_2$, $D_4$ | 1 |
| $D_3$, $D_4$ | 1 |
| $c_B$ | 6 |
| $x_B$ | 6 |
| $p_B$ | 100% |

$c_B$ is the number of "usable" paired-comparisons; $x_B$ is the relevant total score; $p_B$ is the ratio $x_B/c_B$.

**Table 7**
Fictitious decision-making problem, concerning the aggregation of three preference orderings of five alternatives into a consensus ordering. The two consensus orderings in the last column are obtained by applying the Yager's algorithm (YA) and the Enhanced Yager's Algorithm (EYA), respectively.

| (a) Input | | | | | (b) Output | |
|---|---|---|---|---|---|---|
| Agent | Type of ordering | Preference ordering | Importance rank-ordering | → | Consensus orderings | |
| $D_1$ | Linear | $c>a>b>d>e$ | $D_1>D_2>D_3$ | | YA: | $a>b>d>c>e$ |
| $D_2$ | Linear | $c>b>(a\sim d)>e$ | | | EYA: | $c>b>a>d>e$ |
| $D_3$ | Linear | $b>(a\sim d)>e>c$ | | | | |

**Table 8**
Paired-comparison data, related to the three agents' preference orderings and the two consensus orderings in Table 7.

| Paired comparison | From agents' preference orderings | | | From consensus orderings | |
|---|---|---|---|---|---|
| | $D_1$ | $D_2$ | $D_3$ | YA | EYA |
| $a, b$ | $a>b$ | $b>a$ | $b>a$ | $a>b$ | $b>a$ |
| $a, c$ | $c>a$ | $c>a$ | $a>c$ | $a>c$ | $c>a$ |
| $a, d$ | $a>d$ | $a\sim d$ | $a\sim d$ | $a>d$ | $a>d$ |
| $a, e$ | $a>e$ | $a>e$ | $a>e$ | $a>e$ | $a>e$ |
| $b, c$ | $c>b$ | $c>b$ | $b>c$ | $b>c$ | $c>b$ |
| $b, d$ | $b>d$ | $b>d$ | $b>d$ | $b>d$ | $b>d$ |
| $b, e$ | $b>e$ | $b>e$ | $b>e$ | $b>e$ | $b>e$ |
| $c, d$ | $c>d$ | $c>d$ | $d>c$ | $d>c$ | $c>d$ |
| $c, e$ | $c>e$ | $c>e$ | $e>c$ | $c>e$ | $c>e$ |
| $d, e$ | $d>e$ | $d>e$ | $d>e$ | $d>e$ | $d>e$ |

**Table 9**
Consistency table concerning the solution provided by the YA and the preference orderings in Table 8; scores are assigned according to the scoring system in Table 4.

| Paired comparison | Scores | | | |
|---|---|---|---|---|
| | $D_1$ | $D_2$ | $D_3$ | |
| $a, b$ | 1 | 0 | 0 | |
| $a, c$ | 0 | 0 | 1 | |
| $a, d$ | 1 | 0.5 | 0.5 | |
| $a, e$ | 1 | 1 | 1 | |
| $b, c$ | 0 | 0 | 1 | |
| $b, d$ | 1 | 1 | 1 | |
| $b, e$ | 1 | 1 | 1 | |
| $c, d$ | 0 | 0 | 1 | |
| $c, e$ | 1 | 1 | 0 | |
| $d, e$ | 1 | 1 | 1 | |
| $c_j$ | 10 | 10 | 10 | |
| $x_j$ | 7 | 5.5 | 7.5 | |
| $p_j$ | 70.0% | 55.0% | 75.0% | $p_A=66.7\%$ |

and agents' preference orderings. This impression is confirmed by the relatively low value of the global indicator $p_A=66.7\%$, obtained by applying Eq. 3. Also, the agents' importance hierarchy is not very well reflected by the resulting $p_j$ ordering (see Table 10), as denoted by the low value of $p_B=33.3\%$. Consequently, $p_O=min(66.7\%, 33.3\%)=33.3\%$.

### 2.2.2. Case of the EYA

The consistency table concerning the consensus ordering by the EYA is shown in Table 11; it is significantly different from that in Table 9, due to several variations in the paired-comparison relationships. The $p_j$ values of the two most important agents are significantly higher than those obtained for the YA. The $p_A$ value (80.0%) is significantly higher than that in the case of the YA, denoting a significant improvement in terms of consistency between consensus ordering and the whole set of preference orderings. Consistency has also improved at the level of the agents' importance hierarchy, as the value of $p_B=66.7\%$ is twice that obtained for the YA (see the construction in Table 12). Consequently, $p_O=min(80.0\%, 66.7\%)=66.7\%$.

**Table 10**
(a) Paired-comparison data relating to the $p_j$ ordering and the agents' importance rank-ordering, in the case of the YA. (b) Consistency string relating to the same orderings; scores are assigned according to the scoring system in Table 5.

| (a) | | | (b) | |
|---|---|---|---|---|
| Paired comparison | From agents' importance rank-ordering | From the $p_j$ ordering | Paired comparison | Scores |
| $D_1, D_2$ | $D_1>D_2$ | $D_1>D_2$ | $D_1, D_2$ | 1 |
| $D_1, D_3$ | $D_1>D_3$ | $D_3>D_1$ | $D_1, D_3$ | 0 |
| $D_2, D_3$ | $D_2>D_3$ | $D_3>D_2$ | $D_2, D_3$ | 0 |
| | | | $c_B$ | 3 |
| | | | $x_B$ | 1 |
| | | | $p_B$ | 33.3% |

**Table 11**
Consistency table concerning the solution provided by the EYA and the preference orderings in Table 8; scores are assigned according to the scoring system in Table 4.

| Paired comparison | Scores | | | |
|---|---|---|---|---|
| | $D_1$ | $D_2$ | $D_3$ | |
| $a, b$ | 0 | 1 | 1 | |
| $a, c$ | 1 | 1 | 0 | |
| $a, d$ | 1 | 0.5 | 0.5 | |
| $a, e$ | 1 | 1 | 1 | |
| $b, c$ | 1 | 1 | 0 | |
| $b, d$ | 1 | 1 | 1 | |
| $b, e$ | 1 | 1 | 1 | |
| $c, d$ | 1 | 1 | 0 | |
| $c, e$ | 1 | 1 | 0 | |
| $d, e$ | 1 | 1 | 1 | |
| $c_j$ | 10 | 10 | 10 | |
| $x_j$ | 9 | 9.5 | 5.5 | |
| $p_j$ | 90.0% | 95.0% | 55.0% | $p_A=80\%$ |

**Table 12**
(a) Paired-comparison data relating to the $p_j$ ordering and the agents' importance rank-ordering, in the case of the EYA. (b) Consistency string relating to the same orderings; scores are assigned according to the scoring system in Table 5.

| (a) | | | (b) | |
|---|---|---|---|---|
| Paired comparison | From agents' importance rank-ordering | From the $p_j$ ordering | Paired comparison | Score |
| $D_1, D_2$ | $D_1>D_2$ | $D_2>D_1$ | $D_1, D_2$ | 0 |
| $D_1, D_3$ | $D_1>D_3$ | $D_1>D_3$ | $D_1, D_3$ | 1 |
| $D_2, D_3$ | $D_2>D_3$ | $D_2>D_3$ | $D_2, D_3$ | 1 |
| | | | $c_B$ | 3 |
| | | | $x_B$ | 2 |
| | | | $p_B$ | 66.7% |

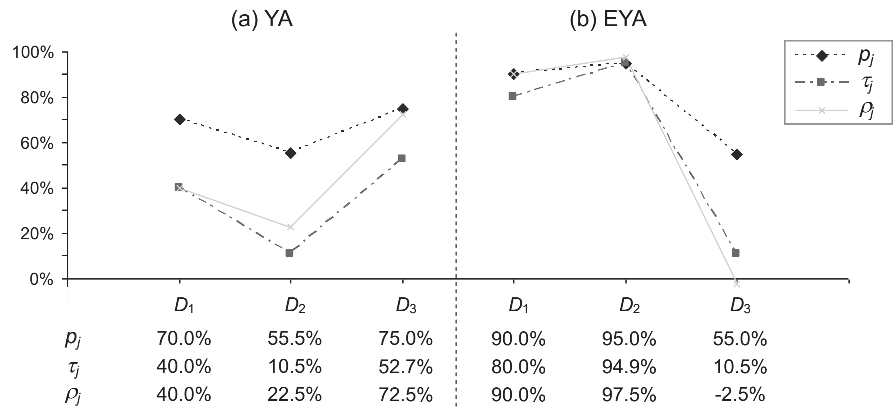


**Fig. 2.** Graphical representation of the $p_j$, $\tau_j$ and $\rho_j$ values, calculated for each combination of preference ordering and consensus ordering.

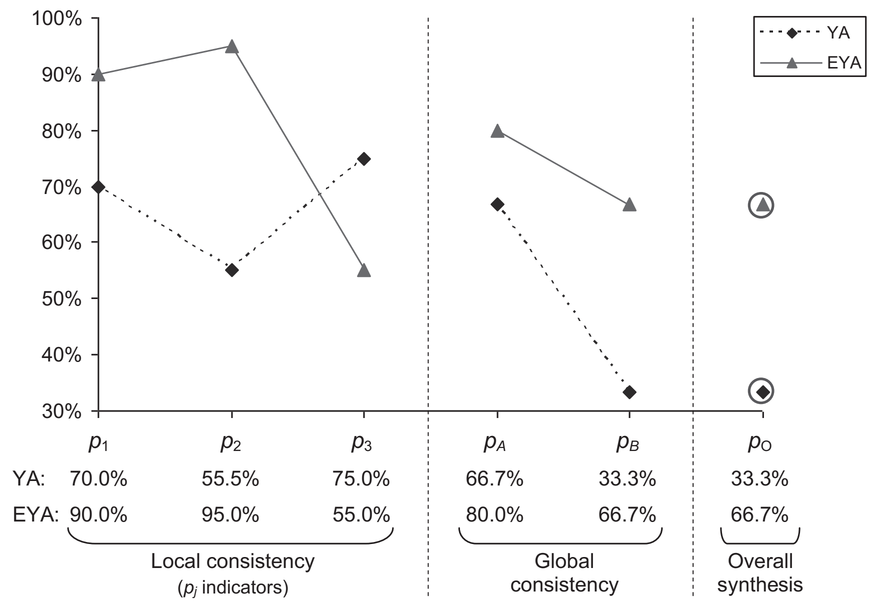


**Fig. 3.** $p_j$-diagram concerning the solutions obtained by the YA and the EYA, when applied to the ordinal semi-democratic decision-making problem in Table 7.

#### 2.2.3. *Comparison between $p_j$ and other correlation measures*

The $p_j$ indicators are the "elementary blocks" of the proposed tool. Their plausibility, as measures of consistency/correlation between a generic $j$-th preference ordering and the consensus ordering, can be assessed through a simple exercise. Since, in the problem examined, agents' preference orderings and consensus orderings are all linear, it is simple to calculate two popular correlation measures – *i.e.*, the Kendall's tau ($\tau_j$) and the Spearman's rho ($\rho_j$) – and compare them with the relevant $p_j$ values[5] [16,23].

Fig. 2 provides a graphical representation of the $p_j$, $\tau_j$ and $\rho_j$ values, calculated combining the three preference orderings related to $D_1$, $D_2$ and $D_3$, and the two consensus orderings obtained by applying the YA and the EYA. The calculation of $\tau_j$ and $\rho_j$ was performed using the relevant functions in the MINITAB® statistical software. Although the three correlation measures are based on different logics, there is a general agreement: in fact, the agent rank-orderings based on the $p_j$, $\tau_j$ and $\rho_j$ values are coincident – *i.e.*, $D_3 > D_1 > D_2$ for the YA and $D_2 > D_1 > D_3$ for the EYA.

The calculation of $p_j$, compared to that of $\tau_j$ and $\rho_j$, is more practical for *partial* preference orderings [12]; also, $p_j$ probably has a more immediate meaning.

[5] We remark that, while $p_j \in [0, 1]$, $\tau_j$ and $\rho_j \in [-1, 1]$.

#### 2.2.4. *Graphical representation through the p-diagram*

The so-called *p*-diagram, in Fig. 3, provides a practical "snapshot" of the comparative analysis between the solution from the YA and that from the EYA. The profile of the EYA solution is significantly higher than that of the YA solution, from almost all points of view, denoting a significantly higher consistency with input data.

## 3. Discussion

The *p* indicators represent a simple and intuitive tool for assessing the consistency between consensus ordering and agents' preference orderings, in ordinal semi-democratic decision-making problems. The $p_j$ indicators show the level of consistency between the solution and the individual preference orderings and can be aggregated into the $p_A$ indicator. The $p_B$ indicator complements the information provided by $p_A$, taking into account the agents' importance hierarchy. Next, $p_A$ and $p_B$ can be synthesized into the overall indicator $p_O$.

These indicators are very versatile since they can be applied in the presence of omissions and/or incomparabilities between alternatives, both in the agents' preference orderings and consensus ordering. Theoretically, they could be applied even in the case in which the agents' judgements and/or the solution of the problem were expressed in other forms – such as measurements/evaluations on ordinal/interval/ratio scales – as long as they could be transformed into paired-comparison relationships. Other advantages of $p$ indicators are (i) the immediate meaning and (ii) the relatively simple calculation.

It was also shown that the $p$ indicators may assist in comparing the solutions generated by two (or more) techniques, in a specific ordinal semi-democratic decision-making problem. To this end, the representation provided by the $p$-diagram is particularly useful and effective.

The $p$ indicators were presented as a *passive* tool for checking the consistency of the solution of one or more aggregation techniques, in a specific decision-making problem. Reversing this perspective, they could be used *actively*, *i.e.*, as parameters to maximize, for identifying the "optimal[6]" consensus ordering(s). A possible approach would be that one based on the maximization of $p_O$, *i.e.*, the overall synthesis indicator. In this case, the optimal consensus ordering – which is that one for which $p_O$ assumes the maximum possible value (*i.e.*, $p_O^{MAX}$) – may be determined by: (i) automatically generating all the possible solutions to a specific problem, through ad hoc software, (ii) determining the corresponding $p_O$ for each of them, and (iii) selecting the solution (s) for which $p_O = p_O^{MAX}$. Knowing the optimal consensus ordering (s) and the corresponding $p_O^{MAX}$ value may be useful for better assessing the performance of a certain technique, in a specific problem. For example, a technique that provides a solution with $p_O = 79\%$, in a decision-making problem where $p_O^{MAX} = 80\%$, will perform certainly better than a technique that provides a solution with $p_O = 80\%$, in a problem where $p_O^{MAX} = 98\%$. In fact, in problems where the agents' preference orderings have a high degree of similarity, the $p_O^{MAX}$ value will tend to increase (since it will be easier to find a consensus ordering, which is compatible with the whole set of agents' preference orderings and their importance hierarchy), while in others in which the differences are greater, it will tend to decrease (since it will be more complicated to find a consensus ordering, which is compatible with the whole set of agents' preference orderings and their importance hierarchy). In this sense, $p_O^{MAX}$ could be also seen as a measure of the degree of "dissimilarity" between the agents' preference orderings.

Future research will aim at deepening this aspect. Also, we will propose new tools for complementing the $p$ indicators and enriching the verification of the solutions generated by different aggregation techniques. Finally, we plan to generalize the $p$ indicators so that they are adaptable to the majority of decision-making problems, in which (i) agents can use different formats to express their preferences and (ii) their importance hierarchy is expressed through a set of weights.

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[6] The adjective "optimal" is related to the conventions adopted when defining the $p$ indicators.