Democracy and the role of minorities in Markov chain models

Non-reversible perturbations of Markov chains models

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Perturbation of dynamical networks

The stability of a complex large-scale dynamical network under localized perturbations is one of the paradigmatic problem of these decades.

Key issues:

- Correlation: understand how local perturbation affect the overall behavior.
- Resilience find bounds on the perturbation 'size' which the network can tolerate.
- Phase transitions

Perturbation of dynamical networks

State of the art:

- Most of the results available in the literature are on connectivity issues.
- Analysis of how the perturbation is altering the degree distribution of the network.
- Degrees are in general not sufficient to study dynamics.
- Example: non-reversible Markov chain models.

Perturbation of dynamical networks

What type of perturbations:

- Failures in nodes or links in sensor or computer networks. Sensor with different technical properties.
- Heterogeneity in opinion dynamics models: minorities, leaders exhibiting a different behavior
- A subset of control nodes in the network...

In this talk:

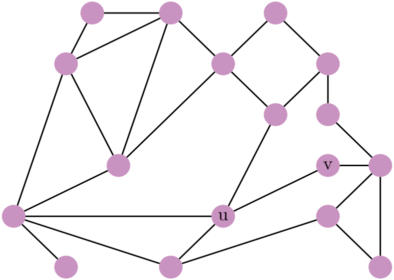
- Non-reversible perturbations of Markov chain models
- Applications to consensus dynamics

Outline

- Perturbation of consensus dynamics.
- The general setting: perturbation of Markov chain models.
- An example: heterogeneous gossip model.
- Results on how the perturbation is affecting the asymptotics.
- Conclusions and open issues.

Consensus dynamics

$G = (V, E)$ connected graph



y_v initial state (opinion) of node v

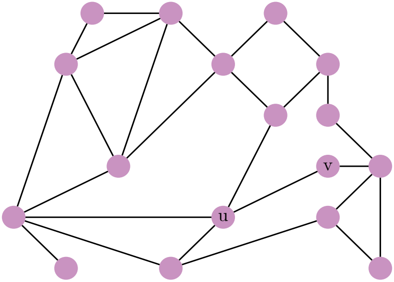
Dynamics: $y(t+1) = Py(t)$, $y(0) = y$
$P \in \mathbb{R}^{V \times V}$ stochastic matrix on G ($P_{uv} > 0 \Leftrightarrow (u, v) \in E$)

Consensus: $\lim_{t \to +\infty} (P^t y)_u = \pi^* y$ for all u ($*$ means transpose)

$\pi \in \mathbb{R}_+^V$, $\pi^* P = \pi^*$, $\sum_u \pi_u = 1$ (invariant probability)

Consensus dynamics

$G = (V, E)$ connected graph



Example: (SRW) $P_{uv} = \frac{1}{d_u}$, d_u degree of node u

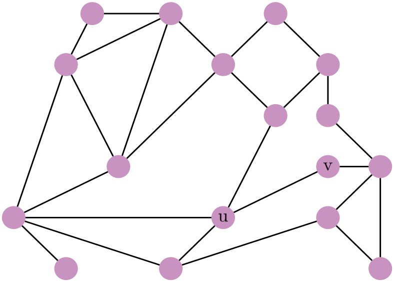
Explicit expression for π: $\pi_u = \frac{d_u}{2|E|}$

π essentially depends on local properties of G.

This holds true for general reversible Markov chains.

Consensus dynamics

$G = (V, E)$ connected graph



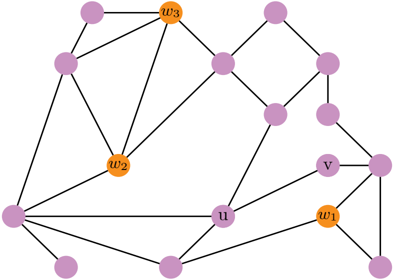
Dynamics: $y(t+1) = Py(t)$, $y(0) = y$
Consensus: $\lim_{t\to+\infty}(P^t y)_u = \pi^* y$ for all u

Two important parameters:

- the invariant probability π responsible for the asymptotics
- the mixing time τ responsible for the transient behavior (speed of convergence)

Perturbation of consensus dynamics

$G = (V, E)$ connected graph

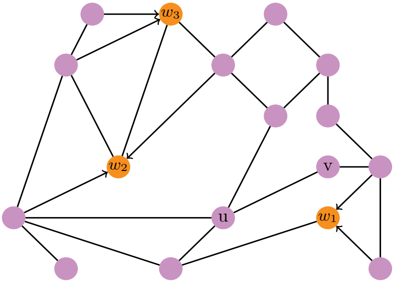


- $P \in \mathbb{R}^{V \times V}$ stochastic matrix on G
- Perturb P in a small set of nodes:

  $\tilde{P}_{uv} = P_{uv}$ if $u \notin W = \{w_1, w_2, w_3\}$.

Perturbation of consensus dynamics

$G = (V, E)$ connected graph



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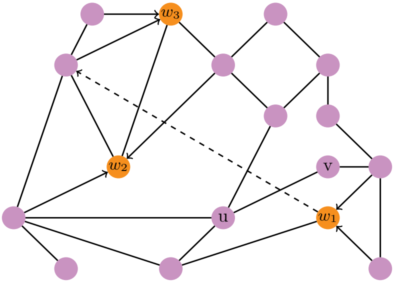
- Perturb P in a small set of nodes:

  $\tilde{P}_{uv} = P_{uv}$ if $u \notin W = \{w_1, w_2, w_3\}$.
- Cut edges

Perturbation of consensus dynamics

$G = (V, E)$ connected graph



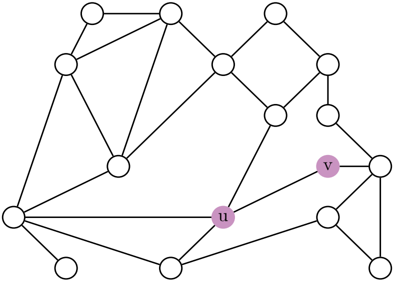
- $P \in \mathbb{R}^{V \times V}$ stochastic matrix on G
- Perturb P in a small set of nodes:

  $\tilde{P}_{uv} = P_{uv}$ if $u \notin W = \{w_1, w_2, w_3\}$.
- Cut edges. Add new edges.

A heterogeneous gossip model

(Acemoglu et al. 2009)

$G = (V, E)$, $W \subset V$ a minority of influent (stubborn) agents

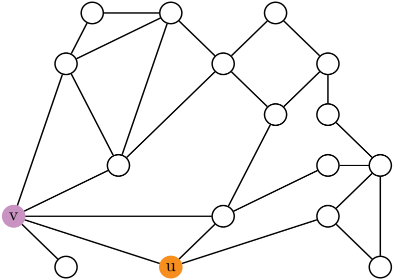


- At each time t choose an edge $\{u, v\}$ at random.
- If $u, v \in V \setminus W$,
  $y_u(t+1) = y_v(t+1) = (x_u(t) + x_v(t))/2$ (reg. interaction)

A heterogeneous gossip model

(Acemoglu et al. 2009)

$G = (V, E)$, $W \subset V$ a minority of influent (stubborn) agents



- At each time t choose an edge $\{u, v\}$ at random.
- If $u \in W$, $v \in V \setminus W$,
  - $y_u(t+1) = y_u(t),\ y_v(t+1) = \varepsilon y_v(t) + (1-\varepsilon) y_u(t)$ with probability p (forceful interaction)
  - $y_u(t+1) = y_v(t+1) = (x_u(t) + x_v(t))/2$ with probability $1-p$ (reg. interaction)

A heterogeneous gossip model

(Acemoglu et al. 2009)

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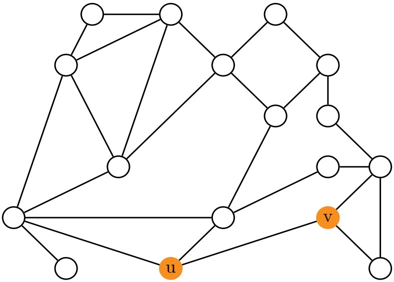


- At each time t choose an edge $\{u, v\}$ at random.
- If $u, v \in W$ nothing happens.

A heterogeneous gossip model

- $y(t+1) = P(t)y(t)$
- $y(t)$ converges to a consensus almost surely if $p \in [0,1)$.
  But what type of consensus?
- If no forceful interaction is present ($p = 0$),
  $y(t)_u \to N^{-1} \sum_v y(0)_v$ for every u.
- $\mathbb{E}(P(t)) = P_p$
- P_p and P_0 only differ in the rows having index in $W \cup \partial W$.

Perturbation of Markov chain models

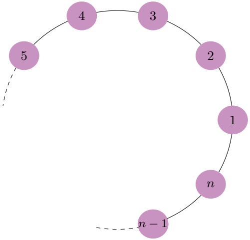
The abstract setting:

- $G = (V, E)$ family of connected graphs. $N = |V| \to +\infty$.
- $W \subseteq V$ perturbation set
- $P, \tilde{P}$ stochastic matrices on G. $\tilde{P}_{uv} = P_{uv}$ if $u \notin W$
- $\pi^* P = \pi^*$, $\tilde{\pi}^* \tilde{P} = \tilde{\pi}^*$.
- Study $||\pi - \tilde{\pi}||_{\mathrm{TV}} := \frac{1}{2} \sum_v |\pi_v - \tilde{\pi}_v|$ (as a function of N)

  Notice that $|\tilde{\pi}^* y - \pi^* y| \leq ||\pi - \tilde{\pi}||_{\mathrm{TV}} ||y||_\infty$

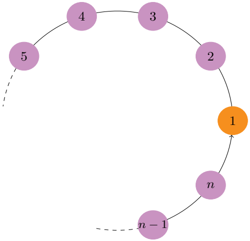
The ideal result: $\pi(W) \to 0 \;\Rightarrow\; ||\tilde{\pi} - \pi||_{\mathrm{TV}} \to 0$

A counterexample



$$P_{u,u+1} = P_{u,u-1} = 1/2, \quad \pi \text{ uniform}$$

A counterexample



$P_{u,u+1} = P_{u,u-1} = 1/2, \quad \pi$ uniform

$\tilde{P}_{1,2} = 1,\ \tilde{P}_{1,n} = 0, \quad \tilde{\pi}_1 = 1/n,\ \tilde{\pi}_j = \frac{2(n-j+1)}{n^2}$ for $j \geq 1$

$||\pi - \tilde{\pi}||_{\mathrm{TV}} \asymp \mathrm{cost}.$

Perturbation of Markov chain models

- $G = (V, E)$ family of connected graphs. $N = |V| \to +\infty$.
- $W \subseteq V$ perturbation set
- $P, \tilde{P}$ stochastic matrices on G. $\tilde{P}_{uv} = P_{uv}$ if $u \notin W$
- $\pi^* P = \pi^*$, $\tilde{\pi}^* \tilde{P} = \tilde{\pi}^*$.

If the chain mixes slowly, the process will pass many times through the perturbed set W before getting to equilibrium. $\tilde{\pi}$ will be largely influenced by the perturbed part.

Consequence: $||\pi - \tilde{\pi}||_{\mathrm{TV}} \not\to 0$

Perturbation of Markov chain models

- $G = (V, E)$ family of connected graphs. $N = |V| \to +\infty$.
- $W \subseteq V$ perturbation set
- $P, \tilde{P}$ stochastic matrices on G. $\tilde{P}_{uv} = P_{uv}$ if $u \notin W$
- $\pi^* P = \pi^*$, $\tilde{\pi}^* \tilde{P} = \tilde{\pi}^*$.

A more realistic result:

$P \text{ mixes suff. fast}, \pi(W) \to 0 \;\Rightarrow\; \tilde{\pi} - \pi \to 0$

Recall: mixing time $\tau := \min\{t \mid ||\mu^* P^t - \pi^*||_{\mathrm{TV}} \leq 1/e \;\forall \mu\}$

SRW on d-grid with N nodes, $\tau \asymp N^{2/d} \ln N$

SRW on Erdos-Renji, small world, configuration model $\tau \asymp \ln N$

Perturbation of Markov chain models: the literature

- $G = (V, E)$ family of connected graphs. $N = |V| \to +\infty$.
- $W \subseteq V$ perturbation set
- $P, \tilde{P}$ stochastic matrices on G. $\tilde{P}_{uv} = P_{uv}$ if $u \notin W$
- $\pi^* P = \pi^*$, $\tilde{\pi}^* \tilde{P} = \tilde{\pi}^*$.

$$||\tilde{\pi} - \pi||_{\mathrm{TV}} \leq C\tau ||\tilde{P} - P||_1 \quad \text{(Mitrophanov, 2003)}$$

To measure perturbations of P, the 1-norm is not good to treat localized perturbations: if P and $\tilde{P}$ differ just in one row u and $|P_{uv} - \tilde{P}_{uv}| = \delta$, then, $||P - \tilde{P}||_1 \geq \delta$ and will not go to 0 for $N \to \infty$. In our context, the bound will always blow up.

Perturbation of Markov chain models

- $G = (V, E)$ family of connected graphs. $N = |V| \to +\infty$.
- $W \subseteq V$ perturbation set
- $P, \tilde{P}$ stochastic matrices on G. $\tilde{P}_{uv} = P_{uv}$ if $u \notin W$
- $\pi^* P = \pi^*$, $\tilde{\pi}^* \tilde{P} = \tilde{\pi}^*$.

A more realistic result:

$P \,\text{mixes suff. fast}, \pi(W) \to 0 \;\Rightarrow\; \tilde{\pi} - \pi \to 0$

There is another problem: if P mixes rapidly, nobody guarantees that $\tilde{P}$ will also do...

Perturbation of Markov chain models: first result

- $G = (V, E)$ family of connected graphs. $N = |V| \to +\infty$.
- $W \subseteq V$ perturbation set
- $P, \tilde{P}$ stochastic matrices on G. $\tilde{P}_{uv} = P_{uv}$ if $u \notin W$
- $\pi^* P = \pi^*$, $\tilde{\pi}^* \tilde{P} = \tilde{\pi}^*$.

Theorem

$$||\tilde{\pi} - \pi||_{TV} \leq \tau \tilde{\pi}(W) \log \frac{e^2}{\tau \tilde{\pi}(W)} . \qquad (1)$$

or, symmetrically,

$$||\tilde{\pi} - \pi||_{TV} \leq \tilde{\tau} \pi(W) \log \frac{e^2}{\tilde{\tau} \pi(W)} . \qquad (2)$$

Proof: Coupling technique.

Perturbation of Markov chain models: first result

Corollary

$$\tau\pi(W) \to 0,\ \tilde{\tau} = O(\tau) \quad \Rightarrow ||\tilde{\pi} - \pi||_{TV} \to 0$$

$$\tau\pi(W) \to 0,\ \tilde{\pi}(W) = O(\pi(W)) \quad \Rightarrow ||\tilde{\pi} - \pi||_{TV} \to 0$$

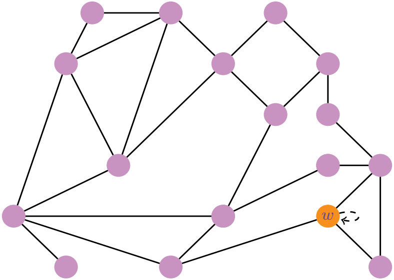
The perturbation, in order to achieve a modification of the invariant probability, necessarily has to

- slow down the chain
- increase the probability on the perturbation subset W.

π and τ are intimately connected to each other!

Perturbation of Markov chain models: first result

Slowing down the chain and putting weight on W look quite connected to each other and essentially amounts to decrease the probability of exiting W:



$$\tilde{P}_{ww} = 1 - 1/N$$

$$\tilde{\pi}_w = \mathbb{E}(\tilde{T}_w^+)^{-1} = \frac{1}{1-N^{-1}+N^{-1}\mathbb{E}(T_w^+)} = \frac{\pi_w}{(1-N^{-1})\pi_w + N^{-1}}$$

$$\pi_w \sim \frac{k}{N} \Rightarrow \tilde{\pi}_w \sim \frac{k}{k+1}$$

A deeper analysis

Lemma

$$\tilde{\pi}(W) \leq \frac{1}{1 + \tilde{\phi}^*_W \tau^*_W} \, ,$$

where

$$\tau^*_W := \min\{\mathbb{E}_v[T_W] : \, v \in V \setminus W\} \, ,$$

minimum entrance time to W

it depends on P

$$\tilde{\phi}^*_W := \frac{\sum_{w \in W} \sum_{v \in V \setminus W} \tilde{\pi}_w \tilde{P}_{wv}}{\tilde{\pi}(W)}$$

bottleneck ratio of W

it depends on $\tilde{P}$

Proof From Kac's lemma

$$\tilde{\pi}(W)^{-1} = \mathbb{E}_{\tilde{\pi}_{\mathcal{W}}}[T^+_{\mathcal{W}}] = 1 + \sum_w \sum_v \frac{\tilde{\pi}_w}{\tilde{\pi}(W)} \tilde{P}_{wv} \mathbb{E}_v[T_W] \geq 1 + \tilde{\phi}^*_W \tau^*_W \, .$$

∎

A deeper analysis

- bottleneck ratio $\longleftrightarrow$ exit probability from W:

  $\mathbb{P}_w(T_{V\setminus W} \leq d) \geq \alpha \; \forall w \in W \;\Rightarrow\; \tilde{\phi}^*_W \geq d/\alpha$

If $\mathbb{P}_w(T_{V\setminus W} \leq d) \geq \alpha$ for fixed $d, \alpha > 0$ and for every $w \in W$, then

$$\tilde{\pi}(W) \leq \frac{1}{1+\tilde{\phi}^*_W \tau^*_W} \asymp (\tau^*_W)^{-1}, \quad \tau\tilde{\pi}(W) = O\left(\frac{\tau}{\tau^*_W}\right)$$

Hence,

$$\frac{\tau}{\tau^*_W} \to 0 \;\Rightarrow\; ||\tilde{\pi} - \pi||_{TV} \to 0$$

A deeper analysis

- minimum entrance time $\longleftrightarrow \pi(W)$:

  $\tau_W^* := \min\{\mathbb{E}_v[T_W]\} \asymp \pi(W)^{-1}$ (Conjecture)

  (Kac's lemma: $\pi(\mathcal{W})^{-1} = 1 + \sum_w \sum_v \frac{\pi_w}{\pi(W)} P_{wv} \mathbb{E}_v[T_W]$
  $\Rightarrow \ \mathbb{E}_v[T_W] \asymp \pi(\mathcal{W})^{-1}$ for some v.....)

$\mathbb{P}_w(T_{V\setminus W} \leq d) \geq \alpha$ for every $w \in W$ plus conjecture imply

$$\frac{\tau}{\tau_W^*} \asymp \tau\pi(W) \to 0 \ \Rightarrow \ ||\tilde{\pi} - \pi||_{TV} \to 0$$

Examples

The conjecture

$$\tau_W^* := \min\{\mathbb{E}_v[T_W]\} \asymp \pi(W)^{-1}$$

holds if P is the simple random walk SRW on

- d-grids with $d \geq 3$, $|W|$ bounded.
- Erdos-Renji, configuration model (w.p. 1) if $|W| = o(N^{1-\epsilon})$

(techniques: electrical network interpretation, effective resistance; locally tree-like graphs)

Recall that

- d-grid, $\tau \asymp N^{d/2} \ln N$
- Erdos-Renji, configuration model (w.p. 1) $\tau = O(\ln N)$

Examples

Theorem

- $G = (V, E)$ family of connected graphs. $N = |V| \to +\infty$.
- $W \subseteq V$ perturbation set
- P SRW on G, $\tilde{P}_{uv} = P_{uv}$ if $u \notin W$
- $\pi^* P = \pi^*$, $\tilde{\pi}^* \tilde{P} = \tilde{\pi}^*$.
- $\mathbb{P}_w(T_{V \setminus W} \leq d) \geq \alpha$ for fixed $d, \alpha > 0$ and for every $w \in W$,

If G and W are:

- d-grids with $d \geq 3$, $|W|$ bounded
- Erdos-Renji, configuration model (w.p. 1), $|W| = o(N^{1-\epsilon})$

then,

$$||\tilde{\pi} - \pi||_{TV} \to 0$$

Application to the heterogeneous gossip model

- $G = (V, E)$, $W \subseteq V$ forceful agents (with prob. p).
- $y(t+1) = P(t)y(t), \quad \mathbb{E}(P(t)) = P_p$
- P_p and P_0 only differ in the rows having index in $W \cup \partial W$.

A specific example: d-regular (toroidal) grid.

$P_0 = (1 - N^{-1})\mathrm{Id} + N^{-1}d^{-1}A_G$ is a lazy simple random walk,

π_0 uniform probability, $\tau_0 \asymp N^{2/d+1} \ln N$, $\tau_W^* \asymp \frac{|W|}{N^2}$

$\frac{\tau_0}{\tau_W^*} \asymp |W| N^{2/d-1} \ln N \to 0$ if $d \geq 3$, $|W|$ bounded.

$$||\pi_p - \pi_0||_{\mathrm{TV}} \to 0$$

- the minority has a vanishing effect on the global population
- $\max_v (\pi_p)_v \to 0$ democracy is preserved ('wise society' in Jackson's terminology)

Gossip with stubborn agents

Take $p = 1$ in the heterogeneous gossip model.

$G = (V, E)$, $W \subset V$ a minority of influent (stubborn) agents

- At each time t choose an edge $\{u, v\}$ at random.
- If $u, v \in V \setminus W$,

$$y_u(t+1) = y_v(t+1) = (x_u(t) + x_v(t))/2$$

- If $u \in W$, $v \in V \setminus W$,

$$y_u(t+1) = y_u(t),\ y_v(t+1) = (y_v(t) + y_u(t))/2$$

Gossip with stubborn agents

(Acemoglu, Como, F., Ozdaglar)

- $y(t) \to y(\infty)$ in distribution. $(y_w(\infty) = y_w(0)\ \forall w \in W)$
- If $\exists w, w' \in W:\ y_w(0) \neq y_{w'}(0)$, then,
  $\mathbb{P}(y_v(\infty) \neq y_{v'}(\infty)) > 0$ (asymptotic disagreement)
- However $\frac{1}{n}\left|\left\{v:\ \left|\mathbb{E}[y_v(\infty)] - \xi\right| \geq \varepsilon\right\}\right| \leq C_\varepsilon \tau\pi(W)$
  If $\tau\pi(W) \to 0$, then approximate consensus!

  This can also be read as a sort of lack of controllability: constraints on the shape of the final state configuration achievable by the global system.

Conclusions and open issues

- Perturbations of Markov chain models and their effect on the invariant probabilities.
- If the mixing time is sufficiently small w.r. to the size of the perturbation, the effect on the invariant probability becomes negligeable in the large scale limit.
- Applications to consensus dynamics

- Find more general estimation of the minimum entrance time parameter τ_W^*.
- Find estimation of type $c_1 \leq \tilde{\pi}_v/\pi_v \leq c_2$. They would permit to obtain estimations of $|\tilde{\tau} - \tau|$.
- Study phase transitions.
- Consider perturbations of non linear models (consensus versus epidemic, threshold models).