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Separable inputs for the identification of block-oriented nonlinear systems

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Abstract— **In this paper, noniterative procedures for the identification of block-oriented nonlinear models, consisting of the interconnection of linear time invariant systems and static nonlinearities are presented. The proposed algorithms are based on the idea of separable input, that is an input which is able to separate the linear and the nonlinear blocks in some sense in identification. Two different block-oriented nonlinear models are considered in this paper, the Hammerstein model and the Wiener model. A simulated example is presented in order to show the effectiveness of the proposed procedures.**

Index Terms— **Wiener systems, Hammerstein systems, separable inputs, Bussgang theorem.**

I. INTRODUCTION

Many nonlinear systems can be successfully described by means of block-oriented nonlinear models which are obtained as combinations of a dynamic linear block and a static nonlinearity. Two different block-oriented nonlinear models are considered in this paper, the Hammerstein model which consists of a static nonlinearity followed by a dynamic linear block (see Fig. 1) and the Wiener model, in which the order of these two blocks is reversed (see Fig. 2).

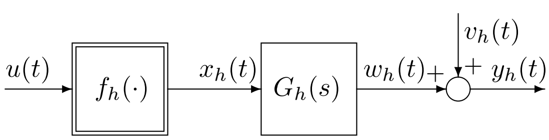

Fig. 1. Single-input single-output Hammerstein model.

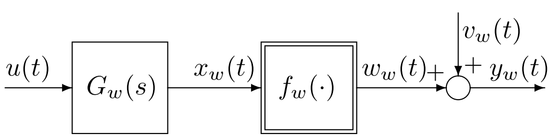

Fig. 2. Single-input single-output Wiener model.

Such cascade systems can be used to model, for example, blind adaptive channel equalization ([1], [2]), control valves ([3]), sensor saturation ([4]), certain biological systems ([5]), bandpass limiters in radio receivers ([6]), echo path ([7]) and traveling-wave tube high power amplifiers ([8]). The identification of such models, which relies solely on input-output measurements, while the inner signal $x(t)$ is not assumed to be available, has attracted the attention of many authors, as can be seen in [9], [10]. In this paper, noniterative procedures for the identification of block-oriented nonlinear models, are presented. The proposed algorithms are based on the idea of separable input, that is an input which is able to separate the linear and the nonlinear blocks in some sense in identification, as will be pointed out in Section II. Identification of block-oriented systems using separable inputs has been addressed, for example, in [5], [11], [12], [13], [14]. The results presented in those papers are based on suitable extensions of the seminal results provided, in the field of random processes, by Bussgang ([15]) and Nuttall ([16]). More precisely, Hunter and Korenberg ([5]) showed that the linear block of Hammerstein and Wiener systems can be identified without any knowledge of the static nonlinearity provided that the input is a white Gaussian noise. In paper [11] by Greblicki and Pawlak, a similar result was presented for the Hammerstein models with white, not necessarily Gaussian, inputs. Ljung and Enqvist ([12], [13]) provided a generalization of the classical Bussgang's theorem which can be applied to the identification of the so called generalized Hammestein and Wiener systems in which the static nonlinear block is replaced by a nonlinear finite impulse response system (NFIR); in particular they showed that the poles of the linear block of such generalized block oriented systems can be estimated without any knowledge of the NFIR. In paper [14], Enqvist exploited the results of Nuttall ([16]) to prove that random multisines signal with a flat amplitude spectrum are separable inputs for the Hammerstein system, that is, when random multisines inputs are used, the identification of the linear block can be performed even if the nonlinear block is completely unknown.

In this paper a general notion of separable input for block-oriented nonlinear system is considered which is related to the ability of an input to separate the linear block and nonlinear block from the identification point of view. Then, it is proven that a number of different inputs often used in identification (e.g., random noises, deterministic sinuoidals, binary signals) are separable ones. Finally, a numerical example is presented to show the effectiveness of using a separable input signal in the identification of a Wiener system with a deadzone nonlinearity.

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II. PROBLEM FORMULATION

Consider the Hammerstein model depicted in Fig. 1, where the nonlinear block maps the input signal $u(t)$ into the unmeasurable inner variable $x_h(t)$ through the nonlinear function

$$x_h(t) = f_h(u(t), \beta), \tag{1}$$

The linear dynamic part transforms $x_h(t)$ into the noise-free output $w_h(t)$ according to

$$W_h(s) = \frac{B(s)}{A(s)} X_h(s) = G_h(s) X_h(s), \tag{2}$$

where $A(s) = 1 + a_1 s + \ldots + a_{na} s^{na}$, $B(s) = b_0 + b_1 s + \ldots + b_{nb} s^{nb}$, $W_h(s)$ and $X_h(s)$ are Laplace transforms of $w_h(t)$ and $x_h(t)$ respectively. Along the same lines, if we consider the Wiener model shown in Fig. 2, we can write

$$X_w(s) = \frac{D(s)}{C(s)} U(s) \tag{3}$$

where $C(s) = 1 + c_1 s + \ldots + c_{nc} s^{nc}$, $D(s) = d_0 + d_1 s + \ldots + d_{nd} s^{nd}$, $X_w(s)$ and $U(s)$ are Laplace transforms of $x_w(t)$ and $u(t)$ respectively. The nonlinear block transforms x_w into the noise-free output w_w according to

$$w_w(t) = f_w(x_w(t), \gamma) \tag{4}$$

In line with the work done by a number of authors, we assume that the linear system is asymptotically stable. Unknown parameter vectors to be identified, $\beta \in R^m$ $\gamma \in R^n$, $\delta \in R^p$ and $\theta \in R^q$ are defined, respectively, as $\beta^T = [\beta_1 \quad \beta_2 \quad \ldots \beta_m]$, $\gamma^T = [\gamma_1 \quad \gamma_2 \quad \ldots \gamma_n]$, $\delta^T = [a_1 \quad \ldots \quad a_{na} \quad b_0 \quad b_1 \quad \ldots b_{nb}]$, where $n_a + n_b + 1 = p$, $\theta^T = [c_1 \quad \ldots \quad c_{nc} \quad d_0 \quad d_1 \quad \ldots d_{nd}]$, where $n_c + n_d + 1 = q$. In this paper we address the problem of identifying the linear dynamics and the static nonlinearity from input and output data, either $\{u(t), y_h(t)\}$ or $\{u(t), y_w(t)\}$, with the underlying assumption that the inner signal, either $x_h(t)$ or $x_w(t)$, is not measurable.

It must be stressed that the main difficulty in block-oriented nonlinear system identification is the coupling of the linear and nonlinear parts. This not only makes identification difficult but also adds an unavoidable degradation in identification performance ([17], [18]). It is desirable to have an input that is able to separate the linear and nonlinearities in some sense in identification. To this end, we introduce the following definition:

Definition — Separable input.

We say that an input $u(t)$ is separable in a general sense if the internal signal $x(t)$ or its properties can be determined based on the available input-output measurements which in turn makes identification of the linear (nonlinear) part possible without knowledge of the nonlinear (linear) part.

III. BUSSGANG-LIKE SEPARABLE INPUTS FOR HAMMERSTEIN MODELS

A. Preliminaries

The Bussgang theorem is a classical result in signal processing. Let $x_h = f_h(u)$ be a smooth nonlinear function. Suppose $u(t)$ is a Gaussian random process with zero mean and finite variance. Let $\mathbf{E}$ be the expectation operator, and $R_u(\tau)$ and $R_{x_h u}(\tau)$ be the auto-correlation and cross-correlation functions respectively. Then, the Bussgang theorem (see, e.g., [19], page 397) states that

$$R_{x_h u}(\tau) = c R_u(\tau) \tag{5}$$

for some constant c. This result makes identification of the linear part in a Hammerstein model possible without knowledge of the unknown nonlinearity $f_h(\cdot)$. More precisely, being $g_h(t)$ the impulse response and $G_h(s)$ the transfer function of the unknown linear part, it follows

$$y_h(t) = \int g_h(\xi) x_h(t-\xi) d\xi \tag{6}$$

and

$$R_{y_h u}(\tau) = \mathbf{E} y_h(t) u(t-\tau) = \int g_h(\xi) \mathbf{E} x_h(t-\xi) u(t-\tau) d\xi$$

$$= c \int g_h(\xi) R_u(\tau - \xi) d\xi. \tag{7}$$

Thus,

$$c G_h(s) = \frac{R_{yu}(s)}{R_u(s)}. \tag{8}$$

From equation (8) it is clear that the linear part can be uniquely estimated, up to a scaling constant, exploiting only input-output data.

Once the linear block has been identified, the nonlinear static map can be estimated either exploiting steady-state operating conditions or recovering the inner unmeasurable signal $x_h(t)$ through proper inversion of the linear system along the line of the paper [18] and [5].

The Bussgang theorem is a foundation for many existing results [20], [11], [12] with various variations. Recently, the results have been extended to some dynamic nonlinearities [13]. We now present some further extension of the Bussgang theorem.

B. A sinusoidal input with a random phase

Let

$$u(t) = A \cos(\omega t + \phi) \tag{9}$$

where the phase is a random variable uniformly distributed in $[0, 2\pi]$. Since $u(t)$ is an even and periodic function with period $T = \frac{2\pi}{\omega}$, the internal signal $x_h(t)$ is also even and periodic with period $T = \frac{2\pi}{\omega}$ and has a Fourier series representation with all the odd terms being zero. Further, because the nonlinearity is static, one gets

$$u(t) = A \cos(\omega t) \rightarrow x_h(t) = \sum r_i \cos(i \omega t) \Longrightarrow \tag{10}$$

$$u(t) = A \cos(\omega t + \phi) = A \cos(\omega (t + \phi/\omega)) \rightarrow \tag{11}$$

$$x_h(t) = \sum r_i \cos(i \omega (t + \phi/\omega)) = \sum r_i \cos(i \omega t + i \phi). \tag{12}$$

It is easily verified that for a sinusoidal signal with a random phase, the auto-correlation function is

$$R_u(\tau) = \mathbf{E} A^2 \cos(\omega t + \phi) \cos(\omega (t - \tau) + \phi) = \frac{A^2}{2} \cos(\omega \tau). \tag{13}$$

On the other hand, the cross-correlation function of $x_h(t)$ and $u(t)$ is

$$R_{x_h u}(\tau) = \mathbf{E}\sum r_i \cos(i\omega t + i\phi)\cos(\omega(t-\tau)+\phi) =$$
$$= \frac{r_1 A}{2}\cos(\omega\tau) = cR_u(\tau). \quad (14)$$

This shows that a sinusoidal signal with a random phase is a separable input. The result is reminiscent of the conclusion in [14] in a discrete time setting and in [16] using a completely different proof.

C. White input

Let $u(t)$ be a random process with zero mean and

$$R_u(\tau) = c_1\delta(\tau). \quad (15)$$

Let $f(\cdot)$ be any static nonlinear function so that the random process $x(t) = f(u(t))$ is well defined. Clearly,

$$R_{xu}(\tau) = \mathbf{E}x(t)u(t-\tau) = c_2\delta(\tau) = cR_u(\tau). \quad (16)$$

Thus, white input is separable.

IV. DETERMINISTIC SINUSOIDAL SEPARABLE INPUTS

A. Wiener model identification

Suppose that $u(t) = A\cos(\omega t)$. Then,

$$x_w(t) = A_1\cos(\omega t + \phi) \quad (17)$$

for some phase shift ϕ. Clearly, if the phase shift is available, the nonlinearity $f_w(\cdot)$ can be uniquely determined based on $x_w(t)$ and $y_w(t)$, up to a scaling constant. We claim the phase shift can be calculated from the input and output measurements; this fact makes a sinusoidal input separable.

To this end, again from the even and periodic properties of $x_w(t)$ as well as the static assumption on the unknown f, it follows

$$x_w(t) = A_1\cos(\omega t) \rightarrow y_w(t) = \sum r_i\cos(i\omega t) \Longrightarrow$$
$$x_w(t) = A_1\cos(\omega t+\phi) = A_1\cos(\omega(t+\phi/\omega)) \rightarrow$$
$$y_w(t) = \sum r_i\cos(i\omega(t+\phi/\omega)) = \sum r_i\cos(i\omega t + i\phi). \quad (18)$$

Now, define

$$\mathcal{H}(L,\alpha) = \frac{1}{LT}\int_0^{LT}\cos(\omega t+\alpha)y_w(t)dt$$
$$= \frac{1}{LT}\int_0^{LT}\cos(\omega t+\alpha)\sum r_i\cos(i\omega t+i\phi)+$$
$$+\frac{1}{LT}\int_0^{LT}\cos(\omega t+\alpha)v_w(t)dt$$
$$= \frac{r_1}{2}\cos(\phi-\alpha) + \frac{1}{LT}\int_0^{LT}\cos(\omega t+\alpha)v_w(t)dt. \quad (19)$$

The second term converges to zero as $L \to \infty$. Moreover, since the scaling constant is not identifiable, we may assume $r_1 = 2$ and this implies

$$\mathcal{H}(L,\alpha) \to \cos(\phi-\alpha). \quad (20)$$

Therefore, the unknown ϕ can be easily calculated from a one dimensional maximization problem of $\max_\alpha \mathcal{H}(L,\alpha)$ for large L. Various efficient algorithms are available in the literature for this purpose.

The above calculation is based on the continuous data collection. It can be easily shown that $\mathcal{H}(L,\alpha)$ can also be calculated from DFTs of the sampled data. Let the cut-off frequency of the anti-aliasing filter be $\bar{i}\omega$ and the sampling period be $T_s = \frac{2\pi}{\bar{i}\omega}\frac{1}{M}$ for some $M > 2$ by the Nyquist frequency. The filtered output can be written as

$$y_f(t) = \sum_{i=0}^{\bar{i}} r_i\cos(i\omega t + i\phi) + v_f(t) \quad (21)$$

Consider the DFT of y_f

$$Y(\frac{2\pi}{\bar{i}M})e^{-j\alpha} = \frac{1}{L\bar{i}M}\sum_{k=0}^{L\bar{i}M-1} y_f(kT_s)e^{-jk2\pi/(\bar{i}M)}e^{-j\alpha}$$
$$= \frac{1}{L\bar{i}M}\sum_{k=0}^{L\bar{i}M-1}\sum_{i=0}^{\bar{i}} r_i\cos(i\omega t+i\phi)e^{-jk2\pi/(\bar{i}M)}e^{-j\alpha}+$$
$$+\frac{1}{L\bar{i}M}\sum_{k=0}^{L\bar{i}M-1} v_f(kT_s)e^{-j\alpha}$$
$$= \frac{r_1}{2}e^{-j\alpha} + \frac{1}{L\bar{i}M}\sum_{k=0}^{L\bar{i}M-1} v_f(kT_s)e^{-j\alpha}. \quad (22)$$

The second term is due to the noise and converges to zero as $L \to \infty$. Again, let $r_1 = 2$ and we have

$$Real[Y(\frac{2\pi}{\bar{i}M})e^{-j\alpha}] = \cos(\phi-\alpha). \quad (23)$$

B. Identification of the linear part of the Wiener model

Once the nonlinear output map f_w has been estimated exploiting the separable input $u(t) = A\cos(\omega t)$, we have to consider the problem of the identification of the linear dynamic subsystem. In order to estimate the parameters of the linear model, one should (i) stimulate the system with a persistently exciting input $u(t)$, (ii) evaluate the corresponding inner signal $x_w(t)$ from the output records $y_w(t)$, (iii) estimate the linear block parameters on the basis of the input signal $u(t)$ and the estimate of the inner signal $x_w(t)$. Unfortunately, one must consider the fact that the nonlinearity f_w is, in general, noninvertible, which means that, given the measured output $y_w(t)$, the inner signal $x_w(t)$ cannot be evaluated uniquely even in the case of exactly known nonlinearity and noise free measurements. In order to overcome such a problem, we propose to exploit the approach presented in [21] which can be summarized as follows (see [21] for details):

(a) Find an *Output Invertibility Interval* for the nonlinearity f_w, i.e., an interval $\mathcal{Y}$ such that $\forall y_w \in \mathcal{Y}$ the equation $f_w(x_w) - y_w = 0$ has only one real root. Such an interval can be easily estimated by inspection of the plot of the previously estimated nonlinearity f_w.

(b) Apply to the Wiener system a suitable input signal $u(t)$ in order to drive the output $y_w(t)$ into the Output invertibility interval $\mathcal{Y}$; such an input signal $u(t)$ should contain a DC component u_{DC} (offset) and a persistently exciting signal $\delta u(t)$ whose amplitudes should be chosen in such a way that $y_w(t) = y_{DC} + \delta y(t)$ belongs to $\mathcal{Y}$ $\forall t$. The value of the offset u_{DC} and the amplitude of the dynamic sequence $\delta u(t)$ can, in general, be tuned by trial and error until the measured output signal satisfies the condition $y_w(t) \in \mathcal{Y}$ $\forall t$. (c) Evaluate the unmeasurable inner signal $x_w(t)$ computing, for each sample of the output signal $y_w(t)$, the unique real root of the equation $f(x_w(t)) - y_w(t) = 0$.

Remark 1 — The results presented in [21] were restricted to the case of noninvertible polynomial nonlinearity. However, the idea can be applied also to other kind of nonlinearity since, as well known, any nonlinear function can be locally approximated with arbitrary accuracy by a polynomial of sufficiently high order.

C. Hammerstein model identification

Let the input $u(t) = A\cos(\omega_k t)$. Then, we have

$$u(t) = A\cos(\omega_k t) \Longrightarrow x(t) = \sum r_i \cos(i\omega_k t). \tag{24}$$

Thanks to the Invariance property presented in [22], we know that the coefficients r_i are input frequency ω_k independent. Assuming, without loss of generality, that $r_1 = 1$, this property leads to

$$x_h(t) = \cos(\omega_k t) + \sum_{i\neq 1} r_i \cos(i\omega_k t) \tag{25}$$

and

$$y_h(t) = |G(j\omega_k)|\cos(\omega_k t + \angle G(j\omega_k)) + \\ + \sum_{i\neq 1} r_i |G(j\omega_k i)| \cos(i\omega_k t + \angle G(j\omega_k i)). \tag{26}$$

Obviously, identification of the linear block $G(s)$ is the same as for a linear system identification if the higher harmonics can be removed. This can be accomplished as follows. Define,

$$V_T(\omega) = \frac{1}{T}\int_0^T v_h(t)e^{-j\omega t}dt,\ U_T(\omega) = \frac{1}{T}\int_0^T u(t)e^{-j\omega t}dt,$$

$$Y_T(\omega) = \frac{1}{T}\int_0^T y(t)e^{-j\omega t}dt. \tag{27}$$

It is verified ([22]) that

$$A\frac{Y_T(\omega_k)}{U_T(\omega_k)} = G(j\omega_k) + 2V_T(\omega_k) \to G(j\omega_k) \tag{28}$$

as $T \to \infty$.
Also, V_T, U_T and Y_T can be computed from DFTs of the sampled data.

V. Binary inputs for Hammerstein models

Let $x_h = f(u)$ be any static function with a binary input $u(t) = \{a, b\}$. Then, the unmeasurable inner signal x_h is determined by

$$x_h = f_h(u) = \eta_0 + \eta_1 u \tag{29}$$

with

$$\begin{pmatrix}\eta_0 \\ \eta_1\end{pmatrix} = \begin{pmatrix}\frac{f_h(b)a - f_h(a)b}{(a-b)} \\ \frac{f_h(a)-f_h(b)}{(a-b)}\end{pmatrix}. \tag{30}$$

The output of the Hammerstein model is given by

$$y_h(t) = g_h(t) * x_h(t) + v_h(t) \tag{31}$$

where $*$ is the convolution integral and $g_h(t)$ is the impulse response of the linear block. By substitution of equation (29) into equation (31), one gets:

$$\begin{aligned} y_h(t) &= \int_0^\infty g_h(\tau)x_h(t-\tau)d\tau + v_h(t) = \\ &= \int_0^\infty g_h(\tau)(\eta_0 + \eta_1 u(t-\tau))d\tau + v_h(t) = \\ &= \int_0^\infty g_h(\tau)\eta_0 + \int_0^\infty \eta_1 g_h(\tau)u(t-\tau)d\tau + v_h(t) = \\ &= c + \eta_1 g_h(t) * u(t) + v_h(t) \end{aligned} \tag{32}$$

where c is a constant offset that can be easily estimated. With a little abuse of notation, this is equivalent to

$$y_h(t) = c + \eta_1 G_h(s)u(t) + v_h(t). \tag{33}$$

Thus, the effects of the unknown nonlinearity is virtually eliminated by the use of binary inputs and identification of the linear part $G_h(s)$ is independent of the unknown nonlinear part since it can be performed directly exploiting the input and output signals $u(t)$ and $y_h(t)$. Therefore, a binary input is separable for Hammerstein systems. In practice, the PRBS input can be used [18].
Once the linear block has been identified, the nonlinear static map can be estimated either exploiting steady-state operating conditions or recovering the inner unmeasurable signal $x_h(t)$ through proper inversion of the linear system along the line of the paper [18] and [5].

VI. Simulated example

In this section an example is presented to show the effectiveness of the proposed approach. Due to lack of space, simulated results are only presented for the case of deterministic sinusoidal inputs for the identification of Wiener system (see Figure 1 for symbols and notations). The considered Wiener system is characterized by the following linear and nonlinear blocks:

$$G(s,\theta) = \frac{\theta_3 s + \theta_4}{s^2 + \theta_1 s + \theta_2} = \frac{5s+1}{s^2+2s+3}, \tag{34}$$

$$f_w(x,d) = \frac{1 - \text{sign}(d - |x|)}{2}(x - d\,\text{sign}(x)), \tag{35}$$

where $d = 0.3$. Thus, the considered nonlinear function is a dead-zone of width $d = 0.3$.

First the nonlinear block can been identified exploiting the ideas presented in the paper for decoupling the nonlinear and linear parts. The following input u can be used:

$$u(t) = \cos(\omega_o t) \tag{36}$$

where $\omega_o = 1$rad/s. Then, the phase φ of the inner signal $x(t)$:

$$x_w(t) = A\cos(\omega_o t + \varphi) \tag{37}$$

can been estimated (through the computation of the DFT of the sampled output) setting $L = 200$, $M = 5$, $\bar{i} = 25$, and the sampling time $T_s = 2\pi/(\bar{i}\omega M) = 0.045$s. Since the parameterization of the structure of Figure 2 is not unique, we can assume $A = 1$ without any loss of generality. That choice implies that the linear system to be estimated, say $\tilde{G}(s)$, must have magnitude equal to 1 for $s = j\omega_o$, i.e.,

$$\tilde{G}(s,\tilde{\theta}) = \frac{G(s)}{\alpha} = \frac{\frac{\theta_3}{\alpha}s + \frac{\theta_4}{\alpha}}{s^2 + \theta_1 s + \theta_2} = \frac{\tilde{\theta}_3 s + \tilde{\theta}_4}{s^2 + \tilde{\theta}_1 s + \tilde{\theta}_2} \tag{38}$$

where $\alpha = |G(j\omega_o)|$. It is immediately verified that $|\tilde{G}(j\omega_o)| = 1$. In order to have the same input-output mapping, the nonlinear block must become, accordingly,

$$f_w(\alpha x, d) = \frac{1 - \text{sign}(d - |\alpha x_w|)}{2}(\alpha x_w - d\,\text{sign}(\alpha x_w)); \tag{39}$$

Now, let us rewrite Equation (39) in the following equivalent way:

$$w(t) = \begin{cases} 0 & \text{for } |x_w| \le \frac{d}{\alpha} \\ \alpha x_w - d = \gamma_1 x_w - \gamma_2 & \text{for } |x_w| \ge \frac{d}{\alpha},\ x_w > 0 \\ \alpha x_w + d = \gamma_1 x_w + \gamma_2 & \text{for } |x_w| \ge \frac{d}{\alpha},\ x_w < 0 \end{cases} \tag{40}$$

Thus, equations (38) and (40) show that the true values of the parameters to be estimated are

$$\gamma = [\gamma_1\ \ \gamma_2]^{\mathrm{T}} = [\alpha\ \ d]^{\mathrm{T}} = [|G(j\omega_o)|\ \ 0.3]^{\mathrm{T}} \tag{41}$$

and

$$\tilde{\theta} = \begin{bmatrix}\tilde{\theta}_1 & \tilde{\theta}_2 & \tilde{\theta}_3 & \tilde{\theta}_4\end{bmatrix}^{\mathrm{T}} = \begin{bmatrix}\theta_1 & \theta_2 & \frac{\theta_3}{\alpha} & \frac{\theta_4}{\alpha}\end{bmatrix}^{\mathrm{T}} = \begin{bmatrix}2 & 3 & \frac{5}{\alpha} & \frac{1}{\alpha}\end{bmatrix}^{\mathrm{T}}. \tag{42}$$

Here the problem of how to identify the nonlinear block arises; in principle one can obtain the estimate solving the following optimization problem, along the lines of paper [23]:

$$\begin{aligned} \hat{\gamma} = [\ \hat{\alpha}\ ,\ \hat{d}\ ] &\doteq \arg\min_{\alpha,d} J_1(\alpha,d) = \\ &= \sum_{m=n}^{L\bar{i}M-1} \arg\min_{\alpha,d}(y(mT_s) - f_w(\alpha,d,\hat{x_w}(mT_s)))^2. \end{aligned} \tag{43}$$

However, the computation of the global minimum of the nonconvex functional $J_1(\alpha,d)$ is not a trivial problem since nonconvex optimization methods can trap in local minima. We propose the following two-stage solution:
(I) approximate localization of the global minimizer $\hat{\gamma}$ through graphical inspection of the two-dimensional level curves plot;
(II) computation of global minimizer $\hat{\gamma}$ through the use of a local optimization method.

Remark 1 — We must stress that, in this example, if we were able to estimate $\alpha = |G(j\omega_o)|$ we would be able to find an estimate of the given true linear system thanks to the fact that the slope of deadzone characteristics is known to be unitary.

Remark 2 — If the slope of deadzone characteristics was not a-priori known, then there would be three parameters to estimate: α, d and, say, m. In this case, however, we wouldn't be able to recover (separately) α and m from the product $m\alpha$, which means that the linear system gain α and the deadzone slope m would be unidentifiable.

As far as the identification of the linear block is concerned, the algorithm proposed in section IV-B has been applied.
In order to drive the output of the Wiener system into an invertibility interval (where the deadzone equals a constant unitary gain), the following input signal has been applied to the system:

$$u(t) = u_{DC} + \delta u(t) \tag{44}$$

where $u_{DC} = 7$ is a constant offset and $\delta u(t)$ can be any persistently exciting signal (here we have chosen a symmetric PRBS signal of amplitude 2). Random uniform distributed measurement errors η have been considered when simulating the collection of data. Three different amplitude of such errors were chosen (0.01, 0.05, 0.1). From the simulated output sequences $\{\overline{w}_w(t)\}$ and $\{\overline{v}_w(t)\}$ for the identification of the nonlinear block, the signal to noise ratio $\overline{SNR}$ has been evaluated through

$$\overline{SNR} = 10\log\frac{\|\overline{w}_w(t)\|_2^2}{\|\overline{v}_w(t)\|_2^2}. \tag{45}$$

Analogously, from the simulated output sequences $\{w_w(t)\}$ and $\{v_w(t)\}$ for the identification of the linear block, the signal to noise ratio SNR has been evaluated through

$$SNR = 10\log\frac{\|w_w(t) - w_{DC}\|_2^2}{\|v_w(t)\|_2^2} \tag{46}$$

where $w_w(t)$ is the noise free output (see Figure 2), $v_w(t)$ is the output noise and w_{DC} is the steady-state value of $w_w(t)$ due to the constant offset u_{DC} . The number of samples of the output used for the identification of the linear block is $N = 1995$. The obtained results, shown in Table 1, Table 2 and Table 3 for the inner signal phase, the nonlinear block parameters and the linear block parameters respectively, are quite satisfactory. The level curves of the functional $J_1(\alpha,d)$ are shown in Figure 3 for the case of noise magnitude equal to 0.01.

VII. CONCLUSION

Noniterative procedures for the identification of Hammerstein and Wiener models have been presented. The proposed algorithms are based on the idea of separable input, that is an input which is able to separate the linear and the nonlinear blocks in some sense in identification. In the paper it has been proven that a number of different inputs often used in identification (e.g., random noise, deterministic sinuoidal, binary signals) are separable ones. The numerical example showed the effectiveness of the proposed procedure for the case of a Wiener system with a deadzone nonlinearity.

Table 1: Inner signal phase estimates ($\hat{\varphi}$).

$\overline{SNR}$ (dB)	True Value	$\hat{\varphi}$	$\|\varphi - \hat{\varphi}\|/\|\varphi\|$
40.2	0.5880	0.5880	0.00e-2
26.1	0.5880	0.5884	0.07e-2
20	0.5880	0.5876	0.07e-2

Table 2: Nonlinear block parameters estimates ($\hat{\gamma}_1 = \hat{\alpha}$, $\hat{\gamma}_2 = \hat{d}$).

$\overline{SNR}$ (dB)	γ_j	True Value	$\hat{\gamma}_j$	$\|\gamma_j - \hat{\gamma}_j\|/\|\gamma_j\|$
40.2	γ_1	1.8028	1.8028	0.00e-2
	γ_2	0.3000	0.3001	0.03e-2
26.1	$\tilde{\gamma}_1$	1.8028	1.8035	0.04e-2
	$\tilde{\gamma}_2$	0.3000	0.3012	0.40e-2
20	$\tilde{\gamma}_1$	1.8028	1.8060	0.18e-2
	$\tilde{\gamma}_2$	0.3000	0.3029	0.97e-2

Table 3: Linear system parameter estimates ($\hat{\theta}_j$).

SNR (dB)	θ_j	True Value	$\hat{\theta}_j$	$\|\theta_j - \hat{\theta}_j\|/\|\theta_j\|$
36.2	θ_1	2.000	1.997	0.15e-2
	θ_2	3.000	2.983	0.56e-2
	θ_3	5.000	4.984	0.32e-2
	θ_4	1.000	0.983	1.70e-2
22.1	θ_1	2.000	1.992	0.40e-2
	θ_2	3.000	2.986	0.46e-2
	θ_3	5.000	4.982	0.36e-2
	θ_4	1.000	0.962	3.80e-2
16.1	θ_1	2.000	2.013	0.65e-2
	θ_2	3.000	2.917	2.76e-2
	θ_3	5.000	5.020	0.40e-2
	θ_4	1.000	0.901	9.90e-2

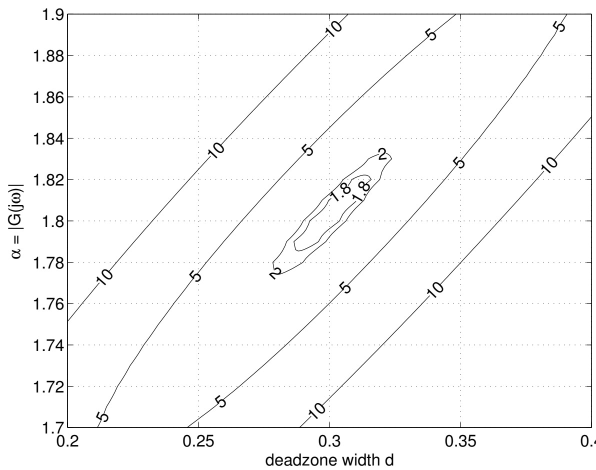

Fig. 3. Inner signal bounds

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