2005 American Control Conference
June 8-10, 2005. Portland, OR, USA

FrB15.4

Bounding the parameters of linear systems with input backlash

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Abstract— **In this paper we present a two-stage procedure for deriving parameters bounds of linear systems with input backlash when the output measurement errors are bounded. First, using steady-state input-output data, parameters of the nonlinear dynamic block are tightly bounded. Then, given a suitable PRBS input sequence we evaluate tight bounds on the unmeasurable inner signal which, together with noisy output measurements are employed for bounding the parameters of the linear dynamic system.**

Index Terms— **Backlash, bounded uncertainty, output errors, errors-in-variable, parameter bounding, linear programming.**

I. INTRODUCTION

Control systems components, such as sensors and actuators, often exhibits backlash which, indeed, is a typical characteristic of gearboxes and, in general, mechanical connections (see, e.g. [1]). Backlash can be classified as a hard (i.e. non-differentiable) and dynamic nonlinearity. It is well known that this kind of nonlinearity may often cause delays, oscillations and inaccuracy which severely limit the performance of control systems (see, e.g. [2]). To cope with these limitations, either robust or adaptive control techniques can be successfully employed (see, e.g., [3], and [4] respectively) which, on the other hand, require the characterization of the nonlinear dynamic block. Amazingly, there are only few contributions in the literature on the identification of systems with nonstatic hard nonlinearities ([5]). Therefore, the identification of control systems with unknown backlash is an open theoretical problem of major relevance to applications.

The configuration we are dealing with in this paper, shown in Fig. 1, closely resembles that of a Hammerstein model which in turn consists of a static nonlinear part $\mathcal{N}$ followed by a linear dynamic system. The identification of such a model relies solely on input-output measurements, while the inner signal x_t, i.e. the output of the nonlinear block, is not assumed to be available. Identification of the Hammerstein structure has attracted the attention of many authors, as can be seen in [6], [7]. It must be stressed that existing identification procedures mostly require that the nonlinearity be static and differantiable, usually a polynomial (see e.g., [8], [9], [10] and the references therein).

In identification, a common assumption is that the measurement error η_t is statistically described. A worthwhile alternative to the stochastic description of measurement errors is the bounded-errors characterization, where uncertainties are assumed to belong to a given set. In the bounding context, all parameter vectors belonging to the *Feasible Parameter Set* (FPS), i.e. parameters consistent with the measurements, the error bounds and the assumed model structure, are feasible solutions of the identification problem. The interested reader can find further details on this approach in a number of survey papers (see, e.g., [11], [12]).

In this paper we present a scheme for the identification of linear systems with input backlash. More precisely, we address the problem of bounding the parameters of a stable single-input single-output SISO discrete time linear system with unknown backlash at the input, that is, in the actuator (see Fig. 1) when the output error is considered to be bounded. Note that the inner signal $x(t)$ is not supposed to be measurable. Results obtained in this paper can be straightly employed in the procedure proposed by [3] where they assume that bounds on the parameters of uncertain backlash are available in order to derive a sliding mode technique for the stabilization of an intrinsically nonlinear plant with an uncertain backlash in the actuator.

To the author's best knowledge, no contribution can be found in the literature which address the above described identification problem. We present a two-stage identification procedure. First, parameters of the nonlinear block are tightly bounded using input-output data collected from the steady-state response of the system to a square wave input. Then, through a dynamic experiment, for all u_t belonging to a suitable Pseudo Random Binary Signal (PRBS) sequence $\{u_t\}$, we compute tight bounds on the inner signal which, together with noisy output measurements are used for bounding the parameters of the linear part.

II. PROBLEM FORMULATION

Consider the SISO discrete-time linear system with input backlash depicted in Fig. 1, where the nonlinear block transforms the input signal u_t into the unmeasurable inner variable x_t according to the following map

$$x_t = \begin{cases} m(u_t + c) & \text{for } u_t \le z_l \\ m(u_t - c) & \text{for } u_t \ge z_r \\ x_{t-1} & \text{for } z_l < u_t < z_r \end{cases} \quad (1)$$

where $m > 0$, $c > 0$ are constant parameters characterizing the backlash and

$$z_l = \frac{x_{t-1}}{m} - c, \qquad z_r = \frac{x_{t-1}}{m} + c \quad (2)$$

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are the u-axis values of intersections of the two m-slope parallel lines with the horizontal inner segment containing x_{t-1}.

The linear dynamic part is modeled by a discrete-time system which transforms x_t into the noise-free output w_t according to the linear difference equation

$$A(q^{-1})w_t = B(q^{-1})x_t, \quad (3)$$

where $A(\cdot)$ and $B(\cdot)$ are polynomials in the backward shift operator q^{-1}, $(q^{-1}w_t = w_{t-1})$,

$$A(q^{-1}) = 1 + a_1 q^{-1} + \ldots + a_{na} q^{-na}, \quad (4)$$
$$B(q^{-1}) = b_0 + b_1 q^{-1} + \ldots + b_{nb} q^{-nb}. \quad (5)$$

In line with the work done by a number of authors, in the contest of identification of block oriented systems, we assume that (i) the linear system is asymptotically stable (see, e.g., [13], [14], [15], [16], [17]); (ii) $\sum_{j=0}^{nb} b_j \neq 0$, that is, the steady-state gain is not zero (see, e.g. [15], [16], [17]); (iii) the only *a priori* information needed is an estimate of the process settling-time (see, e.g., [18]). Let y_t be the noise-corrupted output

$$y_t = w_t + \eta_t. \quad (6)$$

Measurements uncertainty is known to range within given bounds $\Delta\eta_t$, i.e.,

$$\mid \eta_t \mid \leq \Delta\eta_t. \quad (7)$$

Unknown parameter vectors $\gamma \in R^2$ and $\theta \in R^p$ are defined, respectively, as

$$\gamma^T = [\ \gamma_1 \quad \gamma_2\ ] = [\ m \quad c\ ], \quad (8)$$

$$\theta^T = [\ a_1 \quad \ldots \quad a_{na} \quad b_0 \quad b_1 \quad \ldots b_{nb}\ ], \quad (9)$$

where $n_a + n_b + 1 = p$. It is easy to show that the parameterization of the structure of Fig. 1 is not unique. As a matter of fact, any parameters set $\tilde{b}_j = \alpha^{-1} b_j, j = 1, 2, \ldots, nb$, and $\tilde{\gamma}_k = \alpha\gamma_k, k = 1, 2$, for some nonzero and finite constant α, provides the same input-output behaviour. To get a unique parameterization, in this work, we assume, without loss of generality, that the steady-state gain of the linear part be one, that is

$$g = \frac{\sum_{j=0}^{nb} b_j}{1 + \sum_{i=1}^{na} a_i} = 1 \quad (10)$$

In this paper we address the problem of deriving bounds on parameters γ and θ consistently with given measurements, error bounds and the assumed model structure. In Section III, using the steady-state response of the system to a square wave input, parameters of the backlash are tightly bounded, while in Section IV, for a suitable input PRBS sequence we evaluate tight bounds on the unmeasurable inner signal which, together with noisy output measurements are used for bounding the parameters of the linear part. A simulated example is reported in Section V.

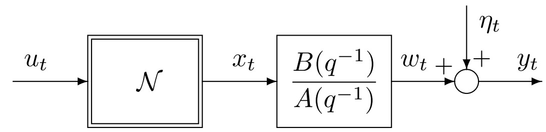

Fig. 1. Single-input single-output discrete-time linear system with input backlash $\mathcal{N}$.

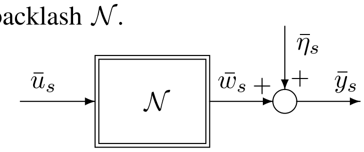

Fig. 2. Steady-state behaviour of the system under consideration when $g = 1$.

III. Assessment of tight bounds on the nonlinear static block parameters

In this work we exploit steady-state operating conditions to bound the parameters of the backlash. The noisy output sequence is collected from the steady-state response of the system to a set of square wave inputs with different amplitudes. Since the nonlinearity of the backlash is assumed to be symmetric, from now on we will focus only on the positive half-wave of the input. Due to the fact that the backlash deadzone is unknown (its evaluation is the main purpose of this section) we suggest to choose the input amplitude in such a way that the output shows any nonzero response. For each value of the input square wave amplitude, only one steady-state value of the noisy output is considered. Thus, given a set of square wave inputs with M different amplitudes, M steady-state values of the output are taken into account. We only assume to have a rough idea of the settling time of the system under consideration, in order to know when steady-state conditions are reached, so that steady-state data can be collected. Indeed, under conditions (i), (ii) and (iii) stated in Section II, combining equations (1), (3), (6) and (10) at steady-state, we get the following input-output description involving only the parameters of the backlash:

$$\bar{y}_s = m(\bar{u}_s - c) + \bar{\eta}_s \quad (11)$$

where $s = 1, \ldots, M$; $\bar{u}$, $\bar{y}$ and $\bar{\eta}$ are steady-state values of the known input signal, output observation and measurement error respectively; $M \geq 2$ is the number of the steady-state samples. A block diagram description of equation (11) is depicted in Fig. 2. The feasible parameter region of the backlash is defined as

$$\begin{aligned}\mathcal{D}_\gamma = \{ & m \in R^+, c \in R^+ : \bar{y}_s = m(\bar{u}_s - c) + \bar{\eta}_s, \\ & \mid \bar{\eta}_s \mid \leq \Delta\bar{\eta}_s; \quad s = 1, \ldots, M\}\end{aligned} \quad (12)$$

where $\{\Delta\bar{\eta}_s\}$ is the sequence of bounds on measurements uncertainty. From definition (12) it can be seen that $\mathcal{D}_\gamma$ is exactly described by the following constraints in the 2-dimensional parameter space

$$\begin{aligned}\bar{y}_s - m(\bar{u}_s - c) &\geq -\Delta\bar{\eta}_s \\ \bar{y}_s - m(\bar{u}_s - c) &\leq \Delta\bar{\eta}_s\end{aligned} \quad (13)$$

where $s = 1, \ldots, M$. The above exact description of $\mathcal{D}_\gamma$ will be used in the next section when deriving tight bounds on the unmeasurable inner signal x_t. Besides, tights bounds on $\gamma_1 = m$ and $\gamma_2 = c$ can be obtained through the computation of the following orthotope-outer bounding set $\mathcal{B}_\theta$ containing $\mathcal{D}_\gamma$:

$$\mathcal{B}_\gamma = \{\gamma \in R^2 : \gamma_j = \gamma_j^c + \delta\gamma_j, \mid \delta\gamma_j \mid \leq \Delta\gamma_j, j = 1, 2\}, \quad (14)$$

where

$$\gamma_j{}^c = \frac{\gamma_j^{min} + \gamma_j^{max}}{2}, \qquad \Delta\gamma_j = \frac{\mid \gamma_j^{max} - \gamma_j^{min} \mid}{2} \quad (15)$$

$$\gamma_j^{min} = \min_{\gamma \in \mathcal{D}_\gamma} \gamma_j, \quad \gamma_j^{max} = \max_{\gamma \in \mathcal{D}_\gamma} \gamma_j. \quad (16)$$

Since constraints (13) which define $\mathcal{D}_\gamma$ are nonlinear in m and c, at least in principle the solution of the above optimization problems (16) requires the use of general nonconvex optimization techniques. However, since $\mathcal{D}_\gamma \in R^2$, extreme values of m and c can be comfortably found by graphical inspection once constraints (13) are drawn on the (m, c) plane.

IV. Bounding the parameters of the linear dynamic model

In the second stage of our procedure we evaluate bounds on the parameters of the linear dynamic block. In this stage, we excite the system to be identified with a pseudo random binary signal (PRBS) which takes the values $\pm u^\star$. We recall that, thanks to its nice properties, a PRBS input is successfully employed in linear system identification ([19], [20]). Although PRBS inputs are not suitable for nonlinear system identification in general ([5], [21]) since it may not adequately excite the unknown nonlinearity, in this paper we show that the use of such a signal is profitable for the identification of linear system with input backlash. The key idea underlying our contribution is based on the following result:

Result 1 Under a PRBS input whose levels are $\pm u^\star$, $u^\star > c$, the output of a backlash, whose deadzone is c, is still a PRBS with levels $\pm x^\star$, $x^\star = m(u^\star - c)$.

The proof of Result 1 is not reported since it is trivial.

Given the exact description of the feasible parameter set (FPS) $\mathcal{D}_\gamma$, tight bounds on the magnitude $x^\star$ of the unmeasurable pseudo random inner signal x_t can be computed $\forall t$ through the following expressions

$$\begin{aligned} x^{\star min} &= \min_{m,c \in \mathcal{D}_\gamma} m(u^\star - c), \quad \text{for } u^\star \geq c \\ x^{\star max} &= \max_{m,c \in \mathcal{D}_\gamma} m(u^\star - c), \quad \text{for } u^\star \geq c \end{aligned} \quad (17)$$

Computation of bounds in equation (17) requires, at least in principle, the solution of 2 nonlinear optimization problems with 2 variables and $4M$ constraints. However, the computational efforts can be dramatically reduced thanks to the results reported below, where we exploit the following definitions:

Definition 1
$h^+(\overline{u}_s)$ and $h^-(\overline{u}_s)$ are the constraints defining the FPS $\mathcal{D}_\gamma$ corresponding to the s-th sets of data:
$h^+(\overline{u}_s) \doteq \{m \in R^+, c \in R^+ : \overline{y}_s + \Delta\eta_s = m(\overline{u}_s - c)\}$
$h^-(\overline{u}_s) \doteq \{m \in R^+, c \in R^+ : \overline{y}_s - \Delta\eta_s = m(\overline{u}_s - c)\}$

Definition 2
Boundary of $\mathcal{D}_\gamma \doteq \mathcal{H}(\mathcal{D}_\gamma)$

Definition 3
The constraints $h^+(\overline{u}_s)$ and $h^-(\overline{u}_s)$ are said to be active constraints if their intersections with $\mathcal{H}(\mathcal{D}_\gamma)$ is not the empty set:

$$h^+(\overline{u}_s) \bigcap \mathcal{H}(\mathcal{D}_\gamma) \neq \emptyset \Longleftrightarrow h^+(\overline{u}_s) \text{ is active.}$$

$$h^-(\overline{u}_s) \bigcap \mathcal{H}(\mathcal{D}_\gamma) \neq \emptyset \Longleftrightarrow h^-(\overline{u}_s) \text{ is active.}$$

Remark 2 — It is trivial to see that a constraint $h^+(\overline{u}_s)$ ($h^-(\overline{u}_s)$) may either (a) intersect $\mathcal{H}(\mathcal{D}_\gamma)$ or (b) be external to $\mathcal{H}(\mathcal{D}_\gamma)$, hence be external to $\mathcal{D}_\gamma$.

Definition 4
$\tilde{h}^+(\overline{u}_s) \doteq h^+(\overline{u}_s) \bigcap \mathcal{D}_\gamma = \{ m, c \in \mathcal{D}_\gamma : \overline{y}_s + \Delta\eta_s = m(\overline{u}_s - c)\}$
$\tilde{h}^-(\overline{u}_s) \doteq h^-(\overline{u}_s) \bigcap \mathcal{D}_\gamma = \{ m, c \in \mathcal{D}_\gamma : \overline{y}_s - \Delta\eta_s = m(\overline{u}_s - c)\}$

Definition 5
x-level curve of the objective function to be optimized:

$$g(\overline{u}_s, x) \doteq \{m \in R^+, c \in R^+ : x = m(\overline{u}_s - c)\}$$

Proposition 1
The constraint $h^+(\overline{u}_s)$ (respectively $h^-(\overline{u}_s)$) is an active constraint if and only if $\tilde{h}^+(\overline{u}_s) \neq \emptyset$ (respectively $\tilde{h}^-(\overline{u}_s) \neq \emptyset$):

$$h^+(\overline{u}_s) \bigcap \mathcal{H}(\mathcal{D}_\gamma) \neq \emptyset \Longleftrightarrow \tilde{h}^+(\overline{u}_s) \neq \emptyset$$

$$h^-(\overline{u}_s) \bigcap \mathcal{H}(\mathcal{D}_\gamma) \neq \emptyset \Longleftrightarrow \tilde{h}^-(\overline{u}_s) \neq \emptyset$$

Proof
<u>Necessity</u> ($\Longrightarrow$):

$h^+(\overline{u}_s) \bigcap \mathcal{H}(\mathcal{D}_\gamma) \neq \emptyset \Longrightarrow$
$\{m, c \in \mathcal{H}(\mathcal{D}_\gamma) : \overline{y}_s + \Delta\eta_s = m(\overline{u}_s - c)\} \neq \emptyset$; we recall that $\mathcal{H}(\mathcal{D}_\gamma) \subset \mathcal{D}_\gamma$, hence $\{m, c \in \mathcal{D}_\gamma : \overline{y}_s + \Delta\eta_s = m(\overline{u}_s - c)\} = \tilde{h}^+(\overline{u}_s) \neq \emptyset$.

<u>Sufficiency</u> ($\Longleftarrow$):

we start from

$$\tilde{h}^+(\overline{u}_s) = \{ m, c \in \mathcal{D}_\gamma : \overline{y}_s + \Delta\eta_s = m(\overline{u}_s - c)\} \neq \emptyset$$

which means that the intersection of $h^+(\overline{u}_s)$ with $\mathcal{D}_\gamma$ is not the empty set. Thanks to *Remark 2*, that intersection take place on $\mathcal{H}(\mathcal{D}_\gamma)$, i.e., the boundary of $\mathcal{D}_\gamma$, which implies that

$$\begin{aligned}&\{\ m,c \in \mathcal{H}(\mathcal{D}_\gamma) : \overline{y}_s + \Delta\eta_s = m(\overline{u}_s - c)\} = \\ &h^+(\overline{u}_s) \bigcap \mathcal{H}(\mathcal{D}_\gamma) \neq \emptyset.\end{aligned} \tag{18}$$

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Proposition 2
Let us assume $u^\star = \overline{u}_s$. The maximum and minimum values of the magnitude $x^\star$ of the inner signal x_t are respectively $\overline{y}_s + \Delta\eta_s$ and $\overline{y}_s - \Delta\eta_s$ if and only if the constraints generated by the measurement $(\overline{u}_s, \overline{y}_s, \Delta\eta_s)$ on the space of parameters (m, c) are both active constraints, i.e.:

$$\begin{cases} x^{\star min} = \overline{y}_s - \Delta\eta_s \\ x^{\star max} = \overline{y}_s + \Delta\eta_s \end{cases} \Longleftrightarrow \begin{cases} \tilde{h}^-(\overline{u}_s) \neq \emptyset \\ \tilde{h}^+(\overline{u}_s) \neq \emptyset \end{cases}$$

Proof
Necessity ($\Longrightarrow$):

$$x^{\star min} = \min_{m,c \in \mathcal{D}_\gamma} m(\overline{u}_s - c) = \overline{y}_s - \Delta\eta_s$$

which imply the existence of some $m, c \in \mathcal{D}_\gamma$ such that the set $g(\overline{u}_s, x^{\star min}) \neq \emptyset$, that is:

$$g(\overline{u}_s, x^{\star min}) \doteq \{m, c \in \mathcal{D}_\gamma : \overline{y}_s - \Delta\eta_s = m(\overline{u}_s - c)\} \neq \emptyset.$$

Now it is easy to note that since $\tilde{h}^-(\overline{u}_s) = \tilde{g}(\overline{u}_s, x^{\star min})$ then $\tilde{h}^-(\overline{u}_s) \neq \emptyset$ which proves the necessity. Along the same line it can be proved that

$$x^{\star max} = \overline{y}_s + \Delta\eta_s \Longrightarrow \tilde{h}^+(\overline{u}_s) \neq \emptyset.$$

Sufficiency ($\Longleftarrow$):
$\tilde{h}^-(\overline{u}_s) \neq \emptyset \Longrightarrow$
$\tilde{h}^-(\overline{u}_s) = \{m, c \in \mathcal{D}_\gamma : \overline{y}_s - \Delta\eta_s = m(\overline{u}_s - c)\} \neq \emptyset$
$\tilde{h}^+(\overline{u}_s) \neq \emptyset \Longrightarrow$
$\tilde{h}^+(\overline{u}_s) = \{m, c \in \mathcal{D}_\gamma : \overline{y}_s + \Delta\eta_s = m(\overline{u}_s - c)\} \neq \emptyset$
which imply that
$m(\overline{u}_s - c) \leq \overline{y}_s + \Delta\eta_s,\ m(\overline{u}_s - c) \geq \overline{y}_s - \Delta\eta_s$, for some $m, c \in \mathcal{D}_\gamma$
which, in turn, means that

$$\max_{m,c \in \mathcal{D}_\gamma} m(\overline{u}_s - c) = \overline{y}_s + \Delta\eta_s = x^{\star max}$$

and

$$\min_{m,c \in \mathcal{D}_\gamma} m(\overline{u}_s - c) = \overline{y}_s - \Delta\eta_s = x^{\star min}$$

which prove sufficiency.

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Remark 3 — Proposition 2 enables the computation of bounds on $x^\star$ provided that one is able to find some $\overline{u}_s$ such that the constraints $h^-(\overline{u}_s)$ and $h^-(\overline{u}_s)$ are both active. As a matter of fact, for large M, it could be quite difficult to find such an $\overline{u}_s$. However, for small M the problem can be easily solved by means of graphical inspection. In particular, when $M = 2$, it is easy to show that all the constraints are active. Indeed, experiment design with $M = 2$, in spite of the small number of samples, provides acceptable results in the simulated example of Section V.

Now, if we define the following quantities

$$x_t^c = \frac{x^{\star min} + x^{\star max}}{2}, \qquad \Delta x_t = \frac{x^{\star max} - x^{\star min}}{2} \tag{19}$$

the following relation can be established between the unknown inner signal x_t and the central value x_t^c:

$$x_t^c = x_t + \delta x_t \tag{20}$$

$$\mid \delta x_t \mid \leq \Delta x_t. \tag{21}$$

We can now formulate the identification of the linear model in terms of the noisy output sequence $\{y_t\}$ and the uncertain inner sequence $\{x_t^c\}$ as shown in Fig. 3. Such a formulation is commonly referred to as an errors-in-variables problem (EIV), i.e. a parameter estimation problem in a linear-in-parameter model where the output and some or all the explanatory variables are uncertain.

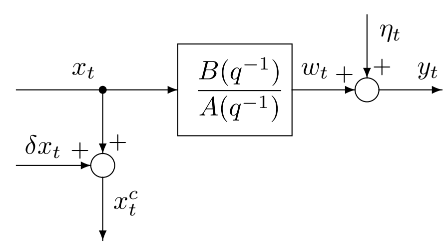

Fig. 3. Errors-in-variables set-up for bounding the parameters of the linear system.

As a matter of fact, combining equations (3), (4), (5), (6), (20) we get

$$y_t = -\sum_{i=1}^{na} (y_{t-i} - \eta_{t-i}) a_i + \sum_{j=0}^{nb} (x_{t-j}^c - \delta x_{t-j}) b_j + \eta_t. \tag{22}$$

The definition of the feasible parameter region for the linear system is:

$$\begin{aligned}\mathcal{D}_\theta = \{\theta \in R^p &: A(q^{-1})[y_t - \eta_t] = B(q^{-1})[x_t^c - \delta x_t]; \\ &g = 1; \mid \eta_t \mid \leq \Delta\eta_t; \mid \delta x_t \mid \leq \Delta x_t; t = 1, \ldots, N\}.\end{aligned} \tag{23}$$

where $g = 1$ takes account of condition (10) on the steady-state gain. From equation (22) it can be seen that consecutive regressions are related deterministically by uncertain output samples and uncertain input samples; that occurrence qualifies the problem as a dynamic EIV. It is referred to as a static EIV problem when the uncertain variables appearing in successive regressions are supposed to vary independently. The relations between successive regressions in the dynamic EIV case give rise to possibly nonlinear exact parameter bounds, which could be not easily and exactly computed [22]. On the other end, in the static EIV case exact parameter bounds are piecewise linear and, although generally non convex, the feasible parameter region is the union of at most 2^p convex sets: each being the intersection of the FPS with a single orthant of the p-dimensional parameter space (a detailed discussion on the geometrical and topological structure of the feasible parameter region for static EIV problems can be found in [23]). Thus, as shown in [22], the FPS of static EIV can be more conveniently handled than the FPS of dynamic EIV. That motivates the use, in this paper, of results from the static EIV [23]; since in model (22) the uncertain variables appearing in successive regressions are deterministically related, only outer approximations of the exact feasible parameter region will be obtained. Thus, in this work, a polytopic outer approximation $\mathcal{D}'_\theta$ of the exact FPS $\mathcal{D}_\theta$, i.e. $\mathcal{D}'_\theta \supset \mathcal{D}_\theta$, will be presented, together with an orthotope-outer bounding set $\mathcal{B}_\theta$ of $\mathcal{D}'_\theta$, which provides parameter uncertainties intervals. When we apply results from [23] to our problem we get the following description of the feasible parameter set $\mathcal{D}'_\theta$ at the single time t

$$\begin{aligned} (\phi_t - \Delta\phi_t)^{\mathrm{T}}\theta \leq y_t + \Delta\eta_t, \\ (\phi_t + \Delta\phi_t)^{\mathrm{T}}\theta \geq y_t - \Delta\eta_t \end{aligned} \tag{24}$$

$$[1 \quad \dots \quad 1 \quad -1 \quad -1 \quad \dots \quad -1]\,\theta = -1 \tag{25}$$

where

$$\phi_t^{\mathrm{T}} = \left[-y_{t-1} \dots - y_{t-na} \; x_t^c \; x_{t-1}^c \dots x_{t-nb}^c\right] \tag{26}$$

$$\begin{aligned} \Delta\phi_t^{\mathrm{T}} = [\Delta\eta_{t-1}\mathrm{sgn}(a_1) \;\; \dots \;\; \Delta\eta_{t-na}\mathrm{sgn}(a_{na}) \;\; \Delta x_t \mathrm{sgn}(b_0) \\ \Delta x_{t-1}\mathrm{sgn}(b_1) \;\; \dots \;\; \Delta x_{t-nb}\mathrm{sgn}(b_{nb})] \end{aligned} \tag{27}$$

Equation (25) takes account of condition (10) on the steady-state gain. The orthotope-outer bounding set $\mathcal{B}_\theta$ is defined as

$$\mathcal{B}_\theta = \{\theta \in R^p : \theta_j = \theta_j^c + \delta\theta_j, \mid \delta\theta_j \mid \leq \Delta\theta_j, j = 1, \dots, p\}, \tag{28}$$

where

$$\theta_j^c = \frac{\theta_j^{min} + \theta_j^{max}}{2}, \qquad \Delta\theta_j = \frac{\mid \theta_j^{max} - \theta_j^{min} \mid}{2} \tag{29}$$

$$\theta_j^{min} = \min_{\theta \in \mathcal{D}'_\theta} \theta_j, \quad \theta_j^{max} = \max_{\theta \in \mathcal{D}'_\theta} \theta_j. \tag{30}$$

Parameter vectors γ^c and θ^c are Chebishev centers in the ℓ_∞ norm of $\mathcal{D}_\gamma$ and $\mathcal{D}'_\theta$ respectively and are commonly referred to as central estimates.

V. A SIMULATED EXAMPLE

In this section we illustrate the proposed parameter bounding procedure through a numerical example. The system considered here is characterized by (1), (3) and (6) with: $m = 1$, $c = 3$; $A(q^{-1}) = (1 + 0.5q^{-1} - 0.1q^{-2})$ and $B(q^{-1}) = (0.2q^{-1} + 1.2q^{-2})$. Thus, the true parameter vectors are $\gamma = [m \; c]^{\mathrm{T}} = [1 \; 3]^{\mathrm{T}}$ and $\theta = [a_1 \; a_2 \; b_1 \; b_2]^{\mathrm{T}} = [0.5 \; -0.1 \; 0.2 \; 1.2]^{\mathrm{T}}$. From the simulated transient sequence $\{w_t, \eta_t\}$, the signal to noise ratio SNR is evaluated through

$$SNR = 10 \log \left\{ \sum_{t=1}^{N} w_t^2 \Big/ \sum_{t=1}^{N} \eta_t^2 \right\}. \tag{31}$$

Bounded relative output errors have been considered when simulating the collection of both steady state and transient data. More precisely, we assumed $\eta_t = \epsilon_t^y y_t$, $\mid \epsilon_t^y \mid \leq \Delta\epsilon_t^y$, $\bar{\eta}_s = \bar{\epsilon}_s^y \bar{y}_s$, $\mid \bar{\epsilon}_s^y \mid \leq \Delta\bar{\epsilon}_s^y$; where $\{\epsilon_t^y\}$ and $\{\bar{\epsilon}_s^y\}$ are random sequences belonging to the uniform distributions $U[-\Delta\epsilon_t^y, +\Delta\epsilon_t^y]$ and $U[-\Delta\bar{\epsilon}_s^y, +\Delta\bar{\epsilon}_s^y]$ respectively. Bounds on steady-state and transient output measurement errors were supposed to have the same value, i.e., $\Delta\epsilon_t^y = \Delta\bar{\epsilon}_s^y \doteq \Delta\epsilon^y$. Five different values of uncertainty bounds were considered: $\Delta\epsilon^y = 0.1\%,\ 1\%,\ 5\%, 10\%, 20\%$. For a given $\Delta\epsilon^y$, the length of steady-state and the transient data are $M = 2$ and $N = [100, 1000]$ respectively. The steady-state input samples are $\{\bar{u}_s\} = \{4, 10\}$, while the transient input sequence $\{u_t\}$ is a PRBS which takes the values ± 4. Results about the nonlinear and the linear block are reported in Table I, Tables III and V respectively. For low noise level ($\Delta\epsilon^y = 0.1\%$) and for all N, the central estimates of both the nonlinear static block and the linear model are consistent with the true parameters. For higher noise level ($\Delta\epsilon^y \geq 1\%$), both γ^c and θ^c give satisfactory estimates of the true parameters. As the number of observations increases (from $N = 100$ to $N = 1000$), parameter uncertainty bounds $\Delta\gamma_j$ and $\Delta\theta_j$ decreases unsurprisingly.

VI. CONCLUSION

A two-stage parameter bounding procedure for linear systems with input backlash in presence of bounded output errors has been outlined. First, using steady-state input-output data the two parameters of the backlash have been tightly bounded. Then, for a given input transient sequence we have computed bounds on the unmeasurable inner signal which, together with output noisy measurements have been used to overbound the parameters of the linear part. The numerical example showed the effectiveness of the proposed procedure.

Table I: Nonlinear block parameter central estimates $\gamma^c = [m^c, c^c]$ and parameter uncertainty bounds $\Delta\gamma = [\Delta m, \Delta c]$ against varying measurements uncertainty ($\Delta\epsilon^y$).

$\Delta\epsilon^y$ (%)	γ_j	True Value	γ_j^c	$\Delta\gamma_j$
0.1	m	1.000	1.001	1.3e-3
	c	3.000	3.001	2.3e-3
1	m	1.000	1.008	1.3e-2
	c	3.000	3.010	2.3e-2
10	m	1.000	1.043	1.4e-1
	c	3.000	2.914	2.5e-1

Table III: Relative Error — Linear system parameter central estimates (θ_j^c) and parameter uncertainty bounds ($\Delta\theta_j$) against varying measurements uncertainty ($\Delta\epsilon^y$) and signal to noise ratio (SNR) when $N = 100$.

$\Delta\epsilon^y$ (%)	SNR (dB)	θ_j	True Value	θ_j^c	$\Delta\theta_j$
0.1	65.1	θ_1	0.500	0.500	1.0e-3
		θ_2	-0.100	-0.100	7.5e-4
		θ_3	0.200	0.200	9.8e-4
		θ_4	1.200	1.200	1.2e-3
1	45.1	θ_1	0.500	0.501	1.1e-2
		θ_2	-0.100	-0.099	6.9e-3
		θ_3	0.200	0.199	9.3e-3
		θ_4	1.200	1.202	1.1e-2
10	23.5	θ_1	0.500	0.460	8.0e-2
		θ_2	-0.100	-0.127	5.8e-2
		θ_3	0.200	0.243	9.1e-2
		θ_4	1.200	1.148	1.0e-1

Table V: Relative Error — Linear system parameter central estimates (θ_j^c) and parameter uncertainty bounds ($\Delta\theta_j$) against varying measurements uncertainty ($\Delta\epsilon^y$) and signal to noise ratio (SNR) when $N = 1000$.

$\Delta\epsilon^y$ (%)	SNR (dB)	θ_j	True Value	θ_j^c	$\Delta\theta_j$
0.1	64.6	θ_1	0.500	0.500	7.2e-4
		θ_2	-0.100	-0.100	5.0e-4
		θ_3	0.200	0.200	8.4e-4
		θ_4	1.200	1.200	9.1e-4
1	44.9	θ_1	0.500	0.502	7.3e-3
		θ_2	-0.100	-0.100	5.2e-3
		θ_3	0.200	0.199	8.1e-3
		θ_4	1.200	1.202	9.2e-3
10	24.9	θ_1	0.500	0.461	7.7e-2
		θ_2	-0.100	-0.130	4.3e-2
		θ_3	0.200	0.241	7.9e-2
		θ_4	1.200	1.145	9.3e-2

ACKNOWLEDGMENT

This research was partly supported by the italian Ministry of Universities and Research in Science and Technology (MURST), under the plan "Robustness and optimization techniques for control of uncertain systems".

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