A kinetic approach to the modeling of crowd dynamics

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Incontro tematico: Safety, panic, traffic and crowd dynamics
Bologna, October 24th, 2013
Complexity features of crowd’s dynamics

- **Ability to express a strategy:** Living entities are capable to develop specific *strategies*, which should include: trend toward the exit; avoiding clusters; avoiding walls and obstacles; perception of signals; etc.
- **Heterogeneity:** The ability described in the first item is *heterogeneously distributed*. Heterogeneity includes, in addition to different walking abilities, also the possible presence of a hierarchy (namely a leader).
- **Interactions:** *Interactions are nonlinearly additive* and are *nonlocal in space*, since pedestrians perceive stimuli at a distance which depends on the geometry of the system where they move, as well as on the general environmental and psychological conditions. For instance, the perception domain of each pedestrian is modified by panic conditions.
The mean distance between pedestrians may be small or large, where pedestrians either fill the whole square or walk in streets without overcrowding phenomena. In some cases these limit situations can occur within the same area.
Hallmarks of the kinetic theory of active particles

- The overall system is subdivided into *functional subsystems* constituted by entities, called *active particles*, whose individual state is called *activity*;
- The state of each functional subsystem is defined by a suitable, time dependent, *distribution function over the microscopic state*;
- Interactions are modeled by tools of games theory, more precisely stochastic games, where the state of the interacting particles and the output of the interactions are known in probability;
- Interactions are delocalized and nonlinearly additive;
- The evolution of the distribution function is obtained by a balance of particles within elementary volumes of the space of the microscopic states, where the dynamics of inflow and outflow of particles is related to interactions at the microscopic scale.
**Toward a kinetic theory of active particles**

<table>
<thead>
<tr>
<th>Crowd dynamics</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Active particles</strong></td>
</tr>
<tr>
<td>Pedestrians</td>
</tr>
<tr>
<td><strong>Microscopic state</strong></td>
</tr>
<tr>
<td>Position</td>
</tr>
<tr>
<td>Velocity</td>
</tr>
<tr>
<td>Activity</td>
</tr>
<tr>
<td><strong>Functional subsystems</strong></td>
</tr>
<tr>
<td>Individuals pursuing different targets etc.</td>
</tr>
</tbody>
</table>

D. Knopoff
A kinetic approach to the modeling of crowd dynamics

Modeling by means of the kinetic theory for active particles

Crowds in unbounded domains

Particles in $P_1$ and $P_2$ move, respectively, toward the directions $T_1$ and $T_2$ identified by the directions $\theta_{v_1}$ and $\theta_{v_2}$ with respect to the horizontal axis.
Polar coordinates with discrete values are used for the velocity variable $v = \{v, \theta\}$:

$$I_\theta = \{\theta_1 = 0, \ldots, \theta_i, \ldots, \theta_n = \frac{n-1}{n}2\pi\}, \quad I_v = \{v_1 = 0, \ldots, v_j, \ldots, v_m = 1\}.$$

$$f(t, x, v, u) = \sum_{i=1}^{n} \sum_{j=1}^{m} f_{ij}(t, x, u) \delta(\theta - \theta_i) \otimes \delta(v - v_j).$$

Some specific cases can be considered. For instance the case of two different groups, labeled with the superscript $\sigma = 1, 2$, which move towards two different targets.

$$f^\sigma(t, x, v, u) = \sum_{i=1}^{n} \sum_{j=1}^{m} f_{ij}^\sigma(t, x) \delta(\theta - \theta_i) \otimes \delta(v - v_j) \otimes \delta(u - u_0),$$

Local density:

$$\rho(t, x) = \sum_{\sigma=1}^{2} \rho^\sigma(t, x) = \sum_{\sigma=1}^{2} \sum_{i=1}^{n} \sum_{j=1}^{m} f_{ij}^\sigma(t, x),$$
Interaction terms

- **Interaction rate**: It is assumed that it increases with increasing local density in the free and congested regimes. For higher densities, when pedestrians are obliged to stop, one may assume that it keeps a constant value, or may decay for lack of interest.

- **Transition probability density**: We assume that particles are subject to two different influences, namely the *trend to the exit point*, and the *influence of the stream* induced by the other pedestrians. A simplified interpretation of the phenomenological behavior is obtained by assuming the factorization of the two probability densities modeling the modifications of the velocity direction and modulus:

\[
\mathcal{A}_{h_k,p_q}(i|j) = B_{h_p}^\sigma(i) \left( \theta_h \rightarrow \theta_i | \rho(t, x) \right) \times C_{k_q}^\sigma(j) \left( v_k \rightarrow v_j | \rho(t, x) \right). 
\]
A kinetic approach to the modeling of crowd dynamics

Modeling by means of the kinetic theory for active particles

Interactions in the table of games

Particle in P moves to a direction $\theta_h$ (black arrow) and interacts with a field particle moving to $\theta_p$ (blue arrow), the direction to the target is $\theta_v$ (red arrow).
A particle can change its velocity direction, in probability, only to an **adjacent state**. (a) A candidate particle with direction \( \theta_h \) interacts with an upper stream with direction \( \theta_p \) and target directions \( \theta_v \) and decides to change its direction to \( \theta_{h+1} \). (b) A candidate particle interacts with an upper stream and lower target directions, and decides to change its direction either to \( \theta_{h+1} \) or \( \theta_{h-1} \).
Transition probability density (example)

- Interaction with an upper stream and target directions, namely $\theta_p > \theta_h$, $\theta_v > \theta_h$:

$$
B_{hp}(i) \begin{cases} 
\alpha u_0(1 - \rho) + \alpha u_0 \rho & \text{if } i = h + 1, \\
1 - \alpha u_0(1 - \rho) - \alpha u_0 \rho & \text{if } i = h, \\
0 & \text{if } i = h - 1.
\end{cases}
$$
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Mathematical structures

Variation rate of the number of active particles

\[
(\partial_t + \mathbf{v}_{ij} \cdot \nabla_{\mathbf{x}}) f_{ij}^\sigma(t, \mathbf{x}) = J[f](t, \mathbf{x})
\]

\[
= \sum_{h,p=1}^n \sum_{k,q=1}^m \int_{\Lambda} \eta[\rho(t, \mathbf{x}^*)] \mathcal{A}_{hk,pq}(ij) [\rho(t, \mathbf{x}^*)] f_{hk}^\sigma(t, \mathbf{x}) f_{pq}^\sigma(t, \mathbf{x}^*) d\mathbf{x}^*
\]

\[
- f_{ij}^\sigma(t, \mathbf{x}) \sum_{p=1}^n \sum_{q=1}^m \int_{\Lambda} \eta[\rho(t, \mathbf{x}^*)] f_{pq}^\sigma(t, \mathbf{x}^*) d\mathbf{x}^*,
\]

(1)

where \( f = \{f_{ij}\} \).
Mild form of the initial value problem

The initial value problem consists in solving Eqs. (1) with initial conditions given by

\[ f_{ij}^\sigma(0, \mathbf{x}) = \phi_{ij}^\sigma(\mathbf{x}). \]

Let us introduce the mild form obtained by integrating along the characteristics:

\[
\hat{f}_{ij}^\sigma(t, \mathbf{x}) = \phi_{ij}^\sigma(\mathbf{x}) + \int_0^t \left( \Gamma_{ij}^\sigma[f, f](s, \mathbf{x}) - \hat{f}_{ij}^\sigma(s, \mathbf{x}) \mathcal{L}[f](s, \mathbf{x}) \right) ds,
\]

\[ i \in \{1, \ldots, n\}, \quad j \in \{1, \ldots, m\}, \quad \sigma \in \{1, 2\}, \]

where the following notation has been used for any given vector \( f(t, \mathbf{x}) \):

\[ f_{ij}^\sigma(t, \mathbf{x}) = f_{ij}^\sigma(t, x + v_j \cos(\theta_i)t, y + v_j \sin(\theta_i)t). \]
Existence theory

**H.1.** For all positive $R$, there exists a constant $C_\eta > 0$ so that
$0 < \eta(\rho) \leq C_\eta$, whenever $0 \leq \rho \leq R$.

**H.2.** Both the encounter rate $\eta[\rho]$ and the transition probability $A_{hk,pq}^\sigma(ij)[\rho]$ are Lipschitz continuous functions of the macroscopic density $\rho$, i.e., that there exist constants $L_\eta, L_A$ is such that

\[
| \eta[\rho_1] - \eta[\rho_2] | \leq L_\eta | \rho_1 - \rho_2 |,
\]

\[
| A_{hk,pq}^\sigma(ij)[\rho_1] - A_{hk,pq}^\sigma(ij)[\rho_2] | \leq L_A | \rho_1 - \rho_2 |,
\]

whenever $0 \leq \rho_1 \leq R$, $0 \leq \rho_2 \leq R$, and all $i,h,p = 1,\ldots,n$ and $j,k,q = 1,\ldots,m$. 
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On the initial value problem

• Let $\phi_{ij} \in L^\infty \cap L^1$, $\phi_{ij} \geq 0$, then there exists $\phi^0$ so that, if $\| \phi \|_1 \leq \phi^0$, there exist $T$, $a_0$, and $R$ so that a unique non-negative solution to the initial value problem exists and satisfies:

$$f \in X_T, \quad \sup_{t \in [0, T]} \| f(t) \|_1 \leq a_0 \| \phi \|_1,$$

$$\rho(t, x) \leq R, \quad \forall t \in [0, T], \quad x \in \Omega.$$  

Moreover, if $\Sigma_{\sigma=1}^2 \Sigma_{i=1}^n \Sigma_{j=1}^m \| \phi_{ij}^\sigma \|_\infty \leq 1$, and $\| \phi \|_1$ is small, one has $\rho(t, x) \leq 1$, $\forall t \in [0, T]$, $x \in \Omega$.

• There exist $\phi^r$, $(r = 1, \ldots, p - 1)$ such that if $\| \phi \|_1 \leq \phi^r$, there exists $a_r$ so that it is possible to find a unique non-negative solution to the initial value problem satisfying for any $r \leq p - 1$ the following $f(t) \in X[0, (p - 1)T]$,

$$\sup_{t \in [0, T]} \| f(t + (r - 1)T) \|_1 \leq a_{r-1} \| \phi \|_1,$$

and $\rho(t + (r - 1)T, x) \leq R, \quad \forall t \in [0, T], \quad x \in \Omega$. Moreover,

$$\rho(t + (r - 1)T, x) \leq 1, \quad \forall t \in [0, T], \quad x \in \Omega.$$
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Numerical Results

Case-study 1
Numerical Results

Case-study 1
Case-study 2

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Numerical Results

Case-study 2

(c)

(d)
Case-study 2 - Top view
Modeling of panic conditions:
- The effective visibility zone becomes larger and signals from large distance become important, while in the case of normal conditions short distance signals appear to be more important;
- Pedestrians appear to be sensitive to crowd concentration, while in normal conditions they attempt to avoid it. Therefore different weights need to be used.
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What’s next?

Questions?