Belief Propagation for combinatorial optimization

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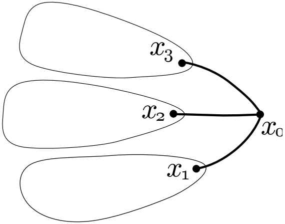
3-COLORING

- Given a (finite) undirected graph $G = (V, E)$
- A proper 3−coloring is $c : V \to \{\color{red}\bullet, \color{green}\bullet, \color{blue}\bullet\}$ such that $c(i) \neq c(j)$ if $(i, j) \in E$
- Finding proper colorings is a hard computational problem (NP-Complete)
- Counting proper colorings is also a hard problem

Belief Propagation on a slide: 3-COLORING



$$N_0\left(\textcolor{red}{\bullet}\right) \quad = \quad N^{(0)}\left(\textcolor{blue}{\bullet\bullet\bullet}\right) + N^{(0)}\left(\textcolor{blue}{\bullet\bullet}\textcolor{green}{\bullet}\right) + N^{(0)}\left(\textcolor{blue}{\bullet}\textcolor{green}{\bullet}\textcolor{blue}{\bullet}\right) + N^{(0)}\left(\textcolor{green}{\bullet}\textcolor{blue}{\bullet\bullet}\right) + \cdots$$

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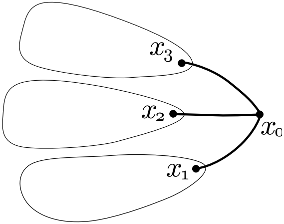

$$
\begin{aligned}
N_0\left(\textcolor{red}{\bullet}\right) &= N^{(0)}\left(\textcolor{blue}{\bullet\bullet\bullet}\right) + N^{(0)}\left(\textcolor{blue}{\bullet\bullet}\textcolor{green}{\bullet}\right) + N^{(0)}\left(\textcolor{blue}{\bullet}\textcolor{green}{\bullet}\textcolor{blue}{\bullet}\right) + N^{(0)}\left(\textcolor{green}{\bullet}\textcolor{blue}{\bullet\bullet}\right) + \cdots \\
&= N_1^{(0)}\left(\textcolor{blue}{\bullet}\right) N_2^{(0)}\left(\textcolor{blue}{\bullet}\right) N_3^{(0)}\left(\textcolor{blue}{\bullet}\right) + N_1^{(0)}\left(\textcolor{blue}{\bullet}\right) N_2^{(0)}\left(\textcolor{blue}{\bullet}\right) N_3^{(0)}\left(\textcolor{green}{\bullet}\right) + \cdots
\end{aligned}
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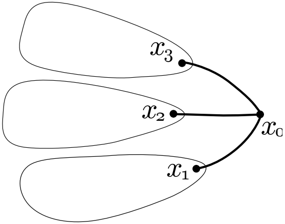

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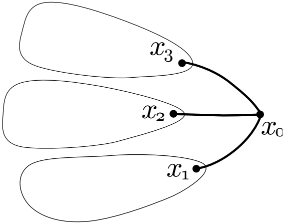

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N_0\left({\color{green}\bullet}\right) &= \left(N_1^{(0)}\left({\color{blue}\bullet}\right) + P_1^{(0)}\left({\color{red}\bullet}\right)\right) \left(N_2^{(0)}\left({\color{blue}\bullet}\right) + N_2^{(0)}\left({\color{red}\bullet}\right)\right) \left(N_3^{(0)}\left({\color{blue}\bullet}\right) + N_3^{(0)}\left({\color{red}\bullet}\right)\right)
\end{aligned}
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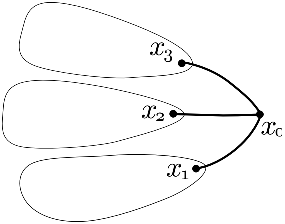

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N_0\left({\color{green}\bullet}\right) &= \left(N_1^{(0)}\left({\color{blue}\bullet}\right) + P_1^{(0)}\left({\color{red}\bullet}\right)\right)\left(N_2^{(0)}\left({\color{blue}\bullet}\right) + N_2^{(0)}\left({\color{red}\bullet}\right)\right)\left(N_3^{(0)}\left({\color{blue}\bullet}\right) + N_3^{(0)}\left({\color{red}\bullet}\right)\right) \\
N_0\left({\color{blue}\bullet}\right) &= \left(N_1^{(0)}\left({\color{red}\bullet}\right) + N_1^{(0)}\left({\color{green}\bullet}\right)\right)\left(N_2^{(0)}\left({\color{red}\bullet}\right) + N_2^{(0)}\left({\color{green}\bullet}\right)\right)\left(N_3^{(0)}\left({\color{red}\bullet}\right) + N_3^{(0)}\left({\color{green}\bullet}\right)\right)
\end{aligned}
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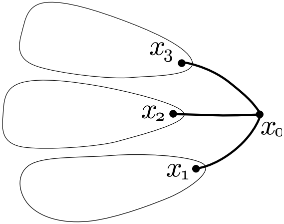

$$
\begin{aligned}
P_0\left(\textcolor{red}{\bullet}\right) &\propto P^{(0)}\left(\textcolor{blue}{\bullet\bullet\bullet}\right) + P^{(0)}\left(\textcolor{blue}{\bullet\bullet}\textcolor{green}{\bullet}\right) + P^{(0)}\left(\textcolor{blue}{\bullet}\textcolor{green}{\bullet}\textcolor{blue}{\bullet}\right) + P^{(0)}\left(\textcolor{green}{\bullet}\textcolor{blue}{\bullet\bullet}\right) + \cdots \\
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P_0\left(\textcolor{green}{\bullet}\right) &\propto \left(P_1^{(0)}\left(\textcolor{blue}{\bullet}\right) + P_1^{(0)}\left(\textcolor{red}{\bullet}\right)\right)\left(P_2^{(0)}\left(\textcolor{blue}{\bullet}\right) + P_2^{(0)}\left(\textcolor{red}{\bullet}\right)\right)\left(P_3^{(0)}\left(\textcolor{blue}{\bullet}\right) + P_3^{(0)}\left(\textcolor{red}{\bullet}\right)\right) \\
P_0\left(\textcolor{blue}{\bullet}\right) &\propto \left(P_1^{(0)}\left(\textcolor{red}{\bullet}\right) + P_1^{(0)}\left(\textcolor{green}{\bullet}\right)\right)\left(P_2^{(0)}\left(\textcolor{red}{\bullet}\right) + P_2^{(0)}\left(\textcolor{green}{\bullet}\right)\right)\left(P_3^{(0)}\left(\textcolor{red}{\bullet}\right) + P_3^{(0)}\left(\textcolor{green}{\bullet}\right)\right)
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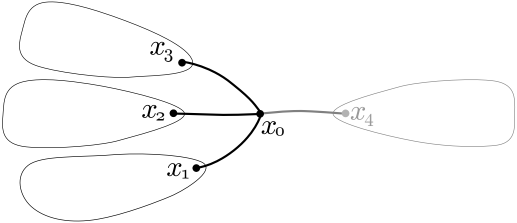

$$
\begin{aligned}
P_0^{(4)}(\textcolor{red}{\bullet}) &\propto P^{(0)}(\textcolor{blue}{\bullet}\textcolor{blue}{\bullet}\textcolor{blue}{\bullet}) + P^{(0)}(\textcolor{blue}{\bullet}\textcolor{blue}{\bullet}\textcolor{green}{\bullet}) + P^{(0)}(\textcolor{blue}{\bullet}\textcolor{green}{\bullet}\textcolor{blue}{\bullet}) + P^{(0)}(\textcolor{green}{\bullet}\textcolor{blue}{\bullet}\textcolor{blue}{\bullet}) + \cdots \\
&= P_1^{(0)}(\textcolor{blue}{\bullet})\, P_2^{(0)}(\textcolor{blue}{\bullet})\, P_3^{(0)}(\textcolor{blue}{\bullet}) + P_1^{(0)}(\textcolor{blue}{\bullet})\, P_2^{(0)}(\textcolor{blue}{\bullet})\, P_3^{(0)}(\textcolor{green}{\bullet}) + \cdots \\
&= \left(P_1^{(0)}(\textcolor{blue}{\bullet}) + P_1^{(0)}(\textcolor{green}{\bullet})\right)\left(P_2^{(0)}(\textcolor{blue}{\bullet}) + P_2^{(0)}(\textcolor{green}{\bullet})\right)\left(P_3^{(0)}(\textcolor{blue}{\bullet}) + P_3^{(0)}(\textcolor{green}{\bullet})\right) \\
P_0^{(4)}(\textcolor{green}{\bullet}) &\propto \left(P_1^{(0)}(\textcolor{blue}{\bullet}) + P_1^{(0)}(\textcolor{red}{\bullet})\right)\left(P_2^{(0)}(\textcolor{blue}{\bullet}) + P_2^{(0)}(\textcolor{red}{\bullet})\right)\left(P_3^{(0)}(\textcolor{blue}{\bullet}) + P_3^{(0)}(\textcolor{red}{\bullet})\right) \\
P_0^{(4)}(\textcolor{blue}{\bullet}) &\propto \left(P_1^{(0)}(\textcolor{red}{\bullet}) + P_1^{(0)}(\textcolor{green}{\bullet})\right)\left(P_2^{(0)}(\textcolor{red}{\bullet}) + P_2^{(0)}(\textcolor{green}{\bullet})\right)\left(P_3^{(0)}(\textcolor{red}{\bullet}) + P_3^{(0)}(\textcolor{green}{\bullet})\right)
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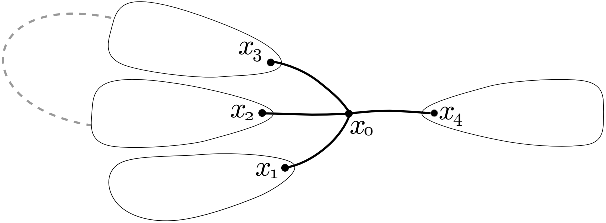

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\begin{aligned}
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&{\color{magenta}\simeq} P_1^{(0)}\left({\color{blue}\bullet}\right) P_2^{(0)}\left({\color{blue}\bullet}\right) P_3^{(0)}\left({\color{blue}\bullet}\right) + P_1^{(0)}\left({\color{blue}\bullet}\right) P_2^{(0)}\left({\color{blue}\bullet}\right) P_3^{(0)}\left({\color{green}\bullet}\right) + \cdots \\
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BP Equations

Given $\Psi_a(\mathbf{x}_a) \geq 0$ with $\mathbf{x}_a = \{x_i\}_{i \in V(a)}$ *($x_i \in X_i$ finite)*, and $P(\mathbf{x}) = \frac{1}{Z} \prod_{a \in A} \Psi_a(\mathbf{x}_a)$, BP Equations are

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$$
\begin{aligned}
m_{ai}(x_i) &\propto \sum_{\{x_j\}_{j \in V(a) \setminus i}} \Psi_a(\mathbf{x}_a) \prod_{j \in V(a) \setminus i} m_{ja}(x_j) \\
m_{ia}(x_i) &\propto \prod_{b \in V(i) \setminus a} m_{bi}(x_i) \\
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- $m_i(x_i) \approx P(x_i) = \sum_{\{x_j\}_{j \in I \setminus i}} P(\mathbf{x})$,
- $m_a(\mathbf{x}_a) \propto \Psi_a(\mathbf{x}_a) \prod_{i \in V(a)} m_{ia}(x_i) \approx P(\mathbf{x}_a) = \sum_{\{x_j\}_{j \in I \setminus V(a)}} P(\mathbf{x})$

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- $m_i(x_i) \approx P(x_i) = \sum_{\{x_j\}_{j \in I \setminus i}} P(\mathbf{x})$,
- $m_a(\mathbf{x}_a) \propto \Psi_a(\mathbf{x}_a) \prod_{i \in V(a)} m_{ia}(x_i) \approx P(\mathbf{x}_a) = \sum_{\{x_j\}_{j \in I \setminus V(a)}} P(\mathbf{x})$
- $-\log Z \approx F_{Bethe} =$ $\sum_a \sum_{\mathbf{x}_a} m_a(\mathbf{x}_a) \log \frac{m_a(\mathbf{x}_a)}{\Psi_a(\mathbf{x}_a)} + \sum_i (1 - |V(i)|) \sum_{x_i} m_i(x_i) \log m_i(x_i)$

BP Algo

- The FP equation $F_{BP}(\mathbf{m}) = \mathbf{m}$ is solved by iteration $\lim_{n \to \infty} F_{BP}^{(n)}(\mathbf{m}_0)$

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$$m_{ia}^{(t+1)}(x_i) \quad \propto \quad \prod_{b\in V(i)\setminus a} \sum_{\{x_j\}_{j\in V(b)\setminus i}} \Psi_b(\mathbf{x}_b) \prod_{j\in V(b)\setminus i} m_{jb}^{(t)}(x_j)$$

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- On a tree BP Equations are exact: there is 1 FP and $m_i(x_i) = P(x_i)\,,\ m_a(\mathbf{x}_a) = P(\mathbf{x}_a),\ F_{Bethe} = -\log Z$

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- Loopy graphs $\implies$ BP solutions are usually *good approximations*

Applications of BP

- **SAMPLER** ($\mathbf{x}^* \sim P(\mathbf{x})$): Note $P(\mathbf{x}) = P(x_1)\, P(x_2, \ldots x_n | x_1)$.
  1. Use BP to estimate $P(x_1)$
  2. Extract $x_1^* \sim P(x_1)$.
  3. Modify the problem by adding a factor $\Psi_1(x_1) = \delta(x_1; x_1^*)$ to P, reiterate

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- **COUNTER** (estimate $\#\{\mathbf{x} : \prod_a \Psi_a(\mathbf{x}) = 1\}$ where $\Psi_a(\mathbf{x}_a) \in \{0, 1\}$): Use BP estimation of $\log Z$

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- **SOLVER** ($\mathbf{x}^* \in \{\mathbf{x} : \prod_a \Psi_a(\mathbf{x}^*) = 1\}$, where $\Psi_a(\mathbf{x}_a) \in \{0, 1\}$):
  1. Run BP.
  2. Find i and x_i^* s.t. $P(x_i^*) = \max\{P(x_j) : j \in V, x_j \in X_j\}$
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  3. Modify the problem by adding a factor $\Psi_1(x_1) = \delta(x_1; x_1^*)$, reiterate
- **OPTIMIZER** (find $\arg\max P(x)$) ... more later!

Crosswords!

■	A	L	M	I	R	A	■	L	E
A	L	A	■	L	A	S	S	E	S
R	I	N	S	E	D	■	H	A	T
I	N	■	P	A	S	S	E	D	■
S	A	R	A	N	■	T	E	S	S
E	■	I	R	A	T	E	R	■	A
S	A	V	E	■	R	E	S	T	S
■	R	E	S	E	E	D	■	A	S
M	I	R	■	R	E	S	A	L	E
B	A	S	T	E	S	■	M	E	D

Crosswords!

- Around 100.000 english words (taken from the aspell dictionary)

	A	L	M	I	R	A		L	E
A	L	A		L	A	S	S	E	S
R	I	N	S	E	D		H	A	T
I	N		P	A	S	S	E	D	
S	A	R	A	N		T	E	S	S
E		I	R	A	T	E	R		A
S	A	V	E		R	E	S	T	S
	R	E	S	E	E	D		A	S
M	I	R		R	E	S	A	L	E
B	A	S	T	E	S		M	E	D

Crosswords!

■	A	L	M	I	R	A	■	L	E
A	L	A	■	L	A	S	S	E	S
R	I	N	S	E	D	■	H	A	T
I	N	■	P	A	S	S	E	D	■
S	A	R	A	N	■	T	E	S	S
E	■	I	R	A	T	E	R	■	A
S	A	V	E	■	R	E	S	T	S
■	R	E	S	E	E	D	■	A	S
M	I	R	■	R	E	S	A	L	E
B	A	S	T	E	S	■	M	E	D

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Crosswords!

	A	L	M	I	R	A		L	E
A	L	A		L	A	S	S	E	S
R	I	N	S	E	D		H	A	T
I	N		P	A	S	S	E	D	
S	A	R	A	N		T	E	S	S
E		I	R	A	T	E	R		A
S	A	V	E		R	E	S	T	S
	R	E	S	E	E	D		A	S
M	I	R		R	E	S	A	L	E
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S	A	R	A	N		T	E	S	S
E		I	R	A	T	E	R		A
S	A	V	E		R	E	S	T	S
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R	I	N	S	E	D	■	H	A	T
I	N	■	P	A	S	S	E	D	■
S	A	R	A	N	■	T	E	S	S
E	■	I	R	A	T	E	R	■	A
S	A	V	E	■	R	E	S	T	S
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- These problems are mathematically easy: all sets here are finite!

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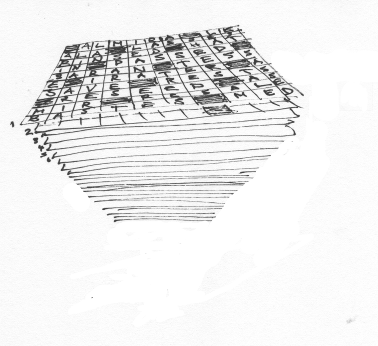

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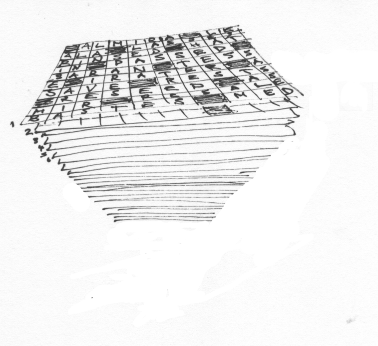

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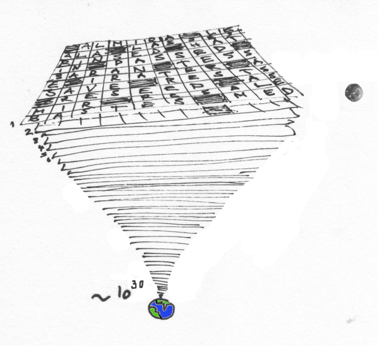

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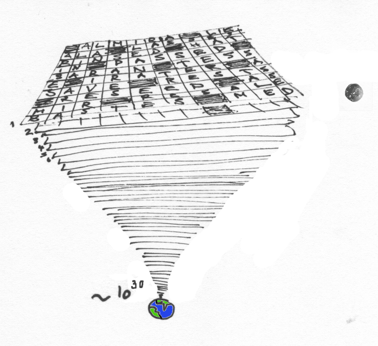

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In Statistical Physics, Gibbs measures are often studied :

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Max Sum (2)

MS for D-bounded Minimum Steiner Tree

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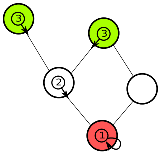

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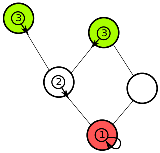

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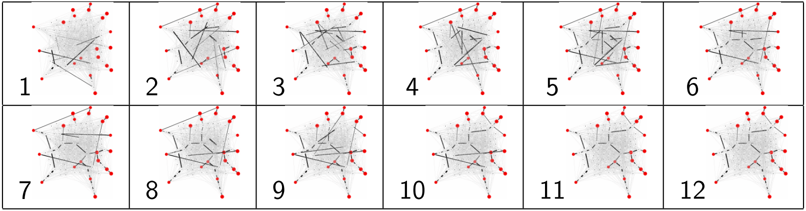

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- T_k is locally isomorphic to G, but T_k is a tree (*covering*)

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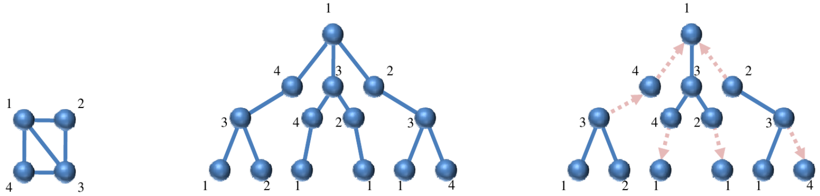
If the min *is unique, in a FP of Max Sum for the Minimum Spanning Tree (i.e. $D = N$, $T = V$) on arbitrary graphs,* $(p_i^*, d_i^*) = \arg\max \phi_i (p_i, d_i)$ *define the* ***optimum*** *tree*

Proof.

(Sketch): Assume FP giving unoptimal solution on G, and consider the *computation tree rooted at* $v \in V$, $T_k = (V_k, E_k)$*:* □

- $V_k = \{v = v_1, \dots v_l : l \leq k, (v_i v_{i+1}) \in E, v_{i+2} \neq v_i\}$
- $E_k = \{(p_1, p_2) \text{ if } p_1 = p_2 v\}$
- T_k is locally isomorphic to G, but T_k is a tree (*covering*)



Max-Sum (4)

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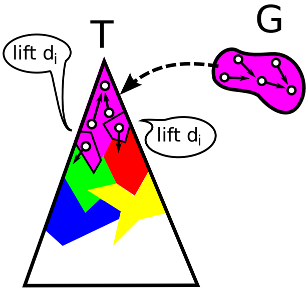
3. On T_k the [lifting of] $\{p_i^*\}$ defines a forest with each connected component not touching the leaves isomorphic to $\{p_i^*\}$ on G.
4. Take the optimal solution $\{q_i\}$ on G and use it to improve the solution on T_k (by replacing the pink CC) $\implies$ contradiction



The End

Thanks, and happy holidays!