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Dimensional analysis approach to study snap back-to-softening-to-ductile transitions in lightly reinforced quasi-brittle materials

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Abstract The analysis of lightly reinforced concrete beams taking into account the nonlinear behavior of concrete in tension is typically addressed by numerical approaches based on the Cohesive Crack Model. In the present paper, Dimensional Analysis is applied to this context, in order to obtain a synthetic description of the phenomenon by reducing the number of the governing parameters. It will be analytically demonstrated that the structural response is more complex than that obtained by the Bridged Crack Model and based on Linear Elastic Fracture Mechanics. In this case, in fact, two dimensionless parameters, N_P and s, are responsible for the brittle-to-softening-to-ductile transitions in the mechanical response. The results of a parametric analysis and of an experimental campaign available in the literature are interpreted in this new light.

Keywords Lightly reinforced concrete beams · Dimensional analysis · Brittleness numbers · Scale effects · Structural instability · Physical similarity

1 Introduction

The Limit Analysis of reinforced concrete (RC) beams usually assumes that stretched concrete is not bearing load and so the cracking phenomenon is not taken into account in the evaluation of the load carrying capacity. This assumption not always yields to a safe design condition, as for instance in the case of lightly RC beams, where the tensile concrete contribution determines a hyper-strength with respect to the ultimate loading condition, with a consequent possible instability in the overall mechanical response. In this case, in fact, the resistant bending moment after the peak cracking moment is a monotonic decreasing function of the crack length, due to an unstable fracture propagation. This aspect is fundamental for evaluating the minimum reinforcement which enables the element to prevent brittle collapse. The failure mode becomes even more complex in the case of higher steel reinforcement percentages, due to the interaction between concrete tensile cracking, concrete compressive crushing, and steel yielding and/or slippage, depending on the mechanical and the geometrical parameters of the RC beams. In this context again, classical approaches, such as the Limit Analysis, cannot predict the ductile-to-brittle transition in the failure mode due to the size-scale effects, commonly observed in experimental tests and usually studied by means of fracture mechanics.

The analysis of lightly RC beams has been addressed in the literature by several authors mainly by means of two different approaches: Linear Elastic Fracture Mechanics (LEFM) and the Cohesive Crack Model. In the former context, the Bridged Crack Model, which is an analytical model, has been originally proposed by Carpinteri (1981a, 1984) and Bosco and Carpinteri

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(1992a) for RC beams with a single reinforcement layer, where the problem of minimum reinforcement amount has also been investigated (Bosco and Carpinteri 1992b; Baluch et al. 1992). Afterwards, the model has been reformulated by Bosco and Carpinteri (1995) and Carpinteri and Massabò (1996) for fiber-reinforced concrete members with cohesive closing stresses and extended to the simultaneous presence of both steel bars and fiber reinforcements (Carpinteri et al. 2003). More recently, it has been further extended by Carpinteri et al. (2007a) in order to describe the shear crack propagation, according to the method proposed by Jenq and Shah (1989), and subsequently improved by So and Karihaloo (1993), for the calculation of the stress-intensity factor at the tip of a shear crack with a nonlinear trajectory. In all the aforementioned applications, this efficient analytical tool, which avoids finite element numerical computations, has permitted to reveal scale effects, instability phenomena and brittle-to-ductile transitions in reinforced structural elements. Clearly, it has to be remarked that LEFM holds only when the size of the fracture process zone at the crack tip is sufficiently small with respect to the size of the crack and the size of the specimen. Otherwise, suitable models, such as the Cohesive Crack Model, have to be used to take into account the nonlinear behavior in the process zone. Applications of such a model to longitudinally reinforced concrete beams have been proposed in the framework of the finite element method since 1985 (Gustafsson 1985; Gustafsson and Hillerborg 1988) for a systematic analysis of the diagonal tension failure mode. Different numerical approaches, based on elastic coefficients computed a-priori by means of a finite element analysis, have been proposed by Hawkings and Hjorsetet (1992), Ruiz et al. (1999), and Brincker et al. (1999). They permitted to investigate the influence of the main parameters on the mechanical response. As an example, Ruiz (2001) found that the bond-slip relation between concrete and reinforcement plays a central role in determining the peak load. The higher the bond stresses, the higher the peak load. On the contrary, it does not influence the value of the ultimate resistant moment corresponding to steel yielding and complete disconnection of the concrete cross-section. Brincker et al. (1999), instead, put into evidence a brittle-to-ductile transition in the overall failure response by increasing the beam size, for constant values of reinforcement percentage.

Dimensional Analysis, based on the Buckingham's Π-Theorem (Buckingham 1915; Barenblatt 1996) turns out to be an effective tool for the interpretation of the results of the aforementioned detailed analysis. The application of this procedure, in fact, permits to clearly connect the mechanical response to dimensionless groups of the variables involved in the phenomenon, rather than to the individual values of them. Dimensional Analysis has been first proposed in the framework of LEFM by Carpinteri, in order to study the stability of progressive cracking in brittle materials and in RC elements (Carpinteri 1980, 1981b), and to evaluate the minimum reinforcement necessary to avoid unstable crack propagation (Bosco and Carpinteri 1992b). In the case of brittle materials, the fracturing phenomenon is governed by the following dimensionless parameter, also called stress brittleness number (Carpinteri 1980, 1981b, 1982):

$$s = \frac{K_{\mathrm{IC}}}{\sigma_{\mathrm{u}}\, h^{1/2}}, \tag{1}$$

where K_{IC} is the material fracture toughness, σ_{u} is its ultimate tensile strength, and h is a characteristic linear size of the specimen. A transition from ductile to brittle failure is predicted by decreasing the brittleness number, s. In the case of lightly RC elements, instead, a different dimensionless parameter has been introduced (Carpinteri 1981a, 1984):

$$N_P = \rho \frac{\sigma_{\mathrm{y}} h^{1/2}}{K_{\mathrm{IC}}}, \tag{2}$$

where ρ_{t} and σ_{y} are, respectively, the percentage and the yielding strength of steel reinforcement. Also in this case, a ductile-to-brittle transition occurs by decreasing the reinforcement brittleness number, N_P. It is worth noting that N_P has been defined in the context of LEFM, and more precisely, it is based on the parameters of the Bridged Crack Model. For this reason, its extension to nonlinear models usually yielded unsatisfactory results. As an example, Brincker et al. (1999), obtained different mechanical responses for beams with different depths and reinforcement ratios, even if the brittleness number was kept constant. Similar remarks can also be done with reference to the experimental results by Bosco et al. (1990).

In the present paper, Dimensional Analysis is applied to the mechanical behavior of lightly RC beams in the case the fracturing process is modeled by a cohesive crack. In particular, this procedure is applied to the algorithm proposed by Carpinteri et al. (2007b, 2009) based on the Cohesive and the Overlapping Crack models. It will be demonstrated that two dimensionless parameters, N_P and s, are responsible for the mechanical response, and not N_P only, as considered so far.

2 The integrated Cohesive and Overlapping Crack Model

Let us consider the RC beam element in Fig. 1 with a rectangular cross-section of thickness b and depth h, a steel reinforcement layer distant c from the lower edge and a crack of length a. The beam segment has a length l equal to the depth and is subjected to the external bending moment M. We assume that the middle cross-section can be considered as representative of the mechanical behavior of the whole element, since all the nonlinearities are localized in this section, whereas the outside parts exhibit an elastic response. The structural behavior of such an element is affected by mechanical nonlinearities due to cracking in tension, steel yielding and crushing of concrete in compression.

The numerical model herein proposed to describe the fracturing behavior of lightly RC beams is the more general algorithm introduced by Carpinteri et al. (2007b, 2009) for modeling the mechanical response of all the possible situations ranging from plain to over-reinforced concrete beams. The proposed model, in fact, permits to correctly describe the relevant nonlinearities. Similar approaches for implementing a cohesive crack have been proposed by Carpinteri (1985); Planas and Elices (1992); Bazant and Beisel (1994); Ruiz et al. (1999), and Brincker et al. (1999).

2.1 Constitutive models

The mechanical response of concrete in tension is described by the Cohesive Crack Model (Hillerborg et al. 1976; Carpinteri 1985), which considers a damaged and microcracked process zone head of the tip of the real crack, where a part of the macrocrack in evolution partially stitched by inclusions, aggregates or fibers can be distinguished and where nonlinear and dissipative microscopic phenomena take place. In particular, a linear-elastic stress–strain relationship is assumed for the undamaged phase (see Fig. 2a), whereas a softening stress–crack opening relationship describes the process zone up to the critical opening, $w^{\mathrm{t}}_{\mathrm{cr}}$, is reached (see Fig. 2b). The softening function, $\sigma = f(w)$, is considered as a material property, as well as the critical value of the crack opening, $w^{\mathrm{t}}_{\mathrm{cr}}$, and the fracture energy, $\mathcal{G}_F$. The shape of $f(w)$ may vary from linear to bilinear or even more complicated relationships depending on the characteristics of the considered material and the analyzed problem. For instance, when plain concrete subjected to an high strain gradient is studied, a simple linear softening law can be sufficient to obtain accurate results (see Carpinteri et al. 1989 for a comparison between different shapes of the softening law). On the other hand, bilinear relationships with long tails are necessary to describe fiber-reinforced concrete elements, taking into account the closing stresses exerted by the fibers for large values of crack opening.

As far as modeling of concrete crushing failure is concerned, the Overlapping Crack Model introduced by Carpinteri et al. (2007b, 2009) is adopted. According to such an approach, based on the original insights from Kotsovos (1983) and van Mier (1984), and related to the pioneering work by Hillerborg (1990), the inelastic deformation in the post-peak regime is modeled by a fictitious interpenetration of the material, while the remaining part of the specimen undergoes an elastic unloading. Such a model differs from others previously proposed, which assume that strain localization takes place within a zone having an extension dependent on the specimen sizes and/or the stress level (Bažant 1989; Markeset and Hillerborg 1995; Jansen and Shah 1997;

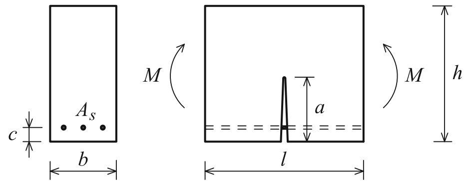

Fig. 1 Scheme of the reinforced concrete element

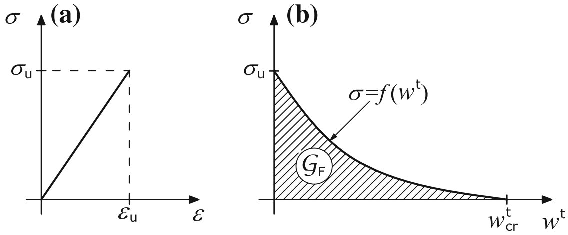

Fig. 2 Cohesive Crack Model: **a** linear-elastic $\sigma - \varepsilon$ law; **b** post-peak softening $\sigma - w$ relationship

Palmquist and Jansen 2001). A pair of constitutive laws for concrete in compression is introduced, in close analogy with the Cohesive Crack Model: a stress–strain relationship until the compressive strength is achieved (Fig. 3a), and a stress–displacement (overlapping) relationship describing the phenomenon of concrete crushing (Fig. 3b). However, it is worth noting that, contrary to the Cohesive Crack Model, that has a direct connection with the actual mechanical behavior of quasi-brittle materials subjected to tension, the Overlapping Crack Model is just an idealization of an extremely complicated failure mechanism that can vary from pure crushing to diagonal shear or to splitting failures, depending on the specimen size-scale and/or slenderness. The fictitious interpenetration modeling the strain localization caused by one of the aforementioned types of failures, is assumed as a global quantity contributing to the total specimen shortening, along with the elastic contribution. In this way, the elaborate description of the kinematics of the failure mode is avoided. The effectiveness of such an approach is proved by the fact that the experimental post-peak stress–displacement curves of specimens with different size or slenderness collapse onto a very narrow band, demonstrating that the σ–w relationship is able to provide not only a slenderness but also a size-independent constitutive law of concrete in compression. Such a law, usually assumed as a linear decreasing function for computational purposes, describes how the stress in the damaged material decreases from its maximum value down to zero as the fictitious interpenetration increases from zero to the critical value, $w_{\mathrm{cr}}^{\mathrm{c}}$. The crushing energy, $\mathcal{G}_{\mathrm{C}}$, which is a dissipated surface energy, defined as the area below the post-peak softening curve in Fig. 3b, can be assumed as a true material property, since it is not affected by the structural size. An empirical equation for calculating the crushing energy has been recently proposed by Suzuki et al. (2006), taking into account the concrete

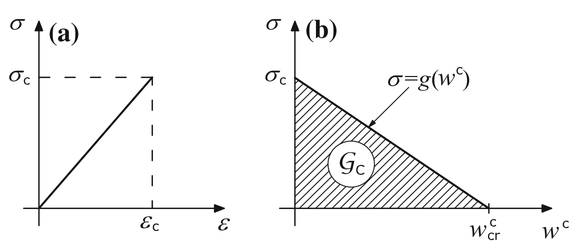

Fig. 3 Overlapping Crack Model: **a** linear-elastic $\sigma - \varepsilon$ law; **b** post-peak softening $\sigma - w$ relationship

confined compressive strength by means of the stirrups yield strength and the stirrups volumetric content. By varying the concrete compressive strength from 20 to 90 MPa, the crushing energy ranges from 30 to 58 N/mm. The critical value for the crushing interpenetration is experimentally found to be approximately equal to 1 mm (see also Jansen and Shah 1997). It is worth noting that this value is a decreasing function of the compressive strength, in agreement with the more brittle response exhibited by high strength concrete. On the contrary, we observe that, in the case of concrete confinement, the crushing energy, and the corresponding critical value for crushing interpenetration, considerably increase.

The steel reinforcement contribution is modeled by a stress versus crack opening relationship obtained by means of preliminary studies carried out on the interaction between the reinforcing bar and the surrounding concrete. On the basis of the bond-slip relationship provided by the Model Code 90 (1993), and by imposing equilibrium and compatibility conditions, it is possible to correlate the reinforcement reaction to the relative slip at the crack edge, which corresponds to half the crack opening displacement. Typically, the obtained relationship is characterized by an ascending branch up to steel yielding, to which corresponds a critical value of the crack opening, w_{y}. After that, the steel reaction is nearly constant. In the present algorithm, this stress-displacement law is introduced in input, together with the cohesive and overlapping constitutive laws.

2.2 Numerical algorithm

A suitable algorithm, based on the finite element method, is herein developed to study the mechanical response of the beam element shown in Fig. 1. The mid-span cross-section is subdivided into n nodes, where cohesive and overlapping stresses are replaced by equivalent nodal forces, F_i, which depend on the corresponding relative nodal displacements according to the cohesive or overlapping post-peak laws (Fig. 4a). These horizontal forces can be computed as follows:

$$\{F\} = [K_w]\{w\} + \{K_M\}\,M, \tag{3}$$

where $\{F\}$ is the vector of the nodal forces, $[K_w]$ is the matrix of the coefficients of influence for the nodal displacements, $\{w\}$ is the vector of the nodal displacements, $\{K_M\}$ is the vector of the coefficients

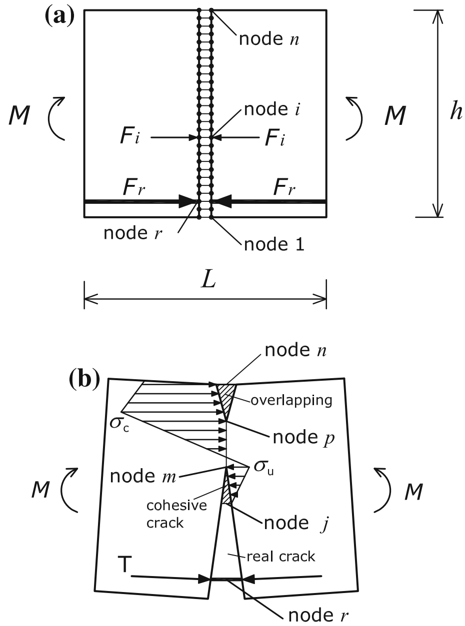

Fig. 4 Finite element nodes (**a**); and force distribution with cohesive crack in tension, crushing in compression and reinforcement closing forces (**b**) across the mid-span cross-section

of influence for the applied moment, M. Equation (3) permits to study the fracturing and crushing phenomena, computing a-priori, by a finite element analysis, all the coefficients of influence. For symmetry, only half-element is discretized through quadrilateral plane stress elements with uniform nodal spacing. Then, horizontal constraints are applied on the nodes along the vertical symmetry edge. Each coefficient of influence, $K_w^{i,j}$, which relates the nodal force, F_i, to the nodal displacement, w_j, is computed by imposing a unitary displacement to the corresponding constrained node. On the other hand, each coefficient K_M^i is computed by imposing a unitary external bending moment.

For the generic situation reported in Fig. 4b, the following equations hold:

$$F_i = 0; \quad \text{for i } = 1, 2, \ldots, (j-1)\,; \quad \text{i} \neq r \tag{4a}$$

$$F_i = f\left(w_\mathrm{i}^\mathrm{t}\right); \quad \text{for i } = j, \ \ldots, (m-1)\,; \quad \text{i} \neq r \tag{4b}$$

$$w_i = 0; \quad \text{for i } = m, \ldots, p \tag{4c}$$

$$F_i = g\left(w_i^\mathrm{c}\right); \quad \text{for i } = (p+1), \ldots, n \tag{4d}$$

$$F_\mathrm{i} = h\left(w_\mathrm{i}\right); \quad \text{for i } = r \tag{4e}$$

where j corresponds to the real crack tip, m represents the fictitious crack tip, p is the fictitious overlapping zone tip, and r corresponds to the reinforcement layer level. Equations (4b), (4d) and (4e) are the constitutive laws of, respectively, cohesive crack, overlapping crack and steel reinforcement.

Equations (3) and (4) constitute a linear algebraic system of $(2n)$ equations in $(2n+1)$ unknowns, namely $\{F\}, \{w\}$ and M. The necessary additional equation derives from the strength criterion adopted to govern the propagation processes. We can set either the force in the fictitious crack tip, m, equal to the ultimate tensile force, F_u, or the force in the fictitious crushing zone tip, p, equal to the ultimate compressive force, F_c. It is important to note that cracking and crushing phenomena are physically independent of each other. As a result, the situation which is closer to one of these two possible conditions is chosen to establish the prevailing phenomenon. The driving parameter of the process is the position of the fictitious tip that in the considered step has reached the limit resistance. Only this tip is moved to the next node, when passing to the next step. Finally, at each step, we can compute the rotation, ϑ, as follows:

$$\vartheta = \{D_w\}^\mathrm{T}\{w\} + D_M M, \tag{5}$$

where D_w^i is the coefficient of influence for the ith nodal displacement given as the rotation of the free edge corresponding to a unitary horizontal displacement of the i-th node, and D_M is the coefficient of influence for the applied bending moment, M.

The size-scale effects are taken into account by relationships of proportionality that affect the coefficients of influence entering Eqs. (3) and (5), so that it is not necessary to repeat the preliminary finite element analysis for any different beam size. In particular, if the depth, h, the span, l, and the thickness, b, are multiplied by a factor k, the coefficients change as follows:

$$K_w^{i,j}(kh) = k K_w^{i,j}(h)\,, \tag{6a}$$

$$K_M^i(kh) = \frac{1}{k} K_M^i(h)\,, \tag{6b}$$

$$D_w^i(kh) = \frac{1}{k} D_w^i(h)\,, \tag{6c}$$

$$D_M(kh) = \frac{1}{k^3} D_M(h)\,. \tag{6d}$$

3 Application of dimensional analysis to lightly RC members

The most relevant applications of Dimensional Analysis in Solids Mechanics have concerned complete and incomplete physical similarity of strength and toughness in disordered materials (Carpinteri 1980, 1981b, 1982, 1984; Carpinteri et al. 2003; Phatak and Dhonde 2003; Phatak and Deshpande 2005), as well as the study of the incomplete self-similarity in fatigue crack growth (Barenblatt and Botvina 1980; Ciavarella et al. 2008).

In any practical physical study, we attempt to obtain relationships among the quantities that characterize the phenomenon being studied. Thus, the problem always reduces to determine relationships of the form:

$$q = \Phi\,(q_1, q_2, \ldots, q_\mathrm{n}; r_1, r_2, \ldots, r_\mathrm{k}) \tag{7}$$

where q is the quantity being determined in the study, q_i and r_i are, respectively, dimensional and nondimensional quantities that are assumed to be given. It is worth noting that, generally, function Φ is not analytically obtainable, although an empirical relationship may be obtained if several results are available, varying the parameters of the problem. Similar best-fitting procedures have been proposed by Phatak and Dhonde (2003) and Phatak and Deshpande (2005) for the prediction of the ultimate torsional capacity of RC beams and the compressive strength of concrete specimens. On the other hand, we can reduce the number of the governing parameters in order to obtain a synthetic description of the problem. The application of Buckingham's Π-Theorem for physical similitude and scale modeling, in particular, permits to minimize the dimension space of the primary variables, in which the physical phenomenon might be studied, by combining them into dimensionless groups.

When the flexural behavior of RC beams is studied, according to the numerical model proposed in the previous section, the functional relationship is the following:

$$M = \Phi\left(\sigma_\mathrm{u}, \mathcal{G}_\mathrm{F}, \sigma_\mathrm{c}, \mathcal{G}_\mathrm{C}, E_\mathrm{c}, \sigma_\mathrm{y}, \rho, h; \frac{b}{h}, \frac{l}{h}, \vartheta\right), \tag{8}$$

where M is the resistant bending moment, σ_u, $\mathcal{G}_\mathrm{F}$, σ_c, $\mathcal{G}_\mathrm{C}$, E_c are, respectively, the tensile strength, the fracture energy, the compressive strength, the crushing energy, and the elastic modulus of concrete, σ_y and ρ represent the yield strength and the percentage of the tensile reinforcement, h is the characteristic size of the body, b/h and l/h define the geometry of the sample according to Fig. 1, and ϑ is the local rotation of the element. Since we are interested in the mechanical response of lightly RC beams, the set of variables can be reduced as follows:

$$M = \Phi(\sigma_\mathrm{u}, \mathcal{G}_\mathrm{F}, E_\mathrm{c}, \sigma_\mathrm{y}, \rho, h; \vartheta), \tag{9}$$

where the parameters describing the behavior of concrete in compression, σ_c and $\mathcal{G}_\mathrm{C}$, are not explicitly considered, since the crushing failure is not involved in the failure mechanism. On the other hand, only the beam depth, h, is considered if the geometrical ratios of the samples, b/h and L/h, are assumed to be constant.

The application of Buckingham's Π-Theorem to Eq. (9) yields the following relationship:

$$\frac{M}{h^{5/2}\sqrt{\mathcal{G}_\mathrm{F} E_\mathrm{c}}} = \Phi_1\left(\frac{\sigma_\mathrm{u}\, h^{1/2}}{\sqrt{\mathcal{G}_\mathrm{F} E_\mathrm{c}}}, \rho\, \frac{\sigma_\mathrm{y}\, h^{1/2}}{\sqrt{\mathcal{G}_\mathrm{F} E_\mathrm{c}}}, \vartheta\, \frac{E_\mathrm{c} h^{1/2}}{\sqrt{\mathcal{G}_\mathrm{F} E_\mathrm{c}}}\right), \tag{10}$$

if h and $\sqrt{\mathcal{G}_\mathrm{F} E_\mathrm{c}}$ are assumed as the dimensionally independent variables. It is worth noting that the former parameter is representative of the size-scale of the specimen, whereas the latter one is a matrix property. In particular, the term $\sqrt{\mathcal{G}_\mathrm{F} E_\mathrm{c}}$ may be replaced by the fracture toughness, K_IC, according to the fundamental relationship proposed by Irwin (1957) for elastic-brittle materials. Herein, it is extended to quasi-brittle materials on the basis of previous investigations. As demonstrated by Carpinteri since 1989 (Carpinteri 1989a,b, 1990), in fact, the application of the Cohesive Crack Model to analyze the crack stability in elastic-softening materials, predicts softening branches with positive slope and cusp catastrophe behaviors when large size-scales are considered. Such bifurcation points can be revealed by the simple LEFM condition $K_\mathrm{I} = K_\mathrm{IC}$. In this case, the Irwin's relationship turns out to be fully valid. On the contrary, it may be assumed just as a definition for small and intermediate size-scales. By admitting such a substitution, the dimensionless functional relation for the proposed model becomes:

$$\tilde{M} = \Phi_2\,(s, N_P, \vartheta_n)\,, \tag{11}$$

where s and N_P are the stress and the reinforcement brittleness number defined in Eqs. (1) and (2), respectively, $\tilde{M}$ is the dimensionless bending moment, and ϑ_n is the normalized local rotation. It is worth noting that the parameter s includes only the mechanical properties of the matrix and the size-scale of the problem, whereas N_P contains the properties of the reinforcement. As a result of the Dimensional Analysis, according to Eq. (11), we expect that the structural response, in terms of dimensionless moment versus normalized rotation, is a function of N_P and s. A single parameter is no more sufficient. Physical similarity is predicted when the two dimensionless parameters are kept constant, although the single mechanical and geometrical properties vary.

4 Numerical results

In this section, the results of a wide parametric analysis, carried out in order to investigate the effects of the main model parameters on the flexural response, are discussed. According to the dimensional analysis proposed in the previous section, the ductile-to-brittle transition in the structural response is analyzed on the basis of the two dimensionless numbers defined in Eqs. (1) and (2). For the sake of simplicity, a linear softening law is assumed for tensile concrete. The yielding strength and the elastic modulus of the steel reinforcement, as well as its relative distance from the tensile edge, c/h, are assumed, respectively, equal to 600 MPa, 200 GPa, and 0.10, for all the numerical simulations. Also the beam width, b, is assumed to be constant and equal to 0.15 m. In this case, since the geometrical ratio b/h is no longer a constant, the expression of the dimensionless moment becomes:

$$\tilde{M} = \frac{M}{bh^{3/2}\sqrt{\mathcal{G}_F E_c}}. \qquad (12)$$

The dimensionless moment versus normalized rotation curves shown in Fig. 5a represent the mechanical behavior of RC beams characterized by values of N_P ranging from 0.00 up to 1.85, with $s = 0.87$. From a practical point of view, these conditions may be obtained either by varying all the mechanical and geometrical parameters in order to satisfy the imposed values for N_P and s, or, in a more simple way, by varying the reinforcement ratio (ρ = 0.00, 0.05, 0.10, 0.20, 0.40, 0.80%) and keeping all the other parameters constant ($\sigma_c = 40$ MPa, $\sigma_u = 3$ MPa, $\mathcal{G}_F = 0.0792$ N/mm, $E_c = 34130$ MPa, $K_{IC} = 51.98$ N/mm$^{3/2}$, $h = 0.4$ m). In general, the obtained curves are characterized by a first ascending branch, according to the elastic material behavior and to the initial stable crack propagation, up to the peak cracking moment, and then by a softening or even snap-back branch, due to unstable crack propagation. Then, depending on the reinforcement amount, the loading process can return to be stable up to steel yielding. Finally, the mechanical response tends asymptotically to the ultimate moment, which corresponds to steel plastic flow and complete disconnection of the concrete cross-section. The diagrams in Fig. 5a evidence an increment in the load-carrying capacity and a more ductile and stable mechanical behavior of the element by increasing N_P. For $N_P = 0.00$, which corresponds to a plain concrete beam, the post-peak softening branch always decreases, whereas a hardening response is obtained for $N_P = 1.85$. More precisely, for the considered value of s, the global response becomes stable, i.e., the asymptotic value of the ultimate moment becomes higher than the peak cracking moment, when N_P is greater than 0.23 approximately. According to the definition given by Bosco et al. (1990), this is the condition corresponding to the minimum reinforcement amount, necessary to guarantee the stability of the mechanical response of RC beams.

The mechanical behavior of the beams shown in Fig. 5a is also described in terms of dimensionless moment versus relative crack depth, $\xi = a/h$, in Fig. 5b, in order to investigate the stability of the fracturing process. In these diagrams, the mechanical response of the element is rigid-elastic until the ultimate tensile strength is reached at the soffit. Then, a tensile crack propagates from the soffit to the extrados of the mid-span cross-section. Beams with different N_P exhibit the same behavior if $\xi \leq c/h$, since the reinforcement contribution is not involved in this stage.

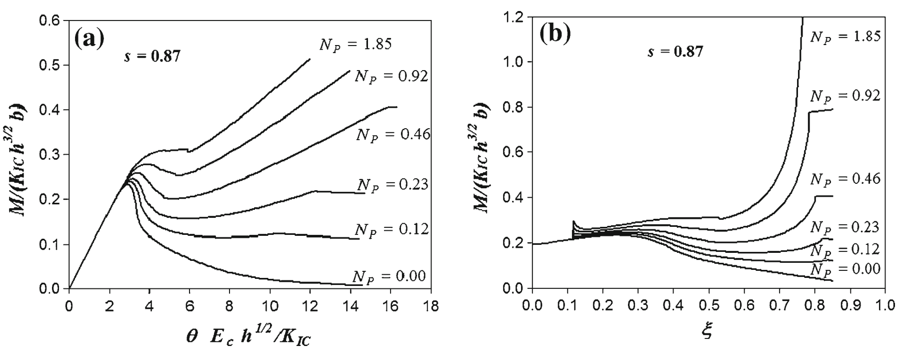

Fig. 5 Numerical dimensionless moment versus (**a**) normalized rotation; **b** relative crack depth, curves by varying N_P and for $s = 0.87$

For values of N_P close to zero, ($N_P \le 0.12$), i.e., for very low reinforced beams, the bending moment is a monotonic decreasing function of ξ. In this case, in fact, an unstable fracture phenomenon occurs, since crack propagates rapidly with lower and lower critical moment values up to complete failure of the cross-section. On the contrary, for values of N_P higher than 0.23, a steeper and steeper stable branch follows the unstable one. In this case, in fact, the model predicts a stable fracturing process for deep cracks, where an increment of the applied moment is necessary to provoke a crack advance.

The diagrams in Fig. 6a show the dimensionless moment versus rotation relationships for different values of the stress brittleness number, s, ranging from 0.43 up to 2.63, with $N_P = 0.28$. More precisely, these curves have been numerically obtained by considering four different values of concrete tensile strength, $\sigma_u = 1, 2, 4$ and 6 MPa, and keeping all the other mechanical and geometrical parameters constant. In particular, it has been assumed a beam depth of 0.4 m and a reinforcement percentage of 0.12%. We can observe that the values of the peak cracking moment, directly correlated to the concrete tensile strength, are a decreasing function of s. On the other hand, the post-peak branches collapse to the same asymptotic value of the ultimate bending moment, which is a function of steel content and yield strength. From a global point of view, a transition in the mechanical behavior from a very ductile and stable response to a brittle and unstable one, with the appearance of a snap-back instability, is evidenced by decreasing s from 2.63 to 0.43. Furthermore, from the dimensionless moment versus relative crack depth diagrams in Fig. 6b, it is possible to highlight the extension of the unstable behavior with respect to the complete crack propagation. The peak cracking moment, in fact, moves toward lower values of ξ by increasing the concrete tensile strength, whereas the steel yielding occurs when ξ is approximately equal to 0.8.

The interpretation of the numerical simulations in terms of the two brittleness numbers herein proposed finds confirmation in the results of the experimental tests carried out by Bosco et al. (1990) to investigate the minimum reinforcement in high-strength concrete beams, shown in Figs. 7 and 8. Such diagrams, in fact, evidence ductile-to-brittle transitions in the mechanical response by varying N_P and s, which are similar to those numerically obtained, and shown, respectively, in Figs. 5a and 6a. It is clear from Figs. 7 and 8 that the mechanical response of reinforced quasi-brittle materials cannot be exhaustively described by a single dimensionless parameter.

Finally, the effect of the fracture energy, $\mathcal{G}_F$, which has been varied from 0.03 to 0.30 N/mm, is investigated in the bending moment versus rotation diagrams shown in Fig. 9. A beam depth $h = 0.4$ m, a concrete tensile strength $\sigma_u = 3$ MPa, and a reinforcement percentage $\rho = 0.12\%$ are assumed in these numerical

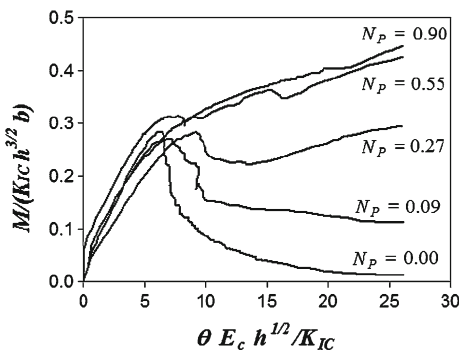

Fig. 7 Experimental dimensionless moment versus normalized rotation curves from Bosco et al. (1990), for different values of N_P and $s = 0.79$

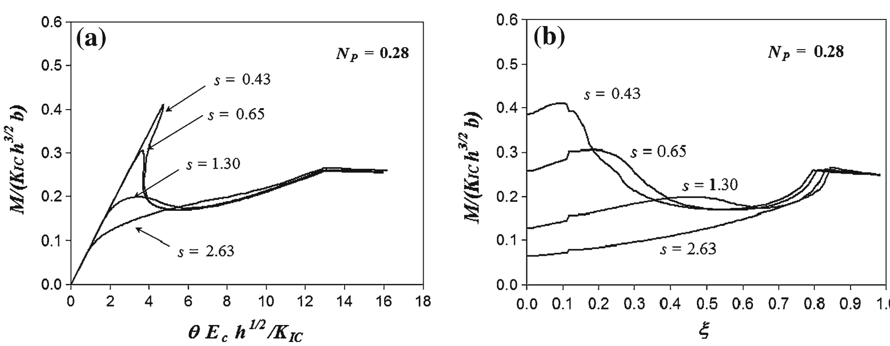

Fig. 6 Numerical dimensionless moment versus (**a**) normalized rotation; **b** relative crack depth, curves by varying s and for $N_P = 0.28$

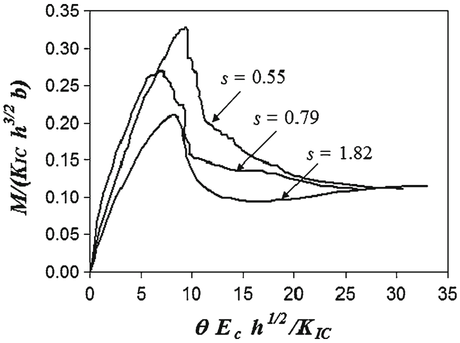

Fig. 8 Experimental dimensionless moment versus normalized rotation curves from Bosco et al. (1990), for different values of s and $N_P = 0.09$

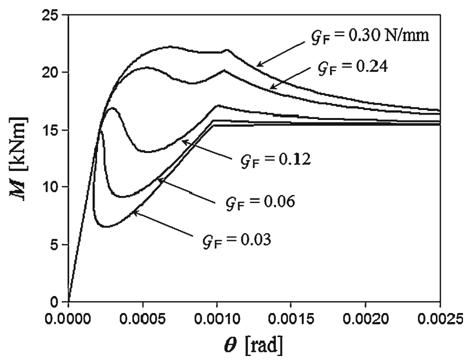

Fig. 9 Numerical moment versus rotation *curves* by varying the fracture energy $\mathcal{G}_F(\sigma_u = 3$ MPa, $h = 0.4$ m and $\rho = 0.12\%)$

simulations. By increasing the fracture energy, the model predicts a more stable behavior of the post-peak branches immediately after the peak loads, with the disappearance of the snap-back phenomenon, expected only for $\mathcal{G}_F = 0.03$. On the other hand, this implies an increment in the value of the reinforcement amount necessary to guarantee a ductile global response. It is worth noting that these diagrams could be assumed as representative of fiber-reinforced concrete. In the context of the Cohesive Crack Model, in fact, the amount of fibers can be introduced as an increase in the matrix toughness.

5 Discussion and conclusions

Dimensional Analysis has been applied to the study of the flexural behavior of RC beams, with particular regard to the stability of the cracking process in the case of low reinforcement percentages. The following main conclusions can be drawn from the analytical and numerical results.

1. The Cohesive Crack Model implemented in the proposed algorithm permits to take into account the nonlinear contribution of concrete in tension. On the other hand, this implies that two dimensionless numbers are responsible for the overall behavior, and not only N_P, as obtained by the Bridged Crack Model. More in detail, a brittle-to-ductile transition is evidenced by increasing N_P (see Fig. 5) and s (see Fig. 6).
2. Physical similitude in the dimensionless moment versus normalized rotation diagram is predicted when both the dimensionless parameters are kept constant. Different behaviors are obtained if only one dimensionless parameter is assumed to be constant (see the experimental results from Bosco et al. 1990 shown in Figs. 7 and 8, and the numerical simulations by Brincker et al. 1999).
3. The obtained results (Figs. 5 and 6) may be extended to shapes of the softening law different from the linear. Moreover, the proposed approach can be applied to fiber-reinforced elements, if tensile strength and fracture toughness of the composite material are included in the dimensionless parameter s.
4. The dimensional analysis proposed in the present paper can be a useful tool to select the minimum reinforcement amount. This will be the subject of a next publication.

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