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Brittle failures at rounded V-notches: a finite fracture mechanics approach

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Abstract Finite Fracture Mechanics (FFM) is applied to investigate the brittle failure behavior of rounded V-notched elements subjected to mode I loading. According to the criterion, fracture does not propagate continuously, but by finite crack extensions, whose value is determined by the contemporaneous fulfilment of a stress requirement and an energy balance. Consequently, the crack advance becomes a structural parameter. By assuming the generalized apparent stress intensity factor as the governing failure parameter, as expected for a brittle structural behavior, the expression of the apparent generalized fracture toughness as a function of the material properties as well as of the notch opening angle and root radius is achieved. FFM predictions are then successfully compared to experimental data available in the Literature and to results provided by other theoretical approaches based on a critical distance.

Keywords Rounded V-notched structures · Mode I · Brittle fracture · FFM

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1 Introduction

Different criteria based on a linear-elastic analysis combined with an internal material length have been proposed to deal with fracture initiation of brittle notched elements subjected to mode I loading.

Approaches based either on stress requirements (Neuber 1958; Novozhilov 1969; Ritchie et al. 1973; Seweryn 1994; Taylor 2004) or energy considerations (Sih and Macdonald 1974; Lazzarin and Zambardi 2001; Pugno and Ruoff 2004; Lazzarin and Berto 2005; Taylor et al. 2005) were firstly introduced, both in their punctual and averaged versions. Nevertheless, the stress and energy conditions remain distinct and the fulfilment of one generally implies the violation of the other one (Carpinteri et al. 2008). More recently, in order to overcome this drawback, a coupled average stress and energy criterion has been put forward, under the heading of FFM (Cornetti et al. 2006). Accordingly, the internal length ceases to be a material constant and becomes a structural parameter, thus able to take into account the interaction between the finite crack advance and the geometry of the specimen. While for U-notched elements it has been shown that failure load predictions related to different approaches are generally very close (Gomez et al. 2006), for what concerns V-notched samples FFM generally provides the most accurate results (Carpinteri et al. 2008, 2009, 2010, 2011).

In the present work, FFM is applied to estimate the failure loads related to rounded V-notched structures subjected to mode I loading (Leguillon and Yoshibah 2003; Picard et al. 2006; Ayatollay and Torabi 2010). How it will be shown in the following section, the analysis involves the characterization of the stress field ahead of the notch tip and of the stress intensity factor (SIF) for a crack stemming from the notch root. While the former is already known from the Literature (Filippi et al. 2002), an analytical expression, consistent with asymptotic expansions and numerical simulations, is herein originally proposed for the latter. On the basis of these results, the relationships providing the critical crack advance and the apparent generalized fracture toughness are then derived. The comparison with experimental data and with theoretical predictions by other criteria (Ritchie et al. 1973; Leguillon 2002; Taylor 2004) concludes the paper.

2 Finite Fracture Mechanics

The FFM criterion (Cornetti et al. 2006; Carpinteri et al. 2008) is based on the hypothesis of a finite crack advance Δ and assumes a contemporaneous fulfilment of two conditions. The former requires that the average stress $\sigma_y(x)$ upon the crack advance Δ is higher than material tensile strength σ_u:

$$\int_0^\Delta \sigma_y(x)\mathrm{d}x \geq \sigma_u \Delta, \tag{1}$$

where (x, y) is the Cartesian coordinate system centered at the notch root (Fig. 1). The latter one ensures that the energy available for a crack increment Δ (obtained by integrating the crack driving force G_I over Δ) is higher than the energy necessary to create the new fracture surface ($G_{Ic}\Delta$):

$$\int_0^\Delta G_I(c)\mathrm{d}c \geq G_{Ic}\Delta. \tag{2}$$

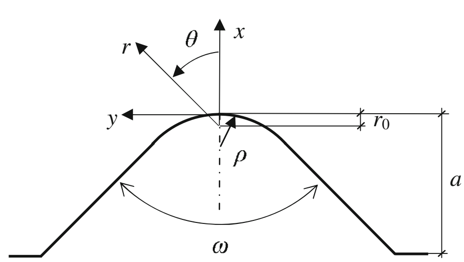

Fig. 1 Rounded V-notch with Cartesian and polar coordinate systems

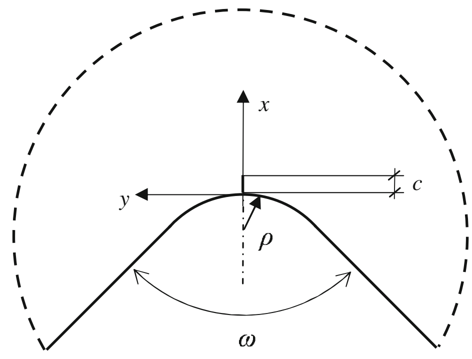

Fig. 2 Crack of length c stemming from a rounded V-notch root

By means of the well-known Irwin's relationship, Eq. (2) can be rewritten as

$$\int_0^\Delta K_I^2(c)\mathrm{d}c \geq K_{Ic}^2\Delta, \tag{3}$$

$K_I(c)$ and K_{Ic} being the SIF related to a crack of length c stemming from the notch root (Fig. 2) and the fracture toughness, respectively. For positive geometries (i.e., for monotonic increasing $K_I(c)$ functions), at incipient failure, Eqs. (1) and (3) can be grouped into a system of two equations in two unknowns: the critical crack advancement Δ_c and the failure load, implicitly embedded in the functions $\sigma_y(x)$ and $K_I(c)$.

From a physical point of view, it can be stated that, according to FFM, fracture is energy driven, but a sufficiently high stress must act over the incipient crack path in order to trigger crack propagation. The finite extension of the crack is assumed *a priori* (Cornetti et al. 2006; Carpinteri et al. 2008).

How it will be shown in Sect. 3, a similar criterion, although based on a point-wise stress requirement, was proposed in (Leguillon 2002).

2.1 Stress field ahead of a rounded V-notch root

By assuming that the notch tip radius ρ is sufficiently small with respect to the notch depth a, the stress field along the rounded V-notch bisector could be expressed in a polar coordinate system (r, θ) (Fig. 1) as (Lazzarin and Tovo 1996; Filippi et al. 2002):

$$\sigma_\theta(r, 0) = \frac{K_I^{V,\rho}}{(2\pi r)^{1-\lambda}}\left[1 + \left(\frac{r_0}{r}\right)^{\lambda-\mu}\eta_\theta(0)\right], \tag{4}$$

Table 1 Values of different parameters versus notch opening angle ω: μ and $\eta_\theta(0)$-values are taken from Filippi et al. (2002), β-values from Carpinteri et al. (2010)

ω	λ	μ	$\eta_\theta(0)$	β
0°	0.5000	−0.5000	1.000	1.000
30°	0.5015	−0.4561	1.034	1.005
60°	0.5122	−0.4057	0.9699	1.017
90°	0.5445	−0.3449	0.8101	1.059
120°	0.6157	−0.2678	0.5700	1.161
150°	0.7520	−0.1624	0.2882	1.394

where $K_I^{V,\rho}$ is the apparent generalized (or notch) SIF, i.e. the generalized SIF for a vanishing notch root radius. The distance between the notch root (coincident with the origin of the Cartesian coordinate system (x, y)) and the origin of the polar coordinate system (r, θ) is denoted by r_0 (Fig. 1). It depends on the radius ρ and, as well as the eigenvalues λ, μ and the function $\eta_\theta(0)$, whose values are reported in Table 1, on the notch opening angle ω:

$$r_0 = \frac{q-1}{q}\rho, \qquad q = \frac{2\pi - \omega}{\pi}. \tag{5}$$

For a vanishing opening angle ($\omega = 0°$), Eq. (4) reduces to the well-known Creager–Paris' expression valid for slender U-notches (Creager and Paris 1967). Indeed, as observed by Filippi et al. (2002), Eq. (4) gives predictions almost coincident with the Creager–Paris' solution within the range $0° \leq \omega \leq 90°$, since λ does not show significant variations. On the other hand, for a vanishing notch root radius ($\rho = 0$), Eq. (4) provides the asymptotic stress field ahead of a V-notch tip (Williams 1952), $K_I^{V,\rho=0} = K_I^V$ being the generalized SIF.

2.2 Stress intensity factor function

In order to evaluate the SIF function $K_I(c)$ related to a small crack of length c emanating from the notch root ($c \ll a$, Fig. 2), numerical simulations on notched elements under tension have been carried out by means of FRANC2D® code (Wawrzynek and Ingraffea 1991): different opening angles ($\omega = 90°$, 120° and 150°) with different ratios c/ρ ranging from 0.01 to 10 have been taken into account, provided that $\rho/a = a/b = 1/20$, b being the overall dimension of the element. On the basis of the asymptotic conditions and the obtained results (see the analysis below), the following best-fitting expression is proposed:

$$K_I(c) = \frac{\alpha K_I^{V,\rho}\sqrt{c}}{\left[\left(\frac{\alpha}{\beta}\right)^{\frac{1}{1-\lambda}} c + r_0\right]^{1-\lambda}} = \frac{\beta K_I^{V,\rho} c^{\lambda-0.5}}{\left[1 + \frac{r_0}{c}\left(\frac{\beta}{\alpha}\right)^{\frac{1}{1-\lambda}}\right]^{1-\lambda}}, \tag{6}$$

where

$$\alpha = 1.12\sqrt{\pi}\frac{[1+\eta_\theta(0)]}{(2\pi)^{1-\lambda}}, \tag{7}$$

while β is given in Table 1. Predictions by Eq. (6) and numerical results are reported in Figs. 3, 4 and 5.

The expression (6) has been primarily chosen to fulfill the asymptotic limits. In fact, for a very short crack ($c/\rho \ll 1$), the notch-crack problem reverts to an edge crack subjected to the local peak stress. Equation (6) provides coherently

$$K_I(c) = 1.12\sigma_{\max}\sqrt{\pi c}, \tag{8}$$

where

$$\sigma_{\max} = \sigma_\theta(r_0, 0) = \frac{K_I^{V,\rho}}{(2\pi r_0)^{1-\lambda}}[1+\eta_\theta(0)] \tag{9}$$

according to Eq. (4).

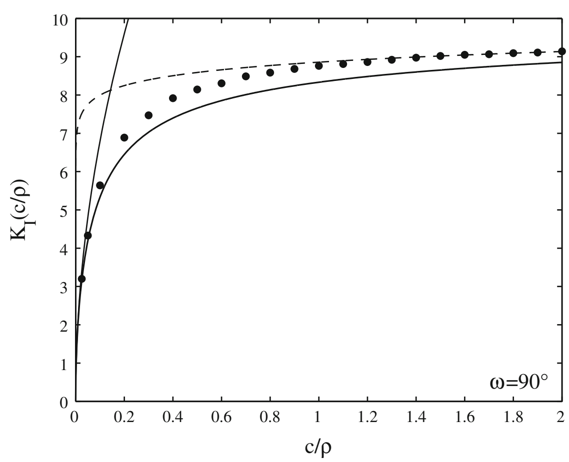

Fig. 3 Crack at the root of a rounded V-notch ($\omega = 90°$): dimensionless plot of the stress intensity factor versus crack advancement. Numerical data (*circles*), analytical predictions (*thick line*, Eq. (6)), asymptotic limits for $c/\rho \to 0$ (*thin line*, Eq. (8)) and for $c/\rho \to \infty$ (*dashed line*, Eq. (10))

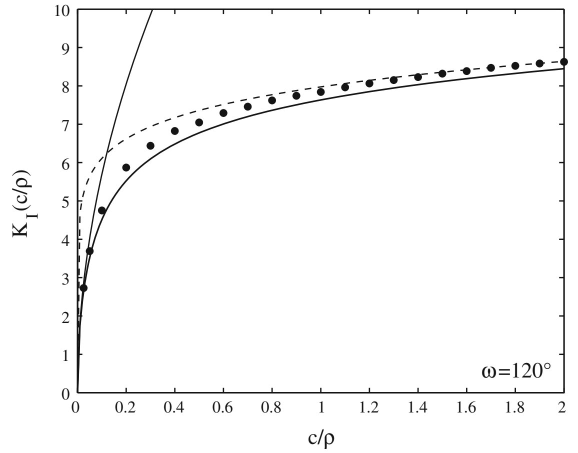

Fig. 4 Crack at the root of a rounded V-notch ($\omega = 120°$): dimensionless plot of the stress intensity factor versus crack advancement. Numerical data (*circles*), analytical predictions (*thick line*, Eq. (6)), asymptotic limits for $c/\rho \to 0$ (*thin line*, Eq. (8)) and for $c/\rho \to \infty$ (*dashed line*, Eq. (10))

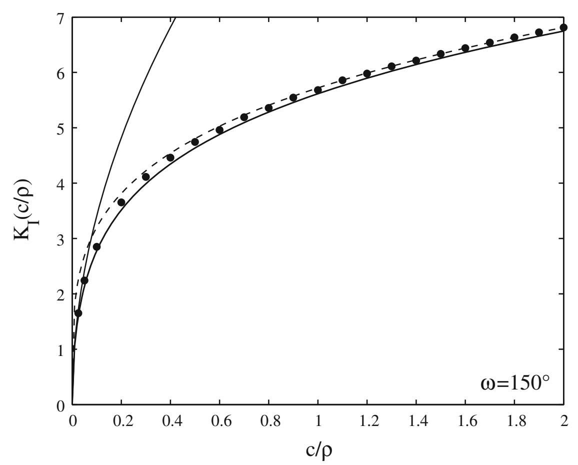

Fig. 5 Crack at the root of a rounded V-notch ($\omega = 150°$): dimensionless plot of the stress intensity factor versus crack advancement. Numerical data (*circles*), analytical predictions (*thick line*, Eq. (6)), asymptotic limits for $c/\rho \to 0$ (*thin line*, Eq. (8)) and for $c/\rho \to \infty$ (*dashed line*, Eq. (10))

On the other hand, for a very long crack ($c/\rho \gg 1$), but still small with respect to the notch depth a, the notch radius effect becomes negligible and Eq. (6) reduces to

$$K_I(c) = \beta K_I^V c^{\lambda-0.5}, \tag{10}$$

i.e., the SIF related to a crack stemming from a V-notch (Hasebe and Iida 1978; Philipps et al. 2008; Livieri and Tovo 2009).

For intermediate c/ρ values, predictions are in good agreement with numerical data: the most significant deviations (below 7%) are related to the case $\omega = 90°$ for $c/\rho \approx 0.4$ (Fig. 3). The accuracy increases as ω increases: for $\omega = 120°$ the maximum relative error is below 5% (Fig. 4), while for $\omega = 150°$ it is lower than 3% (Fig. 5).

Eventually, before proceeding, it is important to point out that, for $\omega = 0°$, Eq. (6) provides

$$K_I(c) = 2.24\sqrt{\frac{c}{5.0c+\rho}}K_I^{V,\rho}. \tag{11}$$

Equation (11) coincides with the relationship for U-notches proposed in (Lukas 1987), the only difference being the factor 5.0 replacing the approximation factor 4.5 introduced by Lukas to improve the fitting for short cracks. It is thus reasonable to expect sufficiently accurate results (errors below 10%) by Eq. (6) also in the range $0° \leq \omega \leq 90°$.

2.3 FFM implementation

By supposing that failure takes place when the apparent generalized SIF reaches its critical conditions $K_I^{V,\rho} = K_{Ic}^{V,\rho}$, as expected within brittle structural behavior, FFM can be implemented by inserting Eqs. (4) and (6) into (1) and (3), respectively (notice that $x = r - r_0$), and integrating. $K_{Ic}^{V,\rho}$ represents the apparent generalized fracture toughness, i.e. the generalized fracture toughness that would be measured if the notch root were sharp. Some analytical manipulations lead to the following two coupled equations:

$$\frac{K_{Ic}^{V,\rho}}{\sigma_u r_0^{1-\lambda}} = f(\overline{\Delta}_c), \tag{12}$$

and

$$\frac{K_{Ic}^{V,\rho}}{K_{Ic} r_0^{0.5-\lambda}} = \sqrt{g(\overline{\Delta}_c)}, \tag{13}$$

where the dimensionless variable $\overline{\Delta}_c = \Delta_c/r_0$ has been introduced. The functions $f(\overline{\Delta}_c)$ and $g(\overline{\Delta}_c)$ read ($0° < \omega \leq 180°$):

$$f(\overline{\Delta}_c) = \frac{(2\pi)^{1-\lambda}\overline{\Delta}_c}{\left[\dfrac{(\overline{\Delta}_c+1)^\lambda - 1}{\lambda} + \eta_\theta(0)\dfrac{(\overline{\Delta}_c+1)^\mu - 1}{\mu}\right]}, \tag{14}$$

and

$$g\left(\overline{\Delta}_c\right)=\frac{\overline{\Delta}_c}{\beta^2}\left[\frac{\left[\overline{\Delta}_c+\left(\frac{\beta}{\alpha}\right)^{\frac{1}{1-\lambda}}\right]^{2\lambda}-\left[\left(\frac{\beta}{\alpha}\right)^{\frac{1}{1-\lambda}}\right]^{2\lambda}}{2\lambda}+\left(\frac{\beta}{\alpha}\right)^{\frac{1}{1-\lambda}}\frac{\left[\overline{\Delta}_c+\left(\frac{\beta}{\alpha}\right)^{\frac{1}{1-\lambda}}\right]^{2\lambda-1}-\left[\left(\frac{\beta}{\alpha}\right)^{\frac{1}{1-\lambda}}\right]^{2\lambda-1}}{2\lambda-1}\right]^{-1}. \tag{15}$$

Equalling Eqs. (12) and (13) with respect to $K_{Ic}^{V,\rho}$ yields:

$$\frac{\sqrt{r_0}\sigma_u}{K_{Ic}}=\frac{\sqrt{g(\overline{\Delta}_c)}}{f(\overline{\Delta}_c)}, \tag{16}$$

from which, for a given material, notch opening angle ω and notch root radius ρ, the value of the critical crack advance Δ_c can be derived. This value could be inserted either into Eqs. (12) or (13) to obtain the apparent generalized fracture toughness $K_{Ic}^{V,\rho}$.

Let us now discuss the results. The dimensionless critical crack advance Δ_c/Δ_c^V, where Δ_c^V is the finite crack extension related to a V-notch according to FFM (Carpinteri et al. 2008)

$$\Delta_c^V=\frac{2}{\lambda\beta^2(2\pi)^{2(1-\lambda)}}\left(\frac{K_{Ic}}{\sigma_u}\right)^2, \tag{17}$$

is plotted as a function of the dimensionless notch root radius $\rho(\sigma_u/K_{Ic})^2$ in Fig. 6. For a fixed material, since $\rho=0$ corresponds to the V-notch case, the curves related to different notch opening angles ω all start at the same point $\Delta_c/\Delta_c^V=1$. As $\rho\to\infty$, the element becomes smooth: each curve approaches its asymptotic value $\Delta_c^V(\omega=\pi)/\Delta_c^V$. The most significant deviations from the V-notch case are observed for $\omega=90°$.

On the other hand, by recalling the expression for K_{Ic}^V:

$$K_{Ic}^V=\xi\frac{K_{Ic}^{2(1-\lambda)}}{\sigma_u^{1-2\lambda}}, \tag{18}$$

with (Carpinteri et al. 2008, 2011)

$$\xi=\lambda^\lambda\left[\frac{(2\pi)^{2\lambda-1}}{\beta^2/2}\right]^{1-\lambda}, \tag{19}$$

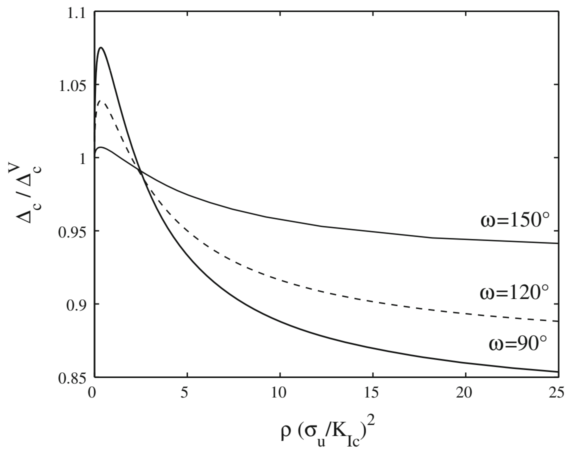

Fig. 6 FFM: dimensionless critical crack advance versus dimensionless notch root radius, for different notch opening angles

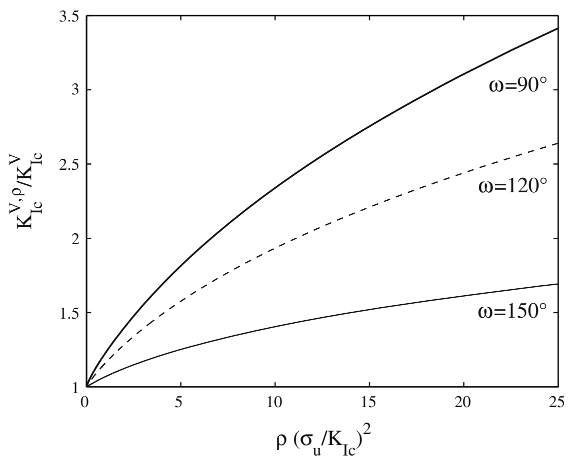

Fig. 7 FFM: dimensionless apparent generalized fracture toughness versus dimensionless notch root radius, for different notch opening angles

and exploiting Eq. (16), the ratio $K_{Ic}^{V,\rho}/K_{Ic}^V$ can be expressed as:

$$\frac{K_{Ic}^{V,\rho}}{K_{Ic}^V}=\frac{1}{\xi}\frac{[g(\overline{\Delta}_c)]^{1-\lambda}}{[f(\overline{\Delta}_c)]^{1-2\lambda}}. \tag{20}$$

By means of Eqs. (16) and (20), the dimensionless apparent fracture toughness can be plotted versus the dimensionless notch root radius, see Fig. 7. It is evident that:

1. the larger the root radius, the higher the apparent generalized fracture toughness;
2. the larger the notch opening angle, the lower the root radius' influence.

Table 2 Material properties of the experimental data considered

Material	σ_u (MPa)	K_{Ic} (MPa$\sqrt{\text{m}}$)
PMMA at $-60°$	130	1.7
$Al_2O_3+7\%Zr$	290	4.1

3 Experimental validation

The theoretical predictions according to FFM (Eq. (16) coupled with either Eqs. (12) or (13)) will be now compared with some experimental results available in the Literature. Two data sets are considered: they refer to three and four point bending tests carried out on single edge notched specimens made of PMMA at $-60°$ (Gomez and Elices 2004) and of Alumina+7%Zirconia ($Al_2O_3+7\%Zr$, Yoshibah et al. 2004), respectively. Different notch opening angles and root radii were taken into account during the experiments. Details of the sample geometry can be found in the quoted references, while the material properties are reported in Table 2.

In order to investigate the potentiality of FFM, the results by the so-called point stress method (Ritchie et al. 1973; Taylor 2004) and Leguillon's criterion (Leguillon 2002) are also considered. While the former is based on the simple failure condition

$$\sigma_y(x = \Delta_c) = \sigma_u, \tag{21}$$

with $\Delta_c = 1/(2\pi) \times (K_{Ic}/\sigma_u)^2$, the latter is based on Eq. (21) coupled with Eq. (3). In such a case, a different $f(\overline{\Delta}_c)$ function (Eq. (14)) coupled with Eq. (15) must be taken into account for implementation, namely:

$$f(\overline{\Delta}_c) = \frac{[2\pi(1+\overline{\Delta}_c)]^{1-\lambda}}{1+\eta_\theta(0)(1+\overline{\Delta}_c)^{\mu-\lambda}}. \tag{22}$$

A similar analysis, although based on an asymptotic matching approach, was performed in (Leguillon and Yoshibah 2003; Picard et al. 2006).

Predictions according to different criteria and experimental data are reported in Fig. 8, where the apparent generalized fracture toughness $K_{Ic}^{V,\rho}$ is plotted versus the notch root radius ρ. As can be seen, FFM generally provides the best results, especially for relatively low notch radii. The maximum percent deviation (nearly 10%) is observed for the 150°-notched PMMA samples with the largest notch root radii ($\rho = 2$ mm and 2.4 mm). Indeed, in these cases, the radius is not small enough with respect to the notch depth ($\rho/a \simeq 0.14$ and 0.17, respectively) and the stress field provided by Eq. (4) slightly overestimates the actual one, as

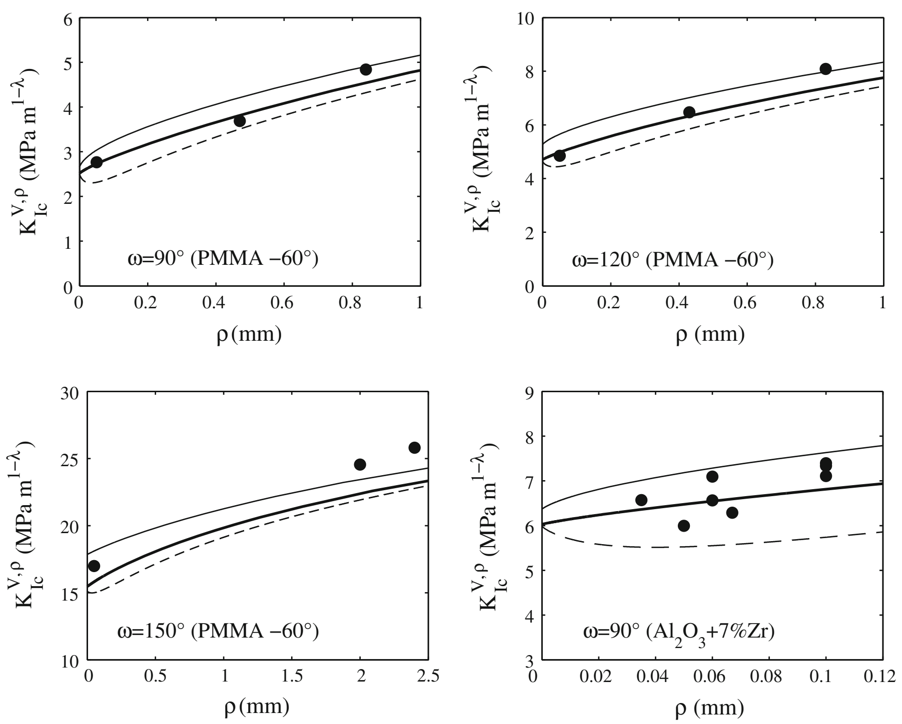

Fig. 8 Apparent generalized fracture toughness $K_{Ic}^{V,\rho}$ versus notch root radius ρ: experimental data (*circle*) and theoretical predictions according to the point method (*dashed line*), Leguillon's criterion (continuous *thin line*) and FFM (continuous *thick line*)

we have easily checked by a finite element analysis. Notice from Fig. 8 that the point method always tends to underestimate the apparent generalized fracture toughness, while Leguillon's criterion provides the highest values. Although not reported in the present analysis, the average stress criterion predictions (Neuber 1958; Novozhilov 1969; Seweryn 1994) are close to those by FFM. However, according to the author's opinion, FFM is more physically sound, being based on an energy balance.

Eventually, with respect to more sophisticated models, such as the cohesive crack model (Carpinteri 1989; Gomez and Elices 2004), the advantage of using FFM is to achieve analytically very similar peak load estimates (Henninger et al. 2007), while avoiding a numerical analysis of the post-peak regime taking place in the fracture process zone.

4 Conclusions

Under the hypothesis that the notch root radius is sufficiently small with respect to the other geometric dimensions, the apparent generalized fracture toughness of brittle elements containing rounded-V notches can be expressed as a function of the root radius, the notch opening angle and the material properties (the tensile strength and the fracture toughness). In the present paper, the problem is faced by means of FFM. The comparison with experimental data and with other fracture criteria available in the Literature confirms the soundness of the present analysis and the potentiality of FFM.

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