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Dimensional Analysis and Fractal Modeling of Fatigue Crack Growth

In the present paper, generalized Paris and Wöhler equations are derived according to dimensional analysis and incomplete similarity concepts. They provide a rational interpretation to a majority of empirical power-law criteria used in fatigue. In particular, they are able to model the effects of the grain size, of the initial crack length, as well as of the size-scale of the tested specimen on the crack growth rate and on the fatigue life. Regarding the important issue of crack-size dependencies of the Paris' coefficient *C* and of the fatigue threshold, an independent approach, based on the application of fractal geometry concepts, is proposed to model such an anomalous behavior. As a straightforward consequence of the fractality of the crack surfaces, the fractal approach provides scaling laws fully consistent with those determined from dimensional analysis arguments. The proposed scaling laws are applied to relevant experimental data related to the crack-size and to the structural-size dependencies of the fatigue parameters in metals and in quasi-brittle materials. Finally, paying attention to the limit points defining the range of validity of the classical Wöhler and Paris power-law relationships, correlations between the so-called cyclic or fatigue properties are proposed, giving a rational explanation to the experimental trends observed in the material property charts.

KEYWORDS: *S-N* curves, fatigue crack growth, short cracks, dimensional analysis, fatigue property charts

Nomenclature

a = crack length (L)
d = microstructural dimension (grain size) (L)
da/dN = crack growth rate (L)
D = fractal dimension (-)
E = elastic modulus (FL^{-2})
h = characteristic structural size (L)
N = number of cycles (-)
R = loading ratio (-)
ΔK = stress-intensity factor range ($FL^{-3/2}$)
ΔK_{th} = fatigue threshold ($FL^{-3/2}$)
$\Delta\sigma$ = stress range (FL^{-2})
$\Delta\sigma_{fl}$ = fatigue limit (FL^{-2})
K_{IC} = fracture toughness ($FL^{-3/2}$)
ω = frequency of the loading cycle (T^{-1})
σ_y = yield strength (FL^{-2})

Introduction

As admitted by Paris in a recent review [1], "a specific accumulation damage model for the computation of damage growth under a wide variety of service loads is still lacking" and "no computational model is entirely satisfactory today," although a general understanding of many aspects of fatigue crack growth was established since the early 1960s. We know that fatigue damage increases with applied cycles in a cumulative way, which may eventually lead to failure. To model this physical phenomenon, the existing approaches for the prediction of fatigue life can be distinguished in two main categories: those related to the cumulative fatigue damage (CFD) approach, which is the traditional framework based on the Wöhler

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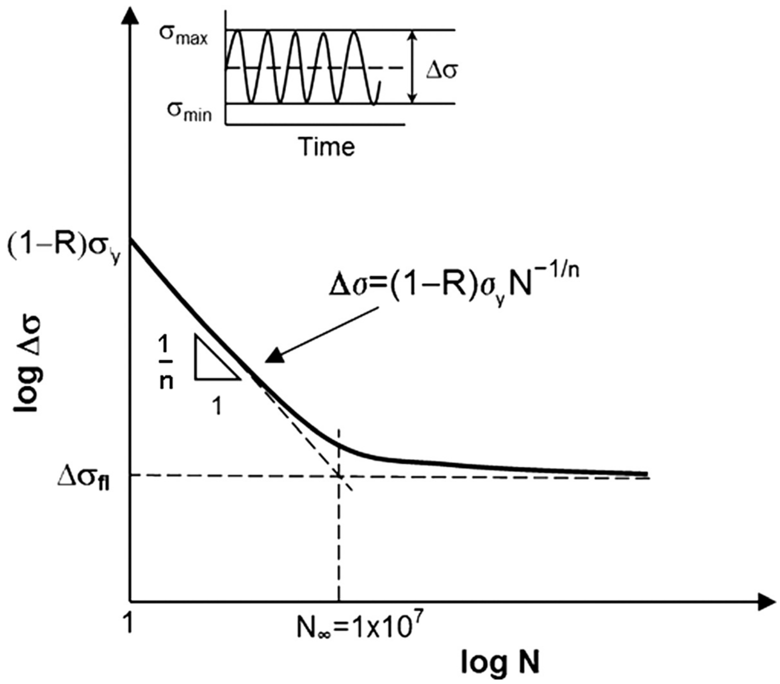

FIG. 1—*Scheme of the Wöhler's curves with the corresponding fatigue parameters.*

or *S-N* curves [2] for fatigue life assessment, and those based on the fatigue crack propagation (FCP) approach, developed since the 1960s after the advent of fracture mechanics and the introduction of the Paris' law [3,4]. In the empirical *S-N* curve, the fatigue life, N, is related to the applied stress range, $\Delta\sigma$ or S, and a reasonable power-law approximation was discovered since 1910 by Basquin [5]. A schematic representation of a typical Wöhler's curve is shown in Fig. 1, where the cyclic stress range, $\Delta\sigma = \sigma_{max} - \sigma_{min}$, is plotted as a function of the number of cycles to failure, N. The loading ratio is the ratio between the minimum and the maximum applied stresses, $R = \sigma_{min}/\sigma_{max}$. In this diagram, we also introduce the range of stress at static failure, $\Delta\sigma_y = \sigma_{max} - \sigma_{min} = \sigma_y - \sigma_{min} = (1-R)\sigma_y$, where σ_y is the material *yield strength*, and we define the *endurance* or *fatigue limit*, $\Delta\sigma_{fl}$, as the stress range that a sample will sustain without fracture for $N_\infty = 1 \times 10^7$ cycles, which is a conventional value that can be thought of as "infinite" life. Fatigue criteria based on the CFD approach have the advantage that can be used for the fatigue life assessment of unnotched or welded specimens, but suffer from the significant deficiency that there is no consistent definition of failure. It may correspond to the appearance of the first detectable crack, although it may also be defined as when the actual failure of the structural component takes place.

With the advent of fracture mechanics, a more ambitious task was undertaken, i.e., to predict, or at least understand, the propagation of cracks. Plotting the crack growth rate, da/dN, as a function of the stress-intensity factor range, $\Delta K = K_{max} - K_{min}$, most of the experimental data can be well-interpreted in terms of a power-law relationship, i.e., according to the so-called Paris' law [3,4] (see Fig. 2). Note that the power-law representation presents some deviations for very high values of ΔK approaching $\Delta K_{cr} = (1-R)K_{IC}$ [6,7], where K_{IC} is the material *fracture toughness*, or for very low values of ΔK approaching the *threshold stress-intensity factor range*, ΔK_{th}. Again, in close analogy with the concept of fatigue limit, the fatigue threshold is defined in a conventional way as the value of ΔK below which the crack grows at a rate of less than 1×10^{-9} m/cycle. The main drawback of this approach relies in the fact that the Paris' law is far from providing a universal representation of fatigue, since several deviations have been noticed in the last decades. Among them, the anomalous behavior of short cracks is probably the most important aspect, which led to the development of more complicated fatigue crack growth criteria (see, e.g., [8–14] for a comprehensive discussion).

For a long time, the CFD and the FCP approaches have been considered as totally independent. The CFD criteria have been mainly confined to the fatigue life assessment of unnotched or welded components, where the elastoplastic nature of damage, crack nucleation and crack initiation are important aspects, whereas the FCP models have been mainly applied to the long-crack regime, when the concept of small scale yielding holds and LEFM applies reasonably well.

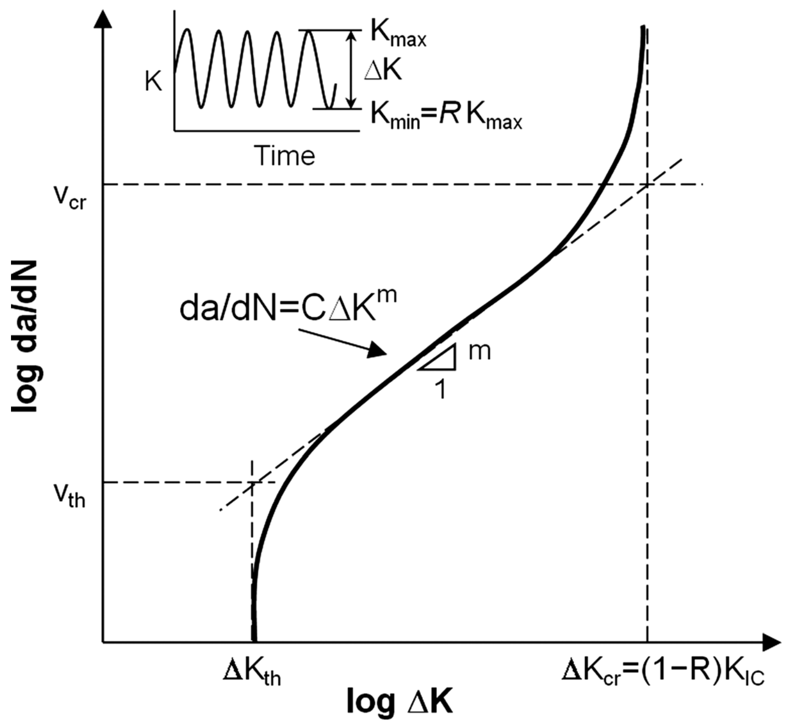

FIG. 2—*Scheme of the Paris' curves with the corresponding fatigue parameters.*

In the last few decades, the researchers have attempted to extend the field of application of the FCP approach. Among the various efforts, it is worth mentioning the contribution by McEvily and co-workers [14], who proposed a modified Paris' law dealing with the elastoplastic behavior of small cracks, and that by Atzori et al. [15], who proposed a method for the fatigue life prediction of welded joints based on the notch stress-intensity factor. The effect of surface roughness was also modeled by Spagnoli [16,17] according to a fractal model, and a unified interpretation of the anomalous scaling laws in fatigue due to short cracks has recently been provided by Paggi and Carpinteri [18,19] according to fractal geometry. These advances in understanding the complex phenomenon of fatigue crack growth shed a new light on the possibility to unify the CFD and the FCP approaches, and to solve the challenging task of interpreting the Paris and Wöhler power-law regimes within a unified theoretical framework. A recent effort in this direction was given by Pugno et al. [20,21], who proposed a generalized Paris' law based on Quantized Fracture Mechanics for a unified treatment of long cracks, short cracks and fully yielded regimes.

In the present paper, we extend the dimensional analysis approach pioneeringly proposed by Barenblatt and Botvina [22,23] to derive generalized mathematical representations of the phenomenon of fatigue. It will be shown that such generalized representations cover almost all the main deviations from the empirical fatigue laws of Wöhler and Paris.

Using an independent approach based on fractal geometry, we also show that the incomplete similarity in the crack length represents the effect of the multiscale fractal roughness of crack profiles. Related implications for the fatigue threshold are discussed.

Finally, analytical correlations between the fatigue properties of engineering materials are determined on a theoretical basis and compared with the empirical trends proposed by Fleck et al. [24], giving a rational interpretation to the fundamental fatigue property charts.

Generalized Cumulative Fatigue Damage Formulation

Let us consider the number of cycles, N, as the parameter representative of fatigue. Following this route, we can consider the following functional dependence

$$N = F\left(\sigma_y, K_{\mathrm{IC}}, \Delta\sigma_{\mathrm{fl}}, \Delta K_{\mathrm{th}}, E, \Delta\sigma, \omega, h, a, d; 1-R\right) \tag{1}$$

where the definitions of the governing variables are summarized in the nomenclature list, along with their physical dimensions expressed in the length-force-time class (LFT). Considering a state with no explicit time dependence, it is possible to apply the Buckingham's Π Theorem [25] to reduce the number of parameters involved in the problem (see, e.g., [26–30] for some relevant applications of this method in Solid Mechanics). As a result, we have

$$N = \Psi\left(\frac{\Delta\sigma_{\mathrm{fl}}}{\sigma_y}, \frac{\Delta K_{\mathrm{th}}}{K_{\mathrm{IC}}}, \frac{E}{\sigma_y}, \frac{\Delta\sigma}{\sigma_y}, \frac{\sigma_y^2}{K_{\mathrm{IC}}^2}h, \frac{\Delta\sigma_{\mathrm{fl}}^2}{\Delta K_{\mathrm{th}}^2}a, \frac{\sigma_y^2}{K_{\mathrm{IC}}^2}d; 1-R\right) = \Psi(\Pi_i), \quad i = 1, \ldots, 8 \tag{2}$$

where Ψ is a nondimensional function. The dimensionless number $\sigma_y^2 h/K_{\mathrm{IC}}^2$ is proportional to the ratio between the structural size h and the critical process zone size r_p, since $r_p = (K_{\mathrm{IC}}/\sigma_y)^2/\pi$ according to Irwin. This number is responsible for the size-scale effects and it is proportional to the square of the number Z introduced by Barenblatt and Botvina [22] and to the inverse of the square of the brittleness number s introduced by Carpinteri [26]. The dimensionless number $\Delta\sigma_{\mathrm{fl}}^2 a/\Delta K_{\mathrm{th}}^2$ is responsible for the crack-size effects, and it is proportional to the ratio between the crack length and the Haddad [8] characteristic size of mechanically short cracks $a_0 = (\Delta K_{\mathrm{th}}/\Delta\sigma_{\mathrm{fl}})^2/\pi$.

At this point, we want to see if the number of the quantities involved in the relationship [2] can be reduced further from eight. This can occur either in the case of *complete* or *incomplete self-similarities* in the corresponding dimensionless numbers. In the former situation, the dependence of the mechanical response on a given dimensionless number, say Π_i, disappears and we can say that Π_i is *nonessential* for the representation of the physical phenomenon. In the latter situation, a power-law dependence on Π_i can be put forward, which usually characterizes a physical situation intermediate between two asymptotic behaviors. To this aim, we assume incomplete self-similarity in Π_4, Π_5, Π_6, Π_7 and Π_8, obtaining

$$N = \left(\frac{\Delta\sigma}{\sigma_y}\right)^{\alpha_1}\left(\frac{h}{r_p}\right)^{\alpha_2}\left(\frac{a}{a_0}\right)^{\alpha_3}\left(\frac{d}{r_p}\right)^{\alpha_4}(1-R)^{\alpha_5}\Psi^*(\Pi_i) \tag{3}$$

The exponents α_i cannot be determined from considerations of dimensional analysis alone and may depend on the nondimensional parameters Π_i. Equation 3 represents a generalized Wöhler relationship of fatigue and encompasses the empirical *S-N* curves as limit cases. For instance, the *S-N* curve in Fig. 1 can be approximated by the Basquin power law in the high cycle fatigue (HCF) regime, stating that

$$1 \times \Delta\sigma_y^n = N_\infty \Delta\sigma_{\mathrm{fl}}^n = N\Delta\sigma^n = k \tag{4}$$

where k is a constant. Equating the first and the third terms in Eq 4, we obtain he following power-law equation

$$N = \left(\frac{\Delta\sigma_y}{\Delta\sigma}\right)^n = \frac{(1-R)^n \sigma_y^n}{\Delta\sigma^n} \tag{5}$$

Comparing the generalized expression of the *S-N* curve in Eq 3 with the empirical one in Eq 5, we find that a perfect correspondence exists when $\alpha_1 = -n$, $\alpha_2 = \alpha_3 = \alpha_4 = 0$ and $\alpha_5 = n$. It is important to notice that the generalization of the *S-N* curve including a power-law dependency on the crack size [31,32] and on the grain size [33,34] permitted to better interpret the experimental trends. Size-scale effects on the *S-N* curves are also observed in concrete, as shown in Fig. 3. The increase in the size of the specimen leads to a lower fatigue life, for a given applied stress-range.

Generalized Fatigue Crack Propagation Formulation

Following the pioneering work by Barenblatt and Botvina [22], we now assume that the mechanical response of the system can be fully represented by the crack growth rate, *da/dN*, which is the parameter to be determined. This output parameter is a function of a number of variables

$$\frac{\mathrm{d}a}{\mathrm{d}N} = F\left(\sigma_y, K_{\mathrm{IC}}, \Delta\sigma_{\mathrm{fl}}, \Delta K_{\mathrm{th}}, E, \Delta K, \omega, h, a, d; 1-R\right) \tag{6}$$

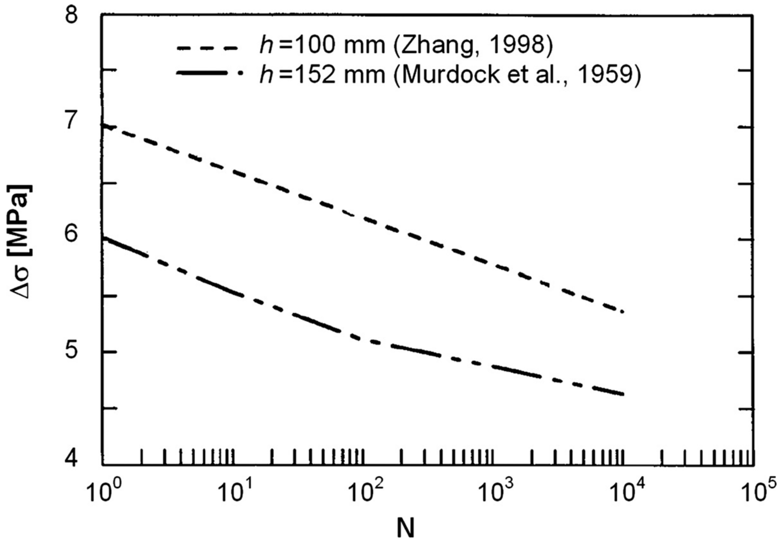

FIG. 3—*Size-scale effects on the S-N curve (experimental data from [35,36]).*

where the governing variables are summarized in the nomenclature, along with their physical dimensions expressed in the length-force-time class (LFT).

Considering a state with no explicit time dependence, it is possible to apply the Buckingham's Π Theorem [25] to reduce the number of parameters involved in the problem. As a result, we have

$$\frac{\mathrm{d}a}{\mathrm{d}N} = \left(\frac{K_{\mathrm{IC}}}{\sigma_y}\right)^2 \Phi\left(\frac{\Delta\sigma_{\mathrm{fl}}}{\sigma_y}, \frac{\Delta K_{\mathrm{th}}}{K_{\mathrm{IC}}}, \frac{E}{\sigma_y}, \frac{\Delta K}{K_{\mathrm{IC}}}, \frac{\sigma_y^2}{K_{\mathrm{IC}}^2}h, \frac{\Delta\sigma_{\mathrm{fl}}^2}{\Delta K_{\mathrm{th}}^2}a, \frac{\sigma_y^2}{K_{\mathrm{IC}}^2}d; 1-R\right) = \\ = \left(\frac{K_{\mathrm{IC}}}{\sigma_y}\right)^2 \Phi\left(\frac{\Delta\sigma_{\mathrm{fl}}}{\sigma_y}, \frac{\Delta K_{\mathrm{th}}}{K_{\mathrm{IC}}}, \frac{E}{\sigma_y}, \frac{\Delta K}{K_{\mathrm{IC}}}, \frac{h}{r_p}, \frac{a}{a_0}, \frac{d}{r_p}; 1-R\right) \tag{7}$$

where the Π_i dimensionless numbers have been rewritten using the same notation as in the Generalized Cumulative Fatigue Damage Formulation section. At this point, we want to see if the number of quantities involved in the relationship [7] can be reduced further from eight. Considering the nondimensional number $\Pi_1 = \Delta K/K_{\mathrm{IC}}$, it has to be noticed that it rules the transition from the asymptotic behavior characterized by the condition of nonpropagating cracks, when $\Delta K \to \Delta K_{\mathrm{th}}$, to the pure Griffith-Irwin instability, when $\Delta K \to \Delta K_{\mathrm{cr}}$. Moreover, incomplete self-similarity in Π_1 would correspond to a power-law dependence of the crack growth rate on the stress-intensity factor range, which is experimentally confirmed by the Paris' law [3,4]. Therefore, complete self-similarity in Π_4 cannot be accepted, whereas incomplete self-similarity gives

$$\frac{\mathrm{d}a}{\mathrm{d}N} = \left(\frac{K_{\mathrm{IC}}}{\sigma_{\mathrm{y}}}\right)^2 \left(\frac{\Delta K}{K_{\mathrm{IC}}}\right)^{\beta_1} \Phi^*(\Pi_i) \tag{8}$$

where the exponent β_1 and the nondimensional function Φ^* cannot be determined from considerations of dimensional analysis alone. Incomplete self-similarity can also be assumed for the nondimensional numbers Π_5, Π_6, Π_7, and Π_8, obtaining the following generalized representation of fatigue crack growth

$$\frac{\mathrm{d}a}{\mathrm{d}N} = \left(\frac{K_{\mathrm{IC}}^{2-\beta_1}}{\sigma_y^2}\right) \Delta K^{\beta_1} \left(\frac{h}{r_p}\right)^{\beta_2} \left(\frac{a}{a_0}\right)^{\beta_3} \left(\frac{d}{r_p}\right)^{\beta_4} (1-R)^{\beta_5} \Phi^{**} \tag{9}$$

The experimentally observed deviations from the simplest power-law regime suggested by Paris ($da/dN = C\Delta K^m$) is therefore the result of incomplete self-similarity which gives us the following expressions for the Paris' law parameters m and C

$$m = \beta_1$$
$$C = \left(\frac{K_{\mathrm{IC}}^{2-m}}{\sigma_y^2}\right)\left(\frac{h}{r_p}\right)^{\beta_2}\left(\frac{a}{a_0}\right)^{\beta_3}\left(\frac{d}{r_p}\right)^{\beta_4}(1-R)^{\beta_5}\Phi^{**} \tag{10}$$

This generalized mathematical representation encompasses several improved versions of the Paris' law proposed in the past to cover specific anomalous deviations from the simplest power-law regime suggested by Paris. For instance, as far as the grain-size dependence of C is concerned, it has recently been demonstrated in [33,34] that the cycles to failure in many alloys is a decreasing function of the grain size, suggesting a power-law dependence of C on d as in Eq 10, with an exponent β_4 related to the parameters of the Hall-Petch relationship.

Modified Paris' laws taking into account the effect of the crack length have been proposed both for metals and quasi-brittle materials. For metals, several researchers have questioned the validity of the similitude hypothesis, which states that "two different sized cracks embedded into two different sized bodies subjected to the same stress-intensity factor range should grow at the same rate." In this context, Molent et al. [37] and Jones et al. [38] have recently proposed a generalized Frost and Dugdale [39] crack growth equation of power-law type on a.

Finally, as far as the loading ratio is concerned, several Authors have proposed to include in the fatigue crack growth criterion both R and ΔK on an empirical basis [40–42]. They obtained the so-called "two-parameters" formulations with an exponent β_5 less than zero, confirming the experimental evidence that the crack propagation rate is an increasing function of the loading ratio.

The Anomalous Crack-Size Dependency of The Paris' Law: An Interpretation According to Fractal Geometry

As shown in the previous section, a crack-size dependency of the Paris' law corresponds to the incomplete similarity in the dimensionless number a/a_0. In this section, we demonstrate that this phenomenon can be ascribed to the multiscale fractal roughness of crack surfaces. An early application to fatigue of the innovative concepts of fractals and multifractal measures, introduced by Mandelbrot in [43], can be traced back to the work by Williford [44,45]. He modeled the fracture surfaces near the crack tip as an invasive fractal and proposed a modified Paris' law where both the Paris' parameters are functions of the surface fractal dimension. In the 1990s, experimental evidences in [46] and [47] pointed out a dependence of the crack growth rate on the specimen size, i.e., a size effect on fatigue crack growth. Thus, exploiting the renormalized quantities related to fractal cracks (whose surfaces can be modeled as invasive fractals according to Carpinteri [48,49]), Spagnoli [17] proposed the following size-independent fatigue crack growth law

$$\frac{\mathrm{d}a^*}{\mathrm{d}N} = C(\Delta K^*)^m \tag{11}$$

where:

$$a^* = a^{1+d_{\mathcal{G}}}$$

$$\Delta K^* = \Delta K a^{-(d_{\mathcal{G}}/2)}$$

and $d_{\mathcal{G}}$ is a parameter related to the fractal dimension of the invasive rough crack profile, $D = 1 + d_{\mathcal{G}}$. A scaling law can be obtained by rewriting Eq 11 in terms of the nominal crack propagation rate, da/dN, and the nominal stress-intensity factor range, ΔK

$$\frac{\mathrm{d}a}{\mathrm{d}N} = \frac{C}{1+d_{\mathcal{G}}} a^{-d_{\mathcal{G}}\left(1+\frac{m}{2}\right)}\Delta K^m \tag{12}$$

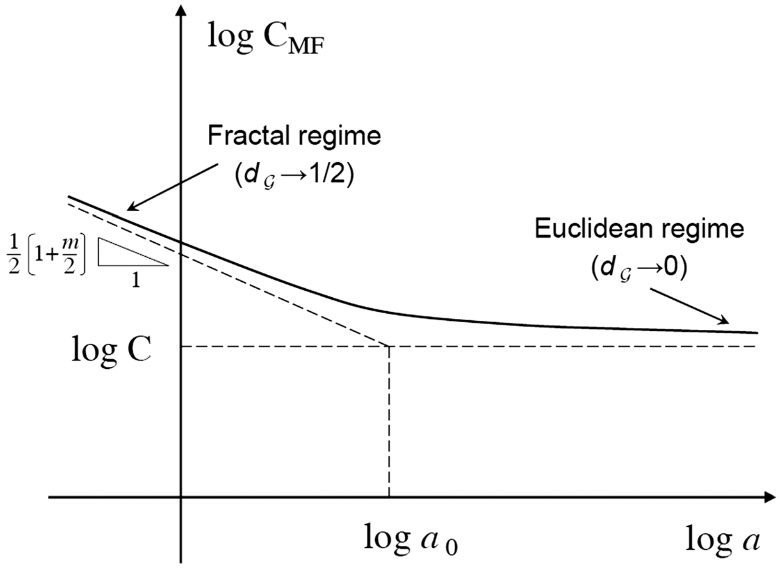

FIG. 4—*Multifractal scaling law for the Paris' law parameter C.*

This model was referred to as *monofractal approach* to size effect on fatigue crack growth [17,50]. Comparing Eq 12 with Eq 10, we note that the incomplete similarity exponent β_3 can be theoretically related to the fractality of the crack profiles, i.e., $\beta_3 = -d_{\mathcal{G}}(1+m/2)$.

The use of a *multifractal approach* was also suggested in [17,50] to model the propagation of cracks over a wider size range. Recently, Paggi and Carpinteri [18] have proposed a multifractal scaling law for the Paris' law parameter C, as an interpolating function between the asymptote for short cracks, where $d_{\mathcal{G}} \rightarrow 1/2$, and that for long cracks, where $d_{\mathcal{G}} \rightarrow 0$, see also Fig. 4

$$C_{\mathrm{MF}}(a) = C\left(1+\frac{a_0}{a}\right)^{\frac{1}{2}\left(1+\frac{m}{2}\right)} \tag{13}$$

Equation 13 has also related consequences on the crack-size dependency of the threshold stress-intensity factor range. In fact, inverting Eq 13 in correspondence of a conventional crack growth rate corresponding to infinite life, v_{th}, we have

$$\Delta K_{th} = \left(\frac{v_{\mathrm{th}}}{C_{\mathrm{MF}}}\right)^{1/m} = \Delta K_{th}^{\infty}\left(1+\frac{a_0}{a}\right)^{-\frac{1}{2}\left(\frac{1}{2}+\frac{1}{m}\right)} \tag{14}$$

where ΔK_{th}^{∞} is the value of the fatigue threshold for long cracks, see Fig. 5.

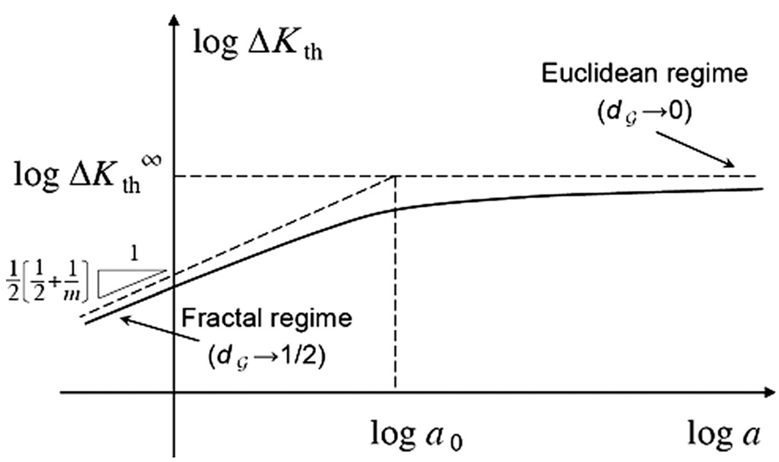

FIG. 5—*Multifractal scaling law for the fatigue threshold.*

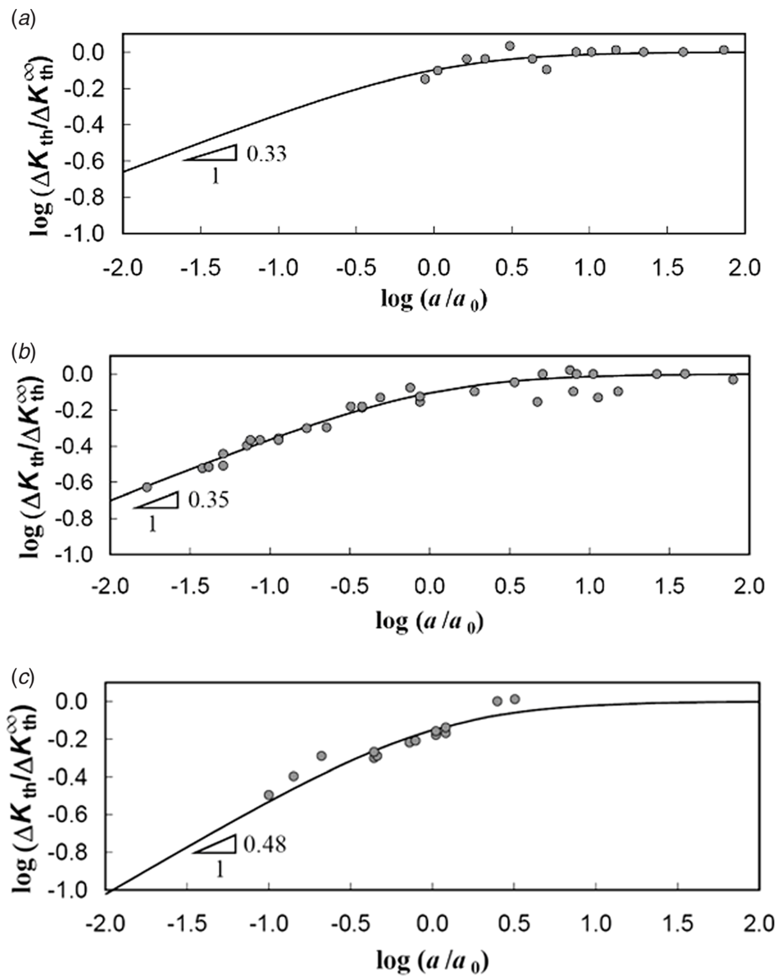

FIG. 6—*Experimental assessment of the proposed multifractal scaling law for the fatigue threshold (experimental data taken from [51]).*

Considering the data collected in [51], an experimental assessment of Eq 14 is proposed in Fig. 6 for different metals. By performing a nonlinear regression analysis on the experimental data, the value of a_0 and the exponent of the multifractal scaling law are determined. The characteristic length a_0 ranges from 1–10 μm for very high strength steels to 100–1000 μm for very low strength steels. The exponent $1/2(1/2+1/m)$ of the scaling law [14] ranges from 0.33 to 0.48.

Analytical Correlations Between the Fatigue Properties of Engineering Materials

Let us consider the limit points of the Paris' curve defining the range of validity of the power-law approximation relating the crack growth rate, da/dN, to the stress-intensity factor range, ΔK. They correspond, respectively, to the points with horizontal coordinates equal to the *fatigue threshold*, ΔK_{th}, and to the fracture instability limit ΔK_{cr}. In this range, the Paris' curve is usually defined in terms of the parameters C and m.

Now, let us consider the construction added with dashed line to Fig. 2, as proposed by Fleck et al. [24]. If a tangent is drawn at the midpoint of the central linear region of the curve and extrapolated, it is found empirically that it intersects the vertical line $\Delta K = \Delta K_{\text{th}}$ in correspondence to a crack growth rate of approximately $v_{\text{th}} = 1\times10^{-9}$ m/cycle, and it intersects the line $\Delta K = \Delta K_{\text{cr}} = (1-R)K_{\text{IC}}$ at about $v_{\text{cr}} = 1\times10^{-5}$ m/cycle. Evaluating the Paris' law in correspondence to the latter point, the following correlation between the parameters C and m of the Paris' curve can be obtained

$$C = \frac{v_{\text{cr}}}{[(1-R)K_{\text{IC}}]^m} \tag{15}$$

Repeating this reasoning for the point defined by the fatigue threshold, we have

$$C = \frac{v_{\rm th}}{(\Delta K_{\rm th})^m} \tag{16}$$

Equating 15 to 16, we express the Paris' law parameter m as a function of the fatigue properties

$$m = \frac{\log v_{\rm th} - \log v_{\rm cr}}{\log \Delta K_{\rm th} - \log[(1-R)K_{\rm IC}]} \tag{17}$$

or, in a bilogarithmic form

$$\log\left(\frac{\Delta K_{\rm th}}{K_{\rm IC}}\right) = \log(1-R) + \frac{1}{m}\log\left(\frac{v_{\rm th}}{v_{\rm cr}}\right) \tag{18}$$

Equation 18 establishes a correspondence between $\Delta K_{\rm th}$, $K_{\rm IC}$ and m in the long-crack regime and was experimentally confirmed by Fleck et al. [24] for a wide range of materials. Considering the fatigue property chart reported in Fig. 7(*a*), we observe a very good agreement between the experimental trend and the proposed correlation, being $R = 0$ and $\log(v_{\rm th}/v_{\rm cr}) \cong \log(1 \times 10^{-9}/1 \times 10^{-5}) = -4$.

A relationship between the fatigue stress-intensity factor threshold and the fatigue limit can be derived by considering the propagation of a Griffith crack of length $2a_0$ in an infinite elastic plate subjected to cyclic loading with $\Delta\sigma = \Delta\sigma_{\rm fl}$ acting at infinity and $R = 0$. The initial crack length is representative of the size of the existing microdefects, i.e., $a_0 = (K_{\rm IC}/\sigma_y)^2/\pi$. If $\Delta\sigma = \Delta\sigma_{\rm fl}$, then $\Delta K_{\rm th} = \Delta\sigma_{fl}\sqrt{\pi a_0}$ and the life of the specimen would tend to infinity. On the other hand, when the applied load $\Delta\sigma = \sigma_y$, the stress-intensity factor at the crack tip reaches the fracture toughness: $K_{\rm IC} = \sigma_y\sqrt{\pi a_0}$. Eliminating a_0 from the previous equations we obtain an important relation

$$\left.\begin{aligned}\Delta K_{\rm th} &= \Delta\sigma_{\rm fl}\sqrt{\pi a_0}\\ K_{\rm IC} &= \sigma_y\sqrt{\pi a_0}\end{aligned}\right\} \Rightarrow \frac{\Delta K_{\rm th}}{K_{\rm IC}} = \frac{\Delta\sigma_{\rm fl}}{\sigma_y}$$

that can be rewritten as follows

$$\log \Delta K_{\rm th} = \log\frac{2K_{\rm IC}}{\sigma_y} + \log\frac{\Delta\sigma_{\rm fl}}{2} \tag{19}$$

A direct comparison between this correlation $\Delta K_{\rm th}$ versus $\Delta\sigma_{\rm fl}/2$ and the experimental trend observed for a wide range of materials and collected in the fatigue property chart by Fleck et al. [24] is proposed in Fig. 7(*b*). A linear relation is correctly reproduced and the intercept depends on the ratio $K_{\rm IC}/\sigma_y$, which is proportional to the square root of the critical process zone size. Engineering ceramics present a lower value of $K_{\rm IC}/\sigma_y$ as compared to steel alloys, and therefore their position in the diagram is shifted downwards.

Conclusions

The Wöhler and Paris curves were originally thought as "universal laws" in the sense that they should have been able to provide a universal description of fatigue. Actually, the experimentally observed deviations led to a proliferation of modified fatigue criteria, very often represented by power laws. Therefore, if on the one hand the research efforts were directed towards the extension of the original fields of application of the Wöhler and Paris representations of fatigue, on the other hand the fundamental problem of finding the link between the cumulative fatigue damage and the fatigue crack propagation approaches remained largely unsolved.

In the present contribution, a dimensional analysis approach and the concepts of complete and incomplete self-similarity have been applied to the Wöhler and Paris' curves. As a main conclusion, it has been shown that the large number of power laws used in fatigue are the result of an incomplete self-similarity in the corresponding dimensionless numbers. This gives a rational interpretation to such empirically-based fatigue criteria, towards a unified description of fatigue and a possible standardization. Special attention

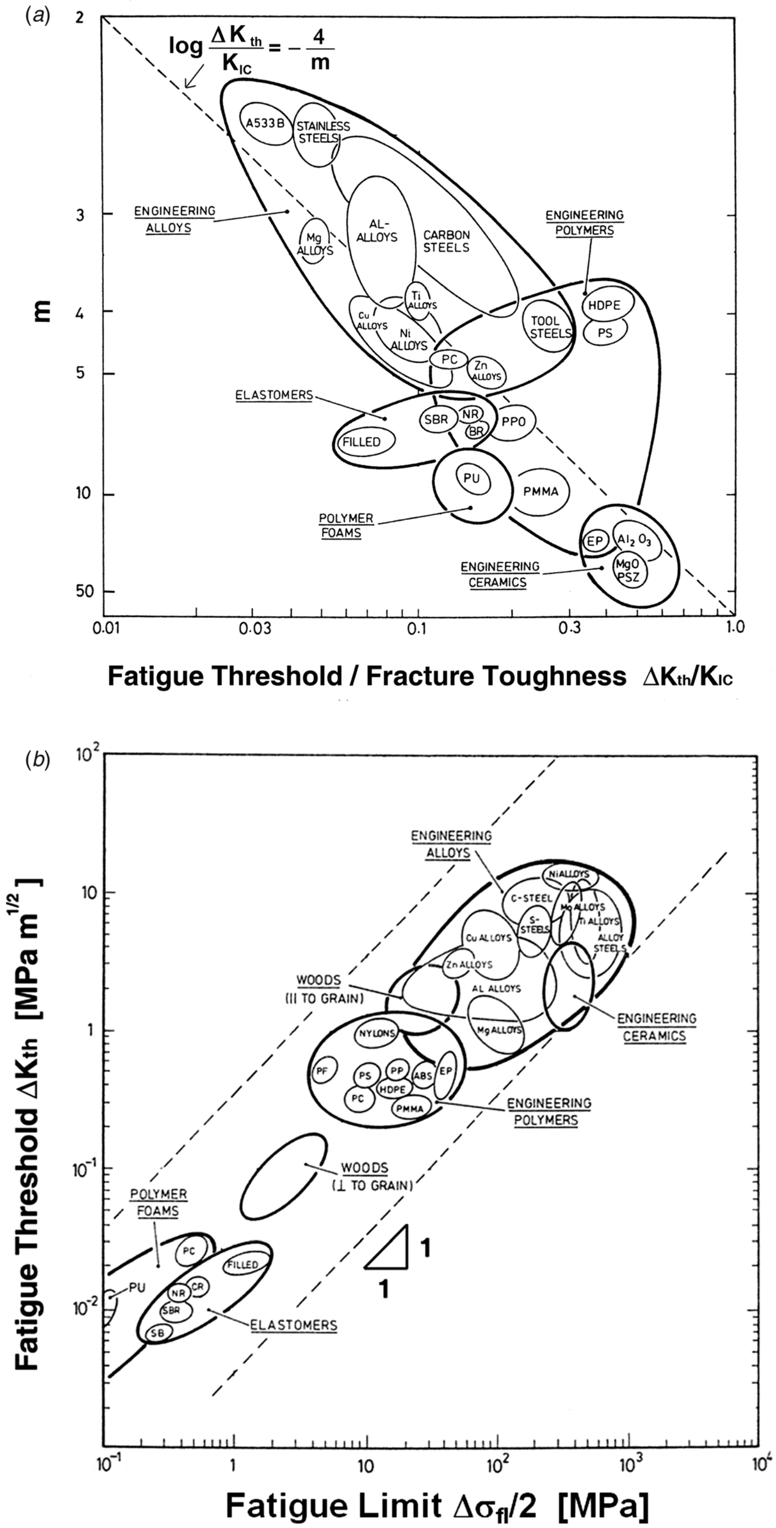

FIG. 7—*Interpretation of the material property charts adapted from [24].*

has been paid to the anomalous crack-size dependencies of C and ΔK_{th}, proposing a model based on the fractality of fatigue crack paths. The results confirmed by experiments provide a way to estimate the incomplete similarity exponent of the crack length on a theoretical basis and to link it to the fractal dimension of the crack profiles.

Finally, analytical correlations between the cyclic properties of engineering materials have been established, providing a rational interpretation to the empirical correlations existing in the Literature and to the well-known fatigue property charts.

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