International Journal of Fatigue 33 (2011) 477–483

Contents lists available at ScienceDirect

International Journal of Fatigue

journal homepage: www.elsevier.com/locate/ijfatigue

A dimensional analysis interpretation to grain size and loading frequency dependencies of the Paris and Wöhler curves

O. Plekhov [a], M. Paggi [b,*], O. Naimark [a], A. Carpinteri [b]

[a] Institute of Continuous Media Mechanics, Russian Academy of Sciences, Ak. Koroleva str 1, 614013 Perm, Russia
[b] Politecnico di Torino, Department of Structural Engineering and Geotechnics, Corso Duca degli Abruzzi 24, 10129 Torino, Italy

ARTICLE INFO

Article history:
Received 4 May 2010
Received in revised form 20 September 2010
Accepted 5 October 2010
Available online 8 October 2010

Keywords:
Dimensional analysis
Fatigue crack growth
Grain size
Loading frequency
Ultra fine grained materials

ABSTRACT

In this paper, a mathematical model based on dimensional analysis and incomplete self-similarity is proposed for the interpretation of the grain size and loading frequency effects on the Paris and Wöhler regimes in metals. In particular, it is demonstrated that these effects correspond to a violation of the physical similitude hypothesis underlying the simplest Paris' and Wöhler power-law fatigue relationships. As a consequence, generalized representations of fatigue have to be invoked. From the physical point of view, the incomplete similarity behaviour can be regarded as the result of the multiscale character of the problem, where the crack length and the grain size are the two length scales interacting together. Moreover, it will be shown that the relationship between strength and grain size (Hall–Petch relationship) has also to be considered in order to consistently interpret the two opposite effects of the grain size on the Paris and Wöhler regimes within a unified framework. The incomplete similarity exponents are suitably quantified according to experimental results for Aluminum, Copper, Titanium and Nickel. The derived scaling laws are expected to be of paramount importance today, especially after the advent of ultra fine grained materials that offer unique mechanical properties owing to their fine microstructure.

1. Introduction

The existing approaches for the prediction of fatigue life can be distinguished in two main categories: those related to the Cumulative Fatigue Damage (CFD) approach, which is the traditional framework for fatigue life assessment, and those based on the Fatigue Crack Propagation (FCP) approach, developed since the 1960s after the advent of fracture mechanics. The CFD analysis is based on the Wöhler or *S–N* curves [1]. A schematic representation of the Wöhler's curve is shown in Fig. 1a, where the cyclic stress range, $\Delta\sigma = \sigma_{\max} - \sigma_{\min}$, is plotted as a function of the number of cycles to failure, *N*. In this diagram, a reasonable power-law relationship between $\Delta\sigma$ and *N* exists [2]. Here, we introduce the range of stress at static failure, $\Delta\sigma_u = (1-R)\sigma_u$, where σ_u is the material *tensile strength* and $R = \sigma_{\min}/\sigma_{\max}$ is the loading ratio, and we define the *endurance* or *fatigue limit*, $\Delta\sigma_{fl}$, as the stress range that a sample will sustain without fracture for $N_\infty = 1 \times 10^7$ cycles, which is a conventional value that can be thought of as "infinite" life.

In the FCP approach, the crack growth rate, d*a*/d*N*, is plotted as a function of the stress-intensity factor range, $\Delta K = K_{\max} - K_{\min}$. Also in this case, most of the experimental data can be interpreted in terms of a power-law relationship, i.e., according to the so-called Paris' law [3,4]. A schematic representation of the Paris' curve is shown in Fig. 1b. The power-law representation presents some deviations for very high values of ΔK approaching $\Delta K_{cr} = (1-R)K_{IC}$ [5], where K_{IC} is the material *fracture toughness*, or for very low values of ΔK approaching the *threshold stress-intensity factor range*, ΔK_{th}. Again, in close analogy with the concept of fatigue limit, the fatigue threshold is defined in a conventional way as the value of ΔK below which the crack grows at a rate of less than 1×10^{-9} m/cycle.

For a long time, the CFD and the FCP approaches have been considered as totally independent. In the last few decades, the researchers have attempted to extend the field of application of the FCP approach [6–11]. These advances in understanding the complex phenomenon of fatigue crack growth shed a new light on the possibility to unify the CFD and the FCP approaches, and to solve the challenging task of interpreting the Paris and Wöhler power-law regimes within a unified theoretical framework [12–16].

In the present paper, we propose an interpretation of grain size and frequency effects on the Paris and Wöhler curves. This will be achieved treating the problem in the framework of dimensional analysis, extending the approach proposed in [16], where the effects of the grain size and of the loading frequency were not

* Corresponding author. Tel.: +39 0115644910; fax: +39 0115644899.
E-mail address: marco.paggi@polito.it (M. Paggi).

0142-1123/$ - see front matter © 2010 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ijfatigue.2010.10.001

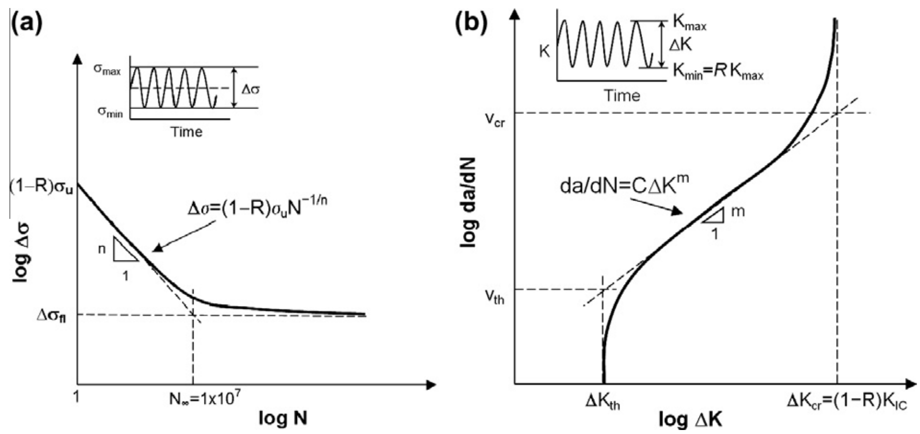

Fig. 1. Schemes of the (a) Wöhler and (b) Paris' curves with the related fatigue parameters.

discussed. The first application of dimensional analysis to fatigue can be traced back to the pioneering paper by Barenblatt and Botvina [6]. More recent developments on this line of research can be found in [8,9,17,18]. In these contributions, the drawbacks of the Paris' law written in its original form, i.e., $da/dN = C\Delta K^m$, have been highlighted. In particular, it was observed that the proportionality constant C is not dimensionless and its physical dimensions cannot be known until the exponent m is estimated. Moreover, the physical dimensions of C are material dependent. The dimensional analysis approach pursued in this paper permits to overcome such drawbacks and it will be used to make explicit the functional dependencies on the grain size and on the load frequency.

The interest in the study of the grain size effect on fatigue properties of metals is motivated by the potential application of a new class of ultra fine grained (ufg) bulk materials. Although it is well-known that these materials exhibit exceptional mechanical and physical properties under quasi-static loading, their fatigue properties require more detailed experimental investigation and the development of a predictive model is the question of the day. A review of recent publications on this subject is given in [19]. Current investigations on the fatigue properties of ufg materials focus primarily on the study of single-phase metals (Copper, Nickel, Titanium, Aluminum). One of the main experimental results concerning ufg materials suggests that, by refining the grain size, the fatigue life of such specimens is higher than that of their conventional counterparts. These trends were found for Aluminum, exploring a grain size range from 46 up to 280 μm [20], for the Al–7Mg alloy, with a grain size range between 0.25 and 100 μm [21], for Ti Grade 2, Ni and Cu, comparing grain sizes less than 100 nm and more than 1 μm [22], as well as for Ti, Ni and Al–Mg, comparing grain sizes less than 0.1 μm and more than 1 μm [23,24]. A qualitatively sketch showing the shift of the Wöhler curve due to the grain size effect is depicted in Fig. 2a. However, more experimental results are indeed required to quantify the variation of the position of the knee point of the curve, as well as of its slope n (see, e.g. [19]).

The effect of the grain size on the fatigue crack growth rate was investigated for Al–7Mg alloy in [21], for Nickel and Titanium in [23], and for Copper, Nickel and Titanium in [25]. As a common trend, it was found that the value of the fatigue threshold, ΔK_{th}, is lower for ufg specimens than for conventional grains. Moreover, for the same value of stress-intensity factor ΔK, the fatigue crack growth rate of ufg is higher than that of conventional grain sizes. This seems to be in contrast with the shift of the Wöhler curve and no explanations have been provided so far to reconciliate such opposite trends. A qualitative evolution of the Paris' curve due to the grain size effect is depicted in Fig. 2b.

An additional experimental result related to the subject of this paper concerns the effect of the loading frequency on the fatigue properties. This effect is not so pronounced as that due to the grain size, but, as we will show in the sequel, they are strongly related to each other and therefore they must be analyzed together. Although it is well-known that the fatigue limit is an increasing function of the loading frequency, the coupled effect of grain size and loading frequency on the fatigue properties is a more recent research field [24,26,27]. A qualitative evolution of the Wöhler curve due to the effect of the loading frequency is shown in Fig. 3a according to

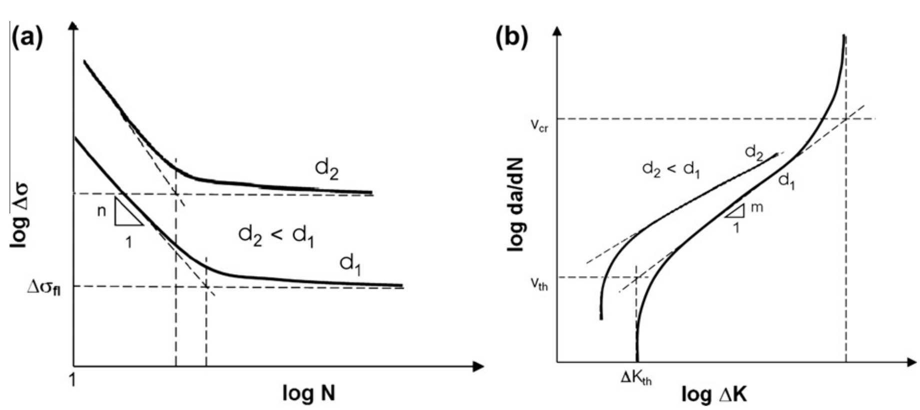

Fig. 2. The effect of the grain size on (a) the Wöhler and (b) the Paris' curves.

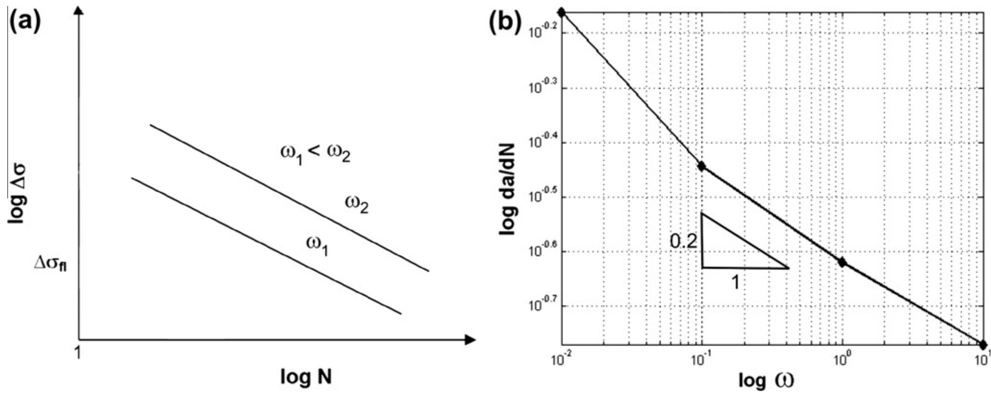

Fig. 3. The effect of loading frequency on (a) the Wöhler curve (according to the results in [24]) and on (b) the fatigue crack growth rate (according to the results in [26]).

the experimental results in [24]. These results were based on the set of Ni specimens with a loading ratio $R = 0.25$. As a general trend, the effect of the loading frequency is the opposite with respect to that due to the grain size. The higher loading frequency corresponds to the higher fatigue life for a given applied stress range (see Fig. 3a). The effect of the loading frequency on the fatigue crack growth rate is also shown in Fig. 3b according to the experimental results in [26]. This data are computed for a given stress-intensity factor range ($\Delta K = 20$ MPa m$^{1/2}$) from a set of Ti6342 alloy specimens at temperature 520 °C with as-received microstructure. Contrary to the grain size effects, these two trends are consistent with each other, since a specimen subjected to a higher loading rate experiences a lower crack growth rate and therefore at the end it has a longer fatigue life.

In Section 2, a generalized mathematical representation of fatigue behaviour in metals based on dimensional analysis and on the concept of incomplete similarity is proposed. The procedure is based on the method proposed in [16], suitably extended by including the grain size and the loading frequency in the formulation. As a result, generalized Wöhler and Paris laws describing the effects of grain size and loading frequency will be derived. Then, in Section 3, we propose an experimental assessment of the predictive capabilities of these new models. To capture the experimental trends and to solve the apparent inconsistencies related to the grain size effect on the Paris and Wöhler regimes, we show that it is necessary to take into account the relationship between strength and grain size, viz. the so-called Hall–Petch relationship, in the mathematical formulation.

2. Generalized mathematical representations of fatigue based on dimensional analysis

According to the work by Barenblatt and Botvina [6] and its more recent generalization by Carpinteri and Paggi [16], the following functional dependence can be considered for the phenomenon of fatigue crack growth:

$$\frac{da}{dN} = F(\sigma_y, \Delta K; K_{IC}, d, a, \omega), \tag{1}$$

where the governing variables are summarized in Table 1, along with their physical dimensions expressed in the Length–Force–Time class (LFT). The applied stress range, $\Delta\sigma$, is not considered in Eq. (1). As demonstrated in the Appendix A, including $\Delta\sigma$ in addition to ΔK in the functional dependency (1) does not add any information to the present analysis.

Considering the Paris law as a manifestation of a self-similar stage of the crack propagation process, we can try to analyze the intermediate asymptotic of the process. Hence, we apply the Buckingham's Π Theorem [26] to reduce the number of parameters involved in the problem (see also [8,13,16,29]). As a result, we have:

Table 1
Main governing variables of the fatigue crack growth phenomenon in metals.

Variable definition	Symbol	Dimensions
Yield strength	σ_y	FL^{-2}
Fracture toughness	K_{IC}	$FL^{-3/2}$
Stress-intensity factor range	ΔK	$FL^{-3/2}$
Stress range	$\Delta\sigma$	FL^{-2}
Grain size	d	L
Crack length	a	L
Loading frequency	ω	T^{-1}
Thermal diffusivity coefficient	χ	$L^2 T^{-1}$

$$\frac{da}{dN} = \left(\frac{\Delta K}{\sigma_y}\right)^2 \Phi\left(\frac{K_{IC}}{\Delta K}, \frac{\sigma_y^2}{K_{IC}^2} d, \frac{\sigma_y^2}{K_{IC}^2} a, \frac{\omega d^2}{\chi}\right) = \left(\frac{K_{IC}}{\sigma_y}\right)^2 \Phi(\Pi_1, \Pi_2, \Pi_3, \Pi_4), \tag{2}$$

where Π_i $(i = 1, \ldots, 4)$ are dimensionless numbers and $\chi = \lambda/(\rho c)$ is the thermal diffusivity coefficient, which is equal to the thermal conductivity, λ (J/(m s K)), divided by the product between the material density, ρ (kg/m^3), and the heat capacity, c (J/(kg K)). According to physical idea by Chan [30], we assume that fatigue crack growth is the result of sequence of low-cycle fatigue failure process of a material volumes located near the crack-tip. The size of the volumes correspond to the characteristic length of the material heterogeneity. In our particular case, the characteristic size of the volume corresponds to dislocation barriers spacing. To take into account the frequency effect on the crack growth rate, we consider the thermal diffusion process in the crack-tip zone. The higher the frequencies, the higher the temperatures in that zone, with a direct influence on the crack growth rate. In this context, the ratio d^2/χ has the physical dimensions of time and corresponds to the characteristic heat conductivity time governing the heat conduction process in the crack-tip zone.

In the derivation of Eq. (2), the material fracture toughness, K_{IC}, is used to make dimensionless the grain size and the crack length. Alternatively, the stress-intensity factor range, ΔK, could be used. However, this choice has no effect on the final scaling law, see the demonstration in the Appendix A.

At this point, we want to see if the number of quantities involved in relationship (2) can be reduced further from four. This can occur either in the case of *complete* or *incomplete self-similarities* in the corresponding dimensionless variables. In the former situation, the dependence of the mechanical response on a given dimensionless number, say Π_i, disappears and we can say that Π_i is *non essential* for the representation of the physical phenomenon. In the latter situation, a power-law dependence on Π_i can be postulated, which usually characterizes a physical situation intermediate between two asymptotic behaviours.

Considering incomplete self-similarity in the dimensionless variables Π_1, Π_2, Π_3 and Π_4, we obtain the following generalized representation of fatigue crack growth:

$$\frac{\mathrm{d}a}{\mathrm{d}N} = \left(\frac{\Delta K}{\sigma_y}\right)^2 \left(\frac{K_{\mathrm{IC}}}{\Delta K}\right)^{\alpha_1} \left(\frac{\sigma_y^2}{K_{\mathrm{IC}}^2} d\right)^{\alpha_2} \left(\frac{\sigma_y^2}{K_{\mathrm{IC}}^2} a\right)^{\alpha_3} \left(\frac{\omega d^2}{\chi}\right)^{\alpha_4} \Phi_1 = \Delta K^{2-\alpha_1} d^{\alpha_2+2\alpha_4} a^{\alpha_3} \left(\frac{\omega}{\chi}\right)^{\alpha_4} \frac{\Phi_1}{K_{\mathrm{IC}}^{-\alpha_1+2\alpha_2+2\alpha_3} \sigma_y^{2(1-\alpha_2-\alpha_3)}}. \quad (3)$$

Eq. (3) can be considered as a generalized Paris' law (see the classical expression in Fig. 1b), in which the functional dependencies on the grain size, the crack size and loading frequency have been suitably explicated. An experimental assessment showing the predictive capabilities of this model is proposed in the next section. It is interesting to note that the yield strength entering Eq. (3) is usually dependent on the grain size in its turn. For nano- and micro-crystalline materials, a power-law function of the grain size, $\sigma_y \propto d^\gamma$, is usually found from experiments, where γ is a coefficient dependent on the grain size range being considered. Hence, Eq. (3) can be further simplified as follows:

$$\frac{\mathrm{d}a}{\mathrm{d}N} = \Delta K^{2-\alpha_1} d^{\alpha_2+2\alpha_4-2\gamma(1-\alpha_2-\alpha_3)} a^{\alpha_3} \left(\frac{\omega}{\chi}\right)^{\alpha_4} \frac{\Phi_1}{K_{\mathrm{IC}}^{-\alpha_1+2\alpha_2+2\alpha_3}}. \quad (4)$$

For conventional grain metals, we usually have $\gamma = -1/2$ according to the classical Hall–Petch [21,31,32] relationship and therefore we expect a power-law exponent for the grain size equal to $2\alpha_4 - \alpha_3 + 1$. Setting $\alpha_3 = 0$ (as occurs in the case of long cracks) and noting that α_4 is negative and $|\alpha_4| > 1/2$, then the power-law exponent of the grain size should be negative valued in its turn. This result agrees well with the experimental results presented in [26].

From the theoretical point of view, the negative power-law exponent of d could also be observed if we reduce the grain size to the nanoscale. In fact, if we allow the possibility to have an inverse Hall–Petch relationship ($\gamma > 0$), then the exponent for the grain size may become negative valued for $\gamma < \frac{\alpha_2+2\alpha_4}{2(1-\alpha_2-\alpha_3)}$.

So far, the crack growth rate has been chosen as the main output parameter characterizing the phenomenon of fatigue crack growth. Alternatively, we can consider the fatigue life, N, as the parameter representative of fatigue. Following this route, we postulate the following functional dependence:

$$N = F(\sigma_y, K_{\mathrm{IC}}, d; \Delta\sigma, a, \omega), \quad (5)$$

where the definition of the governing variables is provided again in Table 1. As in the previous case, it is possible to apply the Buckingham's Π theorem [28] to reduce the number of parameters involved in the problem:

$$N = \left(\frac{\sigma_y}{K_{\mathrm{IC}}}\right)^2 d\ \Psi\left(\frac{\Delta\sigma}{\sigma_y}, \frac{\sigma_y^2}{K_{\mathrm{IC}}^2} a, \frac{\omega d^2}{\chi}\right) = \left(\frac{\sigma_y}{K_{\mathrm{IC}}}\right)^2 d\ \Psi(\Gamma_1, \Gamma_2, \Gamma_3), \quad (6)$$

where Ψ is a dimensionless function, $\Gamma_1 = \Delta\sigma/\sigma_y$, $\Gamma_2 = \Pi_2$, and $\Gamma_3 = \Pi_3$. At this point, we want to see if the number of quantities involved in relationship (6) can be reduced further from three. In close analogy with the procedure discussed for the Paris' law, we assume incomplete self-similarity in Γ_1, Γ_2, and Γ_3, obtaining:

$$N = \left(\frac{\sigma_y}{K_{\mathrm{IC}}}\right)^2 d \left(\frac{\Delta\sigma}{\sigma_y}\right)^{\beta_1} \left(\frac{\sigma_y^2}{K_{\mathrm{IC}}^2} a\right)^{\beta_2} \left(\frac{\omega d^2}{\chi}\right)^{\beta_3} \Psi_1 = \Delta\sigma^{\beta_1} a^{\beta_2} d^{1+2\beta_3} \sigma_y^{2-\beta_1+2\beta_2} K_{\mathrm{IC}}^{-2(1+\beta_2)} \left(\frac{\omega}{\chi}\right)^{\beta_3} \Psi_1. \quad (7)$$

Eq. (7) provides a generalized Wöhler relationship for fatigue and encompasses the empirical S–N curves approximated by the Basquin power-law and by the Coffin–Manson criterion as limit cases, as shown in [16]. However, since $\beta_3 > 0$ from experiments, Eq. (7) would suggest a positive scaling with the grain size, which is not found in reality. Therefore, to solve such an inconsistency, let us introduce the grain size dependence of the yield strength in Eq. (7). Considering $\sigma_y \propto d^\gamma$, we have:

$$N = \Delta\sigma^{\beta_1} a^{\beta_2} d^{1+2\beta_3+\gamma(2-\beta_1+2\beta_2)} K_{\mathrm{IC}}^{-2(1+\beta_2)} \left(\frac{\omega}{\chi}\right)^{\beta_3} \Psi_1. \quad (8)$$

For conventional grain materials, the Hall–Petch law applies and we have a power-law exponent for the grain size equal to $\beta_1/2 - \beta_2 + 2\beta_3$. Assuming $\beta_2 = 0$ (non-pronounced initial defect size), the negative value of the grain size exponent depends now on the values of β_1 and β_3 and no longer on β_3 only. Since usually $\beta_1 < 0$ and $\beta_3 > 0$ from experiments [24], this gives the relationship $\beta_3 < |\beta_1|/4$, that also confirms the small effect of frequency on the fatigue limit.

A relationship between the CFD (Eq. (7)) and FCP (Eq. (3)) approaches can be determined by integrating the generalized Paris' law in Eq. (3) between an initial defect size, a, and a generic critical (final) crack length, a_f, corresponding to a given fatigue life N [11]. Recalling that $\Delta K = \Delta\sigma\sqrt{\pi a}$ for a Griffith crack, then the integration gives the following result:

$$N = \frac{\Delta\sigma^{\alpha_1-2} d^{-\alpha_2-2\alpha_4} \omega^{-\alpha_4}}{\left(\frac{\alpha_1}{2}+\alpha_3\right) \pi^{(2-\alpha_1)/2} \chi^{-\alpha_4} \frac{\Phi_2}{K_{IC}^{-\alpha_1+2\alpha_2+2\alpha_3} \sigma_y^{2(1-\alpha_2-\alpha_3)}}} \left[a_f^{\left(\frac{\alpha_1}{2}-\alpha_3\right)} - a_0^{\left(\frac{\alpha_1}{2}-\alpha_3\right)}\right], \quad (9)$$

which can be simplified noting that $a_f^{\left(\frac{\alpha_1}{2}-\alpha_3\right)} \ll a_0^{\left(\frac{\alpha_1}{2}-\alpha_3\right)}$, since the exponent of the crack length is negative valued and $a_f \gg a_0$. Under such conditions, the fatigue life can be approximated as follows:

$$N \cong \frac{\Delta\sigma^{\alpha_1-2} d^{-\alpha_2-2\alpha_4} \omega^{-\alpha_4} a_0^{\left(\frac{\alpha_1}{2}-\alpha_3\right)}}{\left(\alpha_3-\frac{\alpha_1}{2}\right) \pi^{(2-\alpha_1)/2} \chi^{-\alpha_4} \frac{\Phi_2}{K_{IC}^{-\alpha_1+2\alpha_2+2\alpha_3} \sigma_y^{2(1-\alpha_2-\alpha_3)}}}. \quad (10)$$

A comparison between Eq. (7), obtained according to dimensional analysis arguments, and Eq. (10), obtained through the integration of the generalized Paris' law in Eq. (3), leads to the following relationships between the powers entering the two representations:

$$\begin{cases} \beta_1 = \alpha_1 - 2 \\ \beta_2 = \frac{\alpha_1}{2} - \alpha_3 \\ 1 + 2\beta_3 = -\alpha_2 - 2\alpha_4 \\ \beta_3 = -\alpha_4 \end{cases} \quad \text{or} \quad \begin{cases} \beta_1 = \alpha_1 - 2, \\ \beta_2 = \frac{\alpha_1}{2} - \alpha_3, \\ \alpha_2 = -1, \\ \beta_3 = -\alpha_4. \end{cases} \quad (11)$$

Relationships (11) permits to relate the effects of stress amplitude, loading frequency and grain size in the two mathematical formulations. According to the experimental data in [24], we can also consider the inequality $\beta_3 > 0$. Hence, according to the relationship $\beta_3 = -\alpha_4$, we expect $\alpha_4 < 0$. Therefore, in perfect agreement with the experimental observation in [26], the effect of loading frequency on fatigue crack growth is opposite with respect to that on the fatigue life. Moreover, due to the concave shape of Wöhler curve, we also have $\beta_1 < 0$. Taking also into account the obvious inequality $\beta_2 < 0$ for the effect of the initial defect size on the fatigue life, we obtain the following restrictions to the values of the exponent α_1:

$$\beta_1 < 0 \Rightarrow \alpha_1 - 2 < 0 \Rightarrow \alpha_1 < 2, \qquad (13a)$$

$$\beta_2 < 0 \Rightarrow \frac{\alpha_1}{2} - \alpha_3 < 0 \Rightarrow \alpha_1 < 2\alpha_3. \qquad (13b)$$

Eq. (13a) implies that the Paris' law exponent $m = 2 - \alpha_1$ must be positive valued, as always found in experiments. Moreover, Eq. (13b) implies that the Paris' law exponent must be greater than $2 + 2\alpha_3$. In the long cracks regime (Region II where the Paris' power-law applies), we have $\alpha_3 = 0$, leading to $m \geqslant 2$, which is the classical lower bound to m considered in the literature.

3. Experimental assessment of the proposed fatigue laws

3.1. The effect of the grain size

Experimental investigation on the effect of the grain size on the fatigue crack growth rate was proposed by Pao et al. [21], Hanlon et al. [23,24], Cavaliere et al. [25], and Sansoz et al. [26]. Pao et al. [21] determined the fatigue crack growth rate for Al–7Mg alloys in correspondence to three different grain sizes. The average grain size of the ufg state was 0.25 μm, the average grain size of P/M (powder metallurgy) Al–7Mg is about 2 μm, and the average grain size of I/M (ingot-metallurgy) Al–7Mg is about 100 μm. The experiments were carried out on air with a loading ratio R = 0.25. For fatigue crack growth studies, 5.08-mm-thick compact-tension fracture mechanics specimens with a width of 38.1 mm and oriented in the L–C direction (the crack plane is perpendicular to the extrusion direction and the crack growth direction is parallel to the circumferential direction) were machined from as-extruded UFG Al–7.5Mg rod. For P/M and I/M Al–7Mg, fatigue specimens were oriented in the T–L direction (the crack plane and the crack growth direction are parallel to the longitudinal direction). Side grooves with depths equal to 5% of the thickness were introduced on both sides of each fatigue specimen to enhance constraint. Plotting the experimentally determined crack growth rates vs. grain size in a bilogarithmic diagram, it is possible to evaluate the incomplete similarity exponent $n = \alpha_2 + 2\alpha_4$ of the grain size (see Eq. (3)) as the slope of the corresponding best-fitting curve. More specifically, we find $n = -0.3$ from Fig. 4a. The value of the incomplete similarity exponent was determined for the stress-intensity factor range which leads to the Paris regime of crack propagation (linear response in the bilogarifmic diagram) both in coarse grain and submicrocrystalline grain states. In the paper by Cavaliere et al. [25], Ti grade 2, Ni and Cu specimens with two different characteristic grain sizes were analyzed: the ufg state (d < 1 μm) and the nanocrystalline state (d < 100 nm). Pure Ti grade 2 were produced by equal-channel-angular pressing with eight passes route Bc by employing a 90° corner channel at the temperature of 425 °C. The extruded billets measured 45 mm in diameter and 100 mm in length. For the fatigue crack propagation tests, different disks of 2 mm thickness were cut in a direction perpendicular to the extrusion one. Different specimens for the compact-tension test were obtained by EDM: from the 2 mm disks, squares of 31.25 mm side were cut, holes measuring 4 mm in diameter were obtained for the grips and finally a 12 × 1 mm notch was cut in the center of the specimen. The fatigue crack propagation tests were performed with a loading ratio R = 0.25 at 10 Hz. For Ti grade 2 and for high fatigue crack growth rates, an approximate value of the incomplete similarity exponent is $n = -0.9$ (see Fig. 4b).

Similarly, Hanlon [24] considered Ti, Ni and Al–Mg specimens. Three states of materials were introduced: nanocrystalline (d < 0.1 μm), ufg (0.1 μm < d < 1 μm), and micro-crystalline (grain size d > 1 μm) states. For Titanium, an approximate value of the incomplete similarity exponent is $n = -0.57$, which is again negative valued and not too different from that found by Cavaliere et al. [25].

According to Eq. (7), we also expect an effect of the grain size on the Wöhler curve. In particular, we should have $N \propto \Delta\sigma_{fl}^{\beta_1} d^k$, where $k = 1 + 2\beta_3$. In this case, the exponent β_3 can be determined using the experimental data in [30], where the effect of the grain size on the fatigue limit of AISI 304 stainless steel specimens was investigated. Samples of AISI 304 stainless steel in a cold rolled state were annealed at 850 °C for 30 s (state 1), 1100 °C for 3 min (state 2) and 10 min (state 3), in order to obtain the three different austenitic microstructures. The mean grain sizes were 1, 17, 47 μm, correspondently. Fatigue tests were carried out under uniaxial sinusoidal loading at a frequency of 10 Hz up to 2×10^6 cycles. In particular, the fatigue limit $\Delta\sigma_{fl}$ can be obtained from Eq. (8) in correspondence to a conventional number of cycles, N_∞. Hence, we theoretically expect $\Delta\sigma_{fl} \propto d^{-k/\beta_1}$ and from the slope of Fig. 5 we find $k/\beta_1 = 1.2$. Due to a lack of experimental data in [33], only a poor approximation for β_1 can be made, i.e. $\beta_1 = -0.35 \pm 0.3$, which gives $k = -0.45$. The significant scatter of β_1 values is caused by small numbers of experimental points used for the calculation of Wöhler curve (5–6 point for each grain size) in [34]. However, our estimation allows us to confirm that β_1 is in any case negative valued. The direct estimation of β_3 based on the determination of $k = -0.45$ ($\beta_3 = (k-1)/2$) leads to $\beta_3 = -0.73$, which contradicts the fact that β_3 must be positive valued (see [26]). The same problem arises if we use the negative value of n (for instance, $n = -0.3$) for the estimation of α_4 ($\alpha_4 = (n - \alpha_2)/2$), which must be negative valued according to experiments.

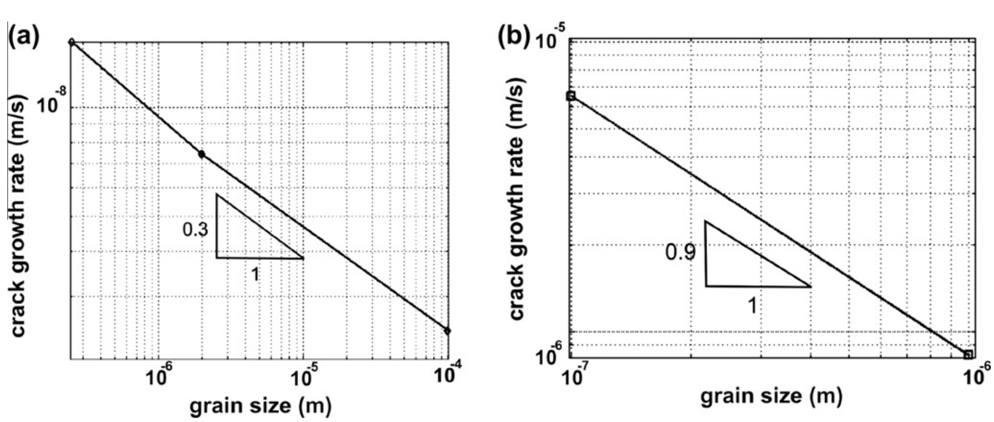

Fig. 4. Fatigue crack growth rate vs. average grain size for (a) Al–7Mg alloys (data from [21]) and for (b) Ti (data from [25]).

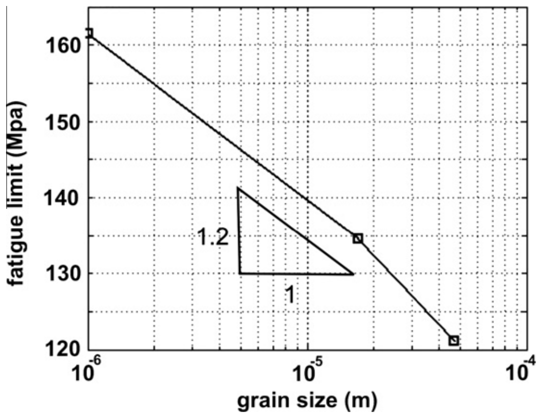

Fig. 5. Fatigue limit vs. average grain size of AISI 304 stainless steel (based on the data [33]).

These inconsistencies can be resolved if the grain size dependence of the material strength is suitably taken into account, viz. $\sigma_y \propto d^{\gamma}$. Hence, using Eq. (4) instead of (3) and Eq. (8) instead of (7), we can redefine the exponents n and k as follows:

$$n = \alpha_2 + 2\alpha_4 - 2\gamma(1 - \alpha_2 - \alpha_3), \tag{14a}$$

$$k = 1 + 2\beta_3 + \gamma(2 - \beta_1 + 2\beta_2). \tag{14b}$$

Using Eqs. (14a) and (14b), the values $n = -0.3$ and $k = -0.45$ we have found earlier, as well as a small positive value of β_3 according to experiments, we get $\alpha_4 = -0.7$, $\beta_2 = -0.51$. The value of $\alpha_4 = -0.7$ leads to the inverse loading frequency effect on fatigue crack growth rate (see Eq. (3)) that qualitatively corresponds to the experimental data presented in [26]. The value of $\beta_2 = -0.51$ allows us to describe the natural inverse initial defect size effect on fatigue life. These exponent values prove the self-consistency of the developed model.

3.2. The effect of the loading frequency

The experimental data in [24] permit also to estimate the dependence of the fatigue limit on the loading frequency. The author used "dog-bone" specimens with length 53 mm, gage length 20 mm, gage width 5 mm, and thickness 100 µm. The tests were carried out with a loading ratio $R = 0.25$ at 0.2 Hz and 10 Hz. In this case, we theoretically expect from Eq. (8) a scaling of the type $N \propto \Delta\sigma^{\beta_1}\omega^{\beta_3}$. Hence, the fatigue limit should scale with the frequency according to the power-law relationship $\Delta\sigma_{fl} \propto \omega^{-\beta_3/\beta_1}$.

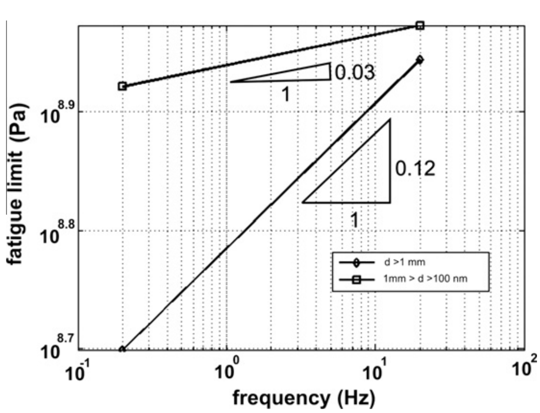

Fig. 6. Fatigues limit of Nickel vs. loading frequency for different grain sizes (based on the data [24]).

For pure Nickel we can determine the exponent β_3/β_1 as the slope of the best-fitting regression curve in the bilogarithmic diagram (see Fig. 6). For grain sizes less than 1 µm and bigger than 100 nm we find $\beta_3/\beta_1 = -0.03$, whereas for grain sizes bigger than 1 µm, we find $\beta_3/\beta_1 = -0.12$. These values were determined using stress amplitude values reported for maximum number of cycles. Negative values of β_1 lead to the common observation that β_3 is a small positive number and that frequency has a small or even negligible effect on the fatigue life (see also [34], where no frequency effects were found for Aluminum 2618).

4. Conclusions

Based on the dimensional analysis approach proposed in [11,16], a self-consistent model describing the effects of grain size and loading frequency on the fatigue properties of metals has been proposed. This model can be useful for comparison and/or prediction of the fatigue properties of ufg metals as compared to their traditional counterparts. As a main outcome, the model describes the effect of the grain size on the Paris' law and on the Wöhler curve. Besides, it establishes important relationships between the coefficients entering the Paris' law and those defining the Wöhler curve.

The theoretical predictions based on dimensional analysis confirm again that the Paris' power-law coefficient m must be larger than two, even in the case of ultra fine grained materials. We have also shown that the grain size and the loading frequency have an opposite effect upon the Wöhler curve: the fatigue life can be increased either by increasing the loading frequency or by reducing the grain size. Although the effect of loading frequency on the Wöhler curve is consistent with the corresponding effect on the Paris curve (the higher fatigue life is the result of a lower crack growth rate), this is not the case for the grain size. In fact, the smaller the grain size, the longer the fatigue life but the higher is the crack growth rate. This apparent inconsistency can be explained by incorporating the Hall–Petch relationship into the mathematical formulation, suggesting that, in fine grained materials, energy dissipation takes place primarily into the volume, rather than over a surface. Finally, the incomplete similarity exponents coming from the theory have been determined for some experimental data available in the literature.

Acknowledgements

The support provided by the Russian Foundation for Basic research (RFBR) and by the E.I.N.S.T.E.IN Consortium to the project "Long-range correlation properties of defects ensembles, criticality of multiscale damage evolution and damage tolerance of materials and constructions" (Joint Grants for Italian and Russian Researchers, 2009–2010) is gratefully acknowledged.

Appendix A

In this appendix, some comments upon the choice of the variables entering the dimensional analysis representation are made.

In the functional dependency (1), the applied stress range $\Delta\sigma$ was not considered. However, this variable does not add any information and the use of the stress-intensity factor range as a loading parameter is sufficient. To show that, let us consider the following functional dependence instead of Eq. (1):

$$\frac{da}{dN} = F(\sigma_y, \Delta K; K_{IC}, d, a, \omega, \Delta\sigma), \tag{A1}$$

where the stress range has been now suitably included. The resulting dimensionless representation would be:

$$\frac{da}{dN} = \left(\frac{\Delta K}{\sigma_y}\right)^2 \Phi\left(\frac{K_{IC}}{\Delta K}, \frac{\sigma_y^2}{K_{IC}^2} d, \frac{\sigma_y^2}{K_{IC}^2} a, \frac{\omega d^2}{\chi}, \frac{\Delta\sigma}{\sigma_y}\right) = \left(\frac{\Delta K}{\sigma_y}\right)^2 \Phi(\Pi_1, \Pi_2, \Pi_3, \Pi_4, \Pi_5), \tag{A2}$$

where a new dimensionless number Π_5 appears. However, introducing the relationship between ΔK and $\Delta\sigma$ ($\Delta K = f\Delta\sigma\sqrt{\pi a}$, where a is the crack length and f is a shape factor depending on the specimen geometry), we note that this new number is not independent of the other dimensionless numbers, but it is a function of Π_1 and Π_3:

$$\Pi_5 = \frac{\Delta\sigma}{\sigma_y} = \frac{\Delta K}{f\sqrt{\pi a}\sigma_y} = \frac{1}{f\sqrt{\pi}} \Pi_1^{-1} \Pi_3^{-1/2}. \tag{A3}$$

Therefore, Π_5 is not essential for the subsequent analysis and it can be omitted. The same reasoning applies to Eq. (5), where $\Delta\sigma$ is used instead of ΔK .

Regarding the derivation of the dimensionless representation of the Paris' law (2), the use of ΔK instead of K_{IC} to made dimensionless the variables with dimension of length, such as the crack length and the grain size, is possible. However, this alternative choice does not change the final result. If we choose ΔK instead of K_{IC}, Eq. (2) would read:

$$\frac{da}{dN} = \left(\frac{\Delta K}{\sigma_y}\right)^2 \Phi\left(\frac{K_{IC}}{\Delta K}, \frac{\sigma_y^2}{\Delta K^2} d, \frac{\sigma_y^2}{\Delta K^2} a, \frac{\omega d^2}{\chi}\right), \tag{A4}$$

and, after the assumption of incomplete similarities in the four dimensionless numbers, we would obtain the following expression:

$$\frac{da}{dN} = \Delta K^{2-\gamma_1-2\gamma_2-2\gamma_3} d^{\gamma_2+2\gamma_4} a^{\gamma_3} \left(\frac{\omega}{\chi}\right)^{\gamma_4} \frac{\Phi_1}{K_{IC}^{-\gamma_1} \sigma_y^{2(1-\gamma_2-\gamma_3)}}, \tag{A5}$$

where the new incomplete similarity exponents, γ_i, can be in principle different from the exponents α_i in Eq. (3). However, since the physical phenomenon is the same, regardless of the choice of ΔK or K_{IC}, the exponents of ΔK, d and a must be the same in both representations. A comparison between Eq. (A5) and (3) leads to $\gamma_1 = \alpha_1 - 2\gamma_2 - 2\gamma_3$, $\gamma_2 = \alpha_2$, $\gamma_3 = \alpha_3$, and $\gamma_4 = \alpha_4$. The choice of K_{IC} instead of ΔK is in any case preferable, since in this way the dimensionless numbers Π_2 and Π_3 solely depend on material properties and not on a parameter related to the applied loading. The introduction of parameters related to the loading conditions in the definition of the numbers Π_2 and Π_3 could in fact lead to difficulties in the assessment of incomplete similarity assumptions.

References

[1] Wöhler A. Versuche über die festiykeit eisenbahnwagenuchsen. Z Bauwesen 1860:10.

[2] Basquin OH. The exponential law of endurance tests. Proc ASTM 1910;10:625–30.

[3] Paris P, Gomez M, Anderson W. A rational analytic theory of fatigue. Trend Eng 1961;13:9–14.

[4] Paris P, Erdogan F. A critical analysis of crack propagation laws. J Basic Eng Trans ASME 1963;58D:528–34.

[5] Carpinteri A, Paggi M. Self-similarity and crack growth instability in the correlation between the Paris' constants. Eng Fract Mech 2007;74:1041–53.

[6] Barenblatt GI, Botvina LR. Incomplete self-similarity of fatigue in the linear range of fatigue crack growth. Fat Fract Eng Mater Struct 1980;3:193–202.

[7] Spagnoli A. Fractality in the threshold condition of fatigue crack growth: an interpretation of the Kitagawa diagram. Chaos, Solitons Fract 2004;22:589–98.

[8] Spagnoli A. Self-similarity and fractals in the Paris range of fatigue crack growth. Mech Mater 2005;37:519–29.

[9] Ciavarella M, Paggi M, Carpinteri A. One, no one, and one hundred thousand crack propagation laws: a generalized Barenblatt and Botvina dimensional analysis approach to fatigue crack growth. J Mech Phys Solids 2008;56:3416–32.

[10] Paggi M, Carpinteri A. Fractal and multifractal approaches for the analysis of crack-size dependent scaling laws in fatigue. Chaos, Solitons Fract 2009;40:1136–45.

[11] Carpinteri A, Paggi M. A unified fractal approach for the interpretation of the anomalous scaling laws in fatigue and comparison with existing models. Int J Fract 2010;161:41–52.

[12] Atzori B, Lazzarin P, Meneghetti G. Fatigue strength assessment of welded joints: from the integration of Paris' law to a synthesis based on the notch stress intensity factors of the uncracked geometries. Eng Fract Mech 2008;75:364–78.

[13] Pugno N, Ciavarella M, Cornetti P, Carpinteri A. A generalized Paris' law for fatigue crack growth. J Mech Phys Solids 2006;54:1333–49.

[14] Ciavarella M, Monno F. On the possible generalizations of the Kitagawa–Takahashi diagram and of the El Haddad equation to finite life. Int J Fatigue 2006;28:1826–37.

[15] Pugno N, Cornetti P, Carpinteri A. New unified laws in fatigue: from the Wöhler's to the Paris' regime. Eng Fract Mech 2007;74:595–601.

[16] Carpinteri A, Paggi M. A unified interpretation of the power laws in fatigue and the analytical correlations between cyclic properties of engineering materials. Int J Fatigue 2009;31:1524–31.

[17] Paggi M. A dimensional approach to fatigue in quasi-brittle materials. Frattura ed Integrità Strutturale 2009;10:43–53.

[18] Ray S, Chandra Kishen JM. Fatigue crack propagation model for plain concrete using self similarity concepts. In: Proceedings of the interquadrennial conference of the international congress on fracture (IQICF), Bangalore, India, 2008.

[19] Mughrabi H, Höppel HW. Cyclic deformation and fatigue properties of very fine-grained metals and alloys. Int J Fatigue 2010;32:1413–27.

[20] Saga J, Hayahi M, Nishio Y. Effect of grain size on fatigue crack propagation in aluminium. J Soc Mater Sci Jpn 1977;26:289–95.

[21] Pao PS, Jones HN, Cheng SF, Feng CR. Fatigue crack propagation in ultrafine grained Al–Mg alloy. Int J Fatigue 2005;27:1164–9.

[22] Cavaliere P. Fatigue properties and crack behavior of ultra-fine and nanocrystalline pure metals. Int J Fatigue 2009;31:1476–89.

[23] Hanlon T, Tabachnikova ED, Suresh S. Fatigue behavior of nanocrystalline metals and alloys. Int J Fatigue 2005;27:1147–58.

[24] Hanlon T. Grain size effect on the fatigue response of the nanocrystalline materials. Ph.D. Thesis in Material Science and Engineering at the Massachusetts University of Technology; 2004. 158p.

[25] Cavaliere P, Panella FW, Suresh S. Fatigue crack behavior of ultra fine grain pure metals via severe plastic deformation. In: Atti del 19 congresso del gruppo italiano frattura (IGF19), Milano, July 2–4, 2007. p. 9–16.

[26] Sansoz F, Ghonem H. Effects of loading frequency on fatigue crack growth mechanisms in α/β Ti microstructure with large colony size. Mater Sci Eng, A 2003;356:81–92.

[27] Kunz L, Lukas P, Svoboda M. Fatigue strength, microstructural stability and strain localization in ultrafine-grained copper. Mater Sci Eng A 2006;424:97–104.

[28] Buckingham E. Model experiments and the form of empirical equations. ASME Trans 1915;37:263–96.

[29] Barenblatt GI. Scaling phenomena in fatigue and fracture. Int J Fract 2006;138:19–35.

[30] Chan K. A scaling law for fatigue crack initiation in steels. Scripta Metal Mater 1995;32:235–40.

[31] Hall EO. The deformation and ageing of mild steel: III. Discussion of results. Proc Phys Soc Lond B 1951;64:747–53.

[32] Petch NJ. The cleavage strength of polycrystals. J Iron Steel Inst Lond 1953;173:25–8.

[33] Di Schino A, Kenny JM. Grain size dependence of the fatigue behaviour of a ultrafine-grained AISI 304 stainless steel. Mater Lett 2003;57:3182–5.

[34] Singh N, Khelawan R, Mathur GN. Effect of stress ratio and frequency on fatigue crack growth rate of 2618 aluminum alloy silicon carbide metal matrix composite. Bull Mater Sci 2007;24:169–71.