Limits to Plastic Analysis due to Size-Scale Effects on the Rotational Capacity of Reinforced Concrete Cross Sections

Mauro Corrado, Post-Doctoral Fellow; **Marco Paggi,** Asst. Professor; **Alberto Carpinteri,** Professor; Department of Structural Engineering and Geotechnics, Politecnico di Torino, Italy. Contact: mauro.corrado@polito.it

Abstract

Reinforced concrete (RC) structures are usually designed to provide a ductile response under bending loadings. To this aim, most codes of practice impose lower and upper limits to the steel ratio in order to prevent unstable crack propagation and to avoid brittle failure due to concrete crushing without steel yielding. Within these limitations, elastic analysis with moment redistribution or even plastic analysis can be adopted for RC structures. In this context, size-scale effects are usually disregarded, leading to unsafe design conditions in the case of large structures.

In the present study, the limitations of the prescriptions provided by the European and American building codes concerning the admissible plastic rotation and moment redistribution are highlighted. In particular, using a numerical algorithm based on the finite element method and on nonlinear fracture mechanics concepts recently developed by the present authors, the mechanical behaviour of the plastic hinge region of RC beams in bending is simulated. The results show that the effect of the structural dimension should be explicitly introduced in the code prescriptions for a safe structural design, by considering different design curves depending on the size-scale of the beams.

Keywords: plastic rotation; moment redistribution; size effects; code prescriptions; nonlinear analysis; nonlinear fracture mechanics.

Introduction

The development of considerable ductility in the ultimate limit state is a key issue in the design of reinforced concrete (RC) beams in bending.[1–6] The interest in ductility was formerly connected with the diffusion of plastic analysis in the design of RC structures.[2,5,6] In this context, in fact, the rotational capacity is required to allow for the bending moment redistribution in statically indeterminate structures. Besides, the ductility contributes to satisfy many other requirements, which are absolutely necessary in order to guarantee the structural safety, as, for example, to give warning of incipient collapse by the development of large deformation prior to collapse and to enable major distortions and energy dissipation during earthquakes.

Due to the complexity of the phenomenon, the first contribution to the study of the rotational capacity came from the experimental program coordinated by the "Indeterminate Structures Commission" of the Comité Européen du Béton[7,8] in the early 1960s. Over 350 tests were performed in different laboratories and countries. A statistical evaluation of the results, proposed by Siviero in 1976,[9] gave the basis for the following hyperbolic relationship between plastic rotation ϑ_{PL} and relative neutral axis depth x/d:

$$\vartheta_{\mathrm{PL}} = \frac{0{,}004}{x/d} \qquad (1)$$

where x is the distance of the neutral axis from the compressed edge at the ultimate condition and d is the effective depth of the beam cross section. This expression was assumed by the Model Code 78 to solve the problem of plastic rotation evaluation for practical purposes.

A second fundamental contribution came from the research carried out in the early 1980s at the University of Stuttgart[3]. In that study, an analytical model was developed to describe the behaviour of plastic hinges, considering that the final collapse can result either from steel rupture or from concrete crushing. The obtained results led to the formulation of the prescriptions of Model Code 90[10] and Eurocode 2.[11] The diagram provided by Eurocode 2 for assessing the admissible rotational capacity of RC beams as a function of the relative neutral axis position x/d is shown in *Fig. 1*. The dashed lines refer to high-ductility steel, while the solid ones to normal-ductility steel. In the case of a step-by-step plastic structural analysis, the designer has to verify that the rotation required for the moment redistribution is lower than the admissible one. To this aim, for a given value of x/d obtained from the application of the ultimate state analysis, *Fig. 1* can be used to determine the admissible plastic rotation as function of the concrete grade. It has to be noted that the size-scale effects on the rotational

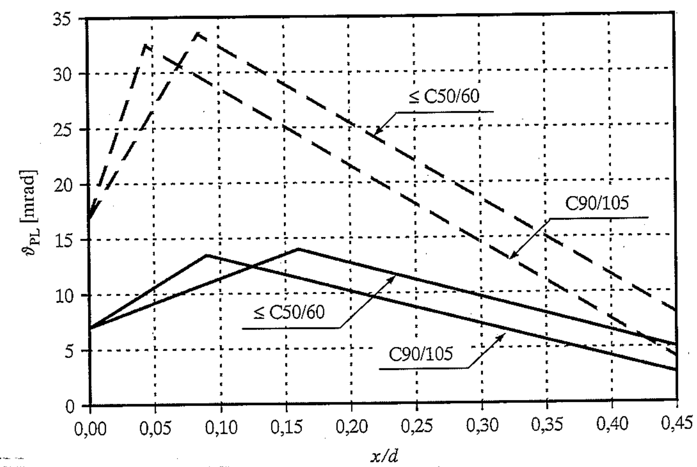

Fig. 1: Plastic rotation versus relative neutral axis position relationships by Eurocode 2[11]

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capacity of RC beams are not considered, although the dependence on the structural dimension was recognized in several experimental tests.[1,4,12,13]

A different approach is proposed in the American Concrete Institute (ACI) 318 Building Code[14] for the linear-elastic analysis with a moment redistribution procedure. In this case, the evaluation of the plastic rotation required by the plastic hinges is avoided by limiting the percentage of moment redistribution. Such a limit can be obtained by entering into the diagram in *Fig. 2* the coefficient of resistance $M_n/(f_c bd^2)$ for a given concrete compressive strength. Such curves have been selected in order to be the most conservative ones with respect to the beam slenderness and to the steel yielding strength.[15] Therefore, as a main difference with respect to the European approach, the ACI prescription introduces a safety factor for the admissible moment redistribution.

From the modelling point of view, the assessment of the available ductility is difficult to achieve because of the simultaneous presence of different nonlinear contributions: crack opening in tension, concrete crushing in compression and steel yielding or slippage. On the other hand, oversimplifications, based on the hypotheses usually assumed for the evaluation of the structural resistance, as, for example, to neglect the concrete contribution in tension and to describe the nonlinear behaviour of concrete in compression and steel in tension by means of $\sigma-\varepsilon$ constitutive laws, do not permit the modelling of all the experimentally observed effects on the ductility of RC beams in bending. In particular, it is impossible to catch the size-scale effects, because the aforementioned constitutive laws consider energy dissipation only within the volume in the nonlinear regime. In this context, significant contributions were given by the original experimental insight by van Mier[16] and by the pioneering constitutive model proposed by Hillerborg in 1990,[17] who introduced the concept of strain localization in concrete in compression. According to the approach by Hillerborg, when the ultimate compressive strength is achieved, a strain localization takes place within a characteristic length proportional to the depth of the compressed zone. This model permits one to address the issue of size effects, although the length over which the strain localization occurs is a free parameter and its value is not defined on the basis of theoretical arguments.

In the present paper, the main features of a numerical method recently developed by the present authors,[18,19] which is able to describe the nonlinear behaviour of RC members during both fracturing and crushing, are briefly outlined. In particular, it will be shown that the tensile cracking phenomenon can be described by means of the well-established *Cohesive Crack Model* (CCM), and that the crushing process can be efficiently analysed according to the *Overlapping Crack Model* (OCM), which considers a fictitious material interpenetration in the post-peak regime.[20] With the proposed algorithm implemented in the finite element method, it is possible to completely capture the moment versus rotation response under monotonic loadings, taking into account the main nonlinearities. As a result of a parametric investigation, plastic rotation versus neutral axis position curves is determined, which is found to be dependent on the structural dimension and on the steel percentage. It can also be seen that the European prescriptions are not conservative in the case of large structural sizes. These results are then used to compute the percentage of moment redistribution versus the coefficient of resistance, which can be directly compared with the ACI Building Code prescriptions. In this case, it is shown that the ACI prescriptions are in general conservative for standard beam depths and slenderness values. However, they become unconservative for very deep and/or slender beams.

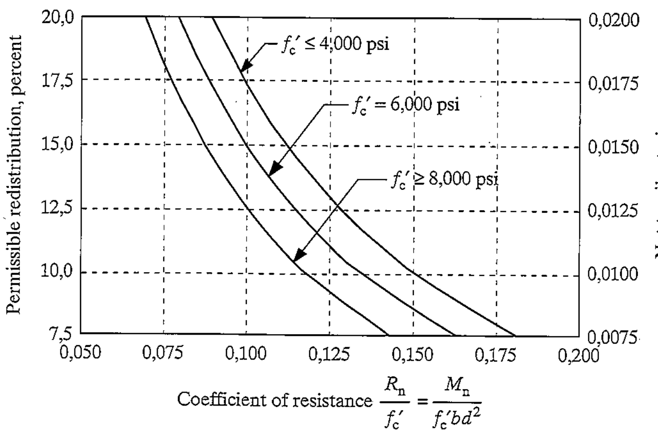

Fig. 2: Admissible moment redistribution versus coefficient of resistance diagram by ACI Building Code[14]

Numerical Approach

In this section, the numerical algorithm proposed by Carpinteri *et al.*[18,19] for the analysis of the behaviour of RC elements in bending is briefly described. This model permits the study of a portion of an RC beam subjected to a constant bending moment M. This element, having a span to depth ratio equal to unity, is representative of the zone of a beam where a plastic hinge formation takes place. It is assumed that fracturing and crushing processes are fully localized along the mid-span cross section of the element. This assumption, which is fully consistent with the physics of the crushing phenomenon, also implies that only one equivalent main tensile crack is considered. The loading process is characterized by crack propagation in tension, steel yielding and/or slippage, as well as concrete crushing in compression.

Constitutive Models

In the proposed algorithm, the behaviour of concrete under tension is described by means of the CCM,[21] which was largely used in the past to study the ductile-to-brittle transition in plain concrete beams in bending.[22,23] According to this model, the adopted constitutive law is a stress–strain linear-elastic relationship up to the achievement of the tensile strength σ_u, for the undamaged zone, and a stress–displacement relationship describing the material behaviour in the process zone. In particular, the cohesive stresses are considered to be linear decreasing functions of the crack opening w^t. The critical crack opening displacement beyond which the transferred stresses vanish is equal to $w^t_{cr} \approx 0{,}1$ mm, and the fracture energy G_F is assumed to vary from 0,05 to 0,15 N/mm depending on concrete strength and maximum aggregate diameter, according to the prescriptions given in the Model Code 90.[10]

As far as modelling of concrete crushing failure is concerned, the OCM introduced by Carpinteri *et al.*[20] is adopted. According to this approach, which has been strongly confirmed by experimental results[16,24] and was related to the pioneering work by Hillerborg,[17] the inelastic deformation in the post-peak regime is described by a fictitious interpenetration of the material, while the remaining part of the specimen undergoes an elastic unloading. As a result, a pair of constitutive laws for concrete in compression is introduced, in close analogy with the CCM: a stress–strain relationship for the undamaged material (*Fig. 3a*), and a stress–displacement (*overlapping*) relationship for the post-peak concrete crushing (*Fig. 3b*). The latter law, which is approximated by a linear softening relationship for modelling purposes,[20] describes how the stress in the damaged material decreases from its maximum value as the fictitious interpenetration increases. It is worth noting that the crushing energy G_C, which is a dissipated surface energy, is defined as the area under the post-peak softening curve in *Fig. 3b*. It can be considered as a true material property, since it is not affected by the structural size, as shown in Ref. [20] in case of plain or fibre-reinforced concretes. In case of lateral confinement exerted by stirrups, an empirical equation to calculate the crushing energy has been recently proposed by Suzuki *et al.*[25] By varying the concrete compressive strength from 20 to 90 MPa, the crushing energy ranges from 30 to 58 N/mm. The critical value for the crushing interpenetration is experimentally found to be approximately equal to 1 mm (see also Jansen and Shah[24]). This value is a decreasing function of the compressive strength, and an increasing function of concrete confinement.

As far as the behaviour of steel reinforcement is concerned, it is impossible to adopt the classical σ–ε laws, since the kinematics of the mid-span cross section of the reinforced concrete member is described by means of displacements, instead of strains. To this aim, constitutive relationships between the reinforcement reaction and the crack opening displacement are obtained by means of preliminary studies carried out on the interaction between the reinforcing bar and the surrounding concrete. In particular, the integration of the differential slips over the transfer length l_{tr} is equal to half the crack opening at the reinforcement level, whereas the integration of the bond stresses gives the reinforcement reaction. A simplified procedure of such an approach have been proposed by Ruiz *et al.*[26] Typically, the obtained relationships are characterized by an ascending branch up to steel yielding, to which corresponds the critical value of the crack opening for steel w_y. After that, the steel reaction is nearly constant.

Numerical Algorithm

A discrete form of the elastic equations governing the mechanical response of the two half-beams is herein introduced. The reinforced concrete member is considered as constituted by two symmetrical elements characterized by an elastic behaviour and connected by means of n pairs of nodes (*Fig. 4a*). In this approach, all the mechanical nonlinearities are localized in the mid-span cross section, where cohesive and overlapping stresses are replaced by equivalent nodal forces F_i by integrating the corresponding stresses over the nodal spacing. Such nodal forces depend on the nodal opening or closing displacements according to the cohesive or overlapping softening laws previously introduced.

With reference to *Fig. 4a*, the horizontal forces F_i acting at the ith node along the mid-span cross section can be computed as follows:

$$\{F\}=[K_w]\{w\}+\{K_M\}M \tag{2}$$

where $\{F\}$ is the vector of nodal forces, $[K_w]$ is the matrix of the coefficients of influence for the nodal displacements, $\{w\}$ is the vector of nodal displacements and $\{K_M\}$ is the vector of the coefficients of influence for the applied moment M.

Equation (2) constitutes a linear algebraic system of n equations and $(2n + 1)$ unknowns, $\{F\}$, $\{w\}$ and M. With reference to the generic situation reported in *Fig. 4b*, n additional equations can be introduced by considering the constitutive laws for concrete in tension and compression and for the reinforcement in the node r (see Refs. [19, 27] for more details). The last additional equation derives from the strength criterion adopted to govern the propagation processes. At each step of the loading process, in fact, either the force in the fictitious crack tip m is set as equal to the ultimate tensile force F_u, or the force in the fictitious crushing tip p is set as equal to the ultimate compressive force F_c. It is important to note that the condition for crack propagation (corresponding to the achievement of the tensile strength at the fictitious crack tip m) does not imply that the compressive strength is reached at the corresponding overlapping crack tip p and vice versa. Hence, the driving parameter of the process is the tip, which in the considered step has reached the limit resistance. Only this tip is moved when passing to the next step. This criterion will ensure the uniqueness of the solution on the basis of physical arguments. Finally, at each step of the algorithm, it is possible to calculate the beam rotation ϑ as follows:

$$\vartheta=\{D_w\}^T\{w\}+D_M M \tag{3}$$

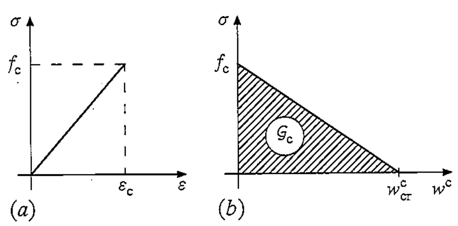

Fig. 3: Overlapping Crack Model for concrete in compression: (a) linear-elastic σ–ε law; (b) post-peak softening σ–w relationship

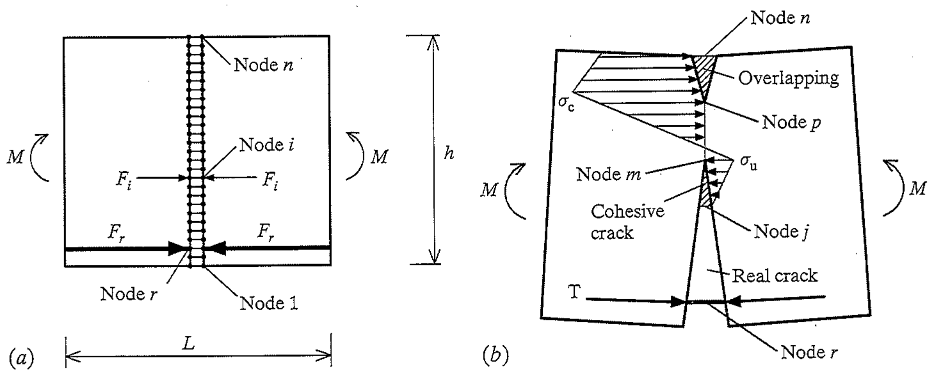

Fig. 4: Finite element nodes: (a) and force distribution with cohesive crack in tension and crushing in compression (b) along the mid-span cross-section

where $\{D_w\}$ is the vector of the coefficients of influence for the nodal displacements and D_M is the coefficient of influence for the applied moment. It is worth noting that Eqs. (2) and (3) permit the analysis of the fracturing and crushing processes of the mid-span cross section, taking into account the elastic behaviour of the reinforced concrete member. To this aim, all the coefficients are computed a priori using a finite element analysis.

Limits of the Prescriptions of the Eurocode 2 Part 1-1 and the New Proposal

In this section, the results of a detailed parametric study carried out to analyse the effect of each parameter to the overall response, with particular regard to the plastic rotational capacity, are presented. With reference to the typical moment versus rotation curve obtained by the application of the proposed algorithm shown in *Fig. 5*, the plastic component of the total rotation can be obtained as the difference between the ultimate rotation and the rotation corresponding to the reinforcement yielding. According to the definition proposed by Hillerborg[17] and Pecce,[28] the ultimate rotation is the rotation beyond which the moment starts descending rapidly.

It is important to note that, since the mechanical nonlinearities are localized along the mid-span cross section, the length of the RC element L influences only the elastic part of the moment versus rotation response. Such an effect is linearly proportional to the applied bending moment. Therefore, its contribution decreases by reducing the tensile reinforcement ratio. On the contrary, it does not influence the nonlinear contribution to the overall beam behaviour; that is, the plastic rotation and the moment redistribution capabilities are not affected by L. This hypothesis has been confirmed by the numerical/experimental comparisons proposed by Carpinteri *et al.*[27,29] in case of beams with different sizes and slenderness.

The results of the parametric analysis can be summarized in a plastic rotation versus relative neutral axis position, x/d, diagram. This is also consistent with the practical prescriptions of the Eurocode 2.[11] The numerical results referred to different beam depths are compared in *Fig. 6* with the curve provided by Eurocode 2 for high-ductility steel and concrete compressive strength less than or equal to 50 MPa. Beams with a depth equal to 0,2 m have a rotational capacity greater than that suggested by the code. On the other hand, by increasing the beam depth up to 0,8 m, the rotations provided by the code appear to be unconservative. It is worth noting that the numerical results for h = 0,4 m are, however, in good agreement with the curve provided by the code, which represents the 5% fractile of the plastic rotations of beams or slabs with depth of about 0,3 m (see Eligehausen *et al.*[30] for more details). In order to improve the code provisions, the effect of the structural dimension should be explicitly taken into account by considering different design curves such as, for instance, those proposed in *Fig. 6*.

The effect of shear cracks on the ductility of plastic hinges (see the conclusive studies by Bachmann[31] and Dilger[32]) is not explicitly considered in the present method since the maximum plastic rotation is not limited by the behaviour of the tensile side. In the existing conventional approaches, where the rotation is obtained by the integration of the axial deformations over a length corresponding to the extension of the region where steel yielding takes place, the shear effect is taken into account by increasing such a plastic length. In the proposed model, where all the nonlinearities are localized in a single cross section, such an effect can be accounted for by increasing the maximum crack opening displacement corresponding to the achievement of a limit value imposed on the steel tensile strain or to the steel bar rupture. In this manner, the equivalence between the single tensile crack that propagates at the mid-span of the RC element and the several cracks developing in the plastic hinge zone will be guaranteed. However, in this respect, it is important to remark that no upper limits have been imposed to the value of the crack opening in the present computations, except for very low steel percentages where, on the other hand, flexural failure takes place without a significant development of shear cracks. It has been found that the rotational capacity is fully limited by concrete crushing, which determines a decrease in the resistant moment at the end of the plastic plateau, and not by the constitutive behaviour in tension. Therefore, the obtained plastic rotations have to be considered as the maximum rotations allowed by the behaviour of the compression side, which is not influenced by shear cracks.

Limits of the Prescriptions of the ACI Building Code and the New Proposal

In this section, the existence of limits to the prescriptions of the ACI Building Code due to size-scale effects is investigated. To this aim, the allowable degree of moment redistribution is determined from the available rotation capacity of plastic hinges necessary

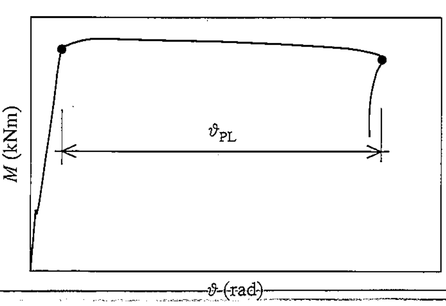

Fig. 5: Definition of plastic rotation

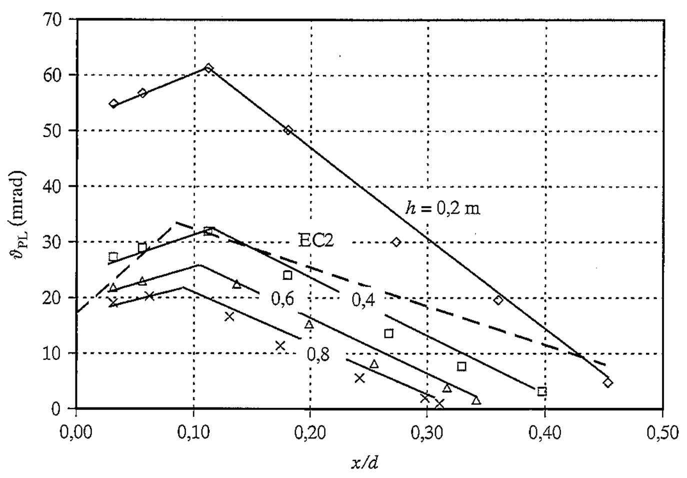

Fig. 6: Predicted plastic rotation for different beam depths (solid lines) compared to the Eurocode 2 prescription (dashed line)

to develop a collapse mechanism in a statically indeterminate RC member. Two limit conditions from the point of view of the structural system are considered: a continuous beam over three supports, and a fixed-end two-span continuous beam. In both cases, a uniformly distributed load q is applied. When the top reinforcement amount is equal to the bottom reinforcement amount, for example, the stiffness for negative moments $(EI)^-$ equals that for positive moments $(EI)^+$, and the elastic bending moment acting at the central support is given by:

$$M = kqL^2 \qquad (4)$$

where k is equal to 0,125 for the continuous beam over three supports and 0,083 for the fixed-end two-span continuous beam. It is worth noting that, strictly speaking, such values for the parameter k are no longer valid when $(EI)^-$ differs from $(EI)^+$. In these cases, in fact, the indeterminate bending moment is influenced by the variation of the flexural stiffness along the beam span. When the ultimate resistant moment M_{PL} is reached, a plastic hinge starts to develop by the central support. At this point, each span is able to bear a load increment Δq, according to the scheme of simply supported beam ($M = \Delta qL^2/8$). From the kinematic point of view, and ignoring the resistant moment capacity of the mid-span cross section, Δq is limited by the rotational capacity of the plastic hinge according to the following expression:

$$\Delta q = \frac{24EI^*}{L^3}\frac{\vartheta_{PL}}{2} \qquad (5)$$

where L is the beam span, E is the concrete elastic modulus, and I^* is a moment of inertia which takes into account the variation of the flexural stiffness along the beam span according to the study performed by Cosenza *et al.*[33] The admissible rotation ϑ_{PL} obtained from the diagrams in *Fig. 6* as a function of x/d and of the beam depth is divided by 2 because of the symmetry of the problem. In this way, it is possible to evaluate the moment redistribution factor as follows:

$$M_{\text{red},\%} = \left(1 - \frac{M_{PL}}{M_{EL}}\right)100 \qquad (6)$$

where M_{EL} is the elastic bending moment at the central support due to the effect of the ultimate load carried out by the beam after redistribution, $(q + \Delta q)$. It is evaluated by applying Eq. (4) in the hypothesis of uncracked cross sections.

The admissible redistribution is shown in *Fig. 7* as a function of the coefficient of resistance defined by ACI Building Code for a concrete compressive strength $f_c = 40$ MPa, a slenderness $L/d = 30$ (d is the effective beam depth) and different beam depths. In this diagram, only the fixed-end two-span continuous beam is analysed. Consistently with the size-scale effects on the rotation capacity evidenced in *Fig. 6*, the allowable moment redistribution is now a decreasing function of the beam depth. It is worth noting that the curve provided by the ACI Building Code for $f_c = 41$ MPa (6000 psi) becomes unconservative for beam depths higher than approximately 1 m.

The effect of slenderness is analysed in *Fig. 8* for $f_c = 40$ MPa, $h = 2$ m and fixed-end two-span continuous beam.

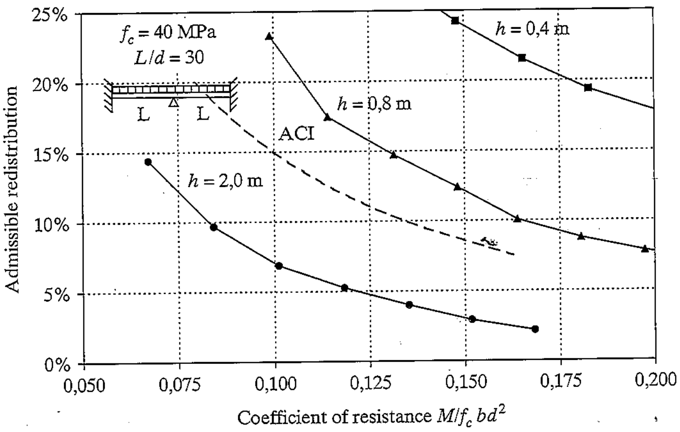

Fig. 7: Predicted admissible redistribution versus coefficient of resistance for different beam depths (solid lines) compared to the ACI prescription (dashed line)

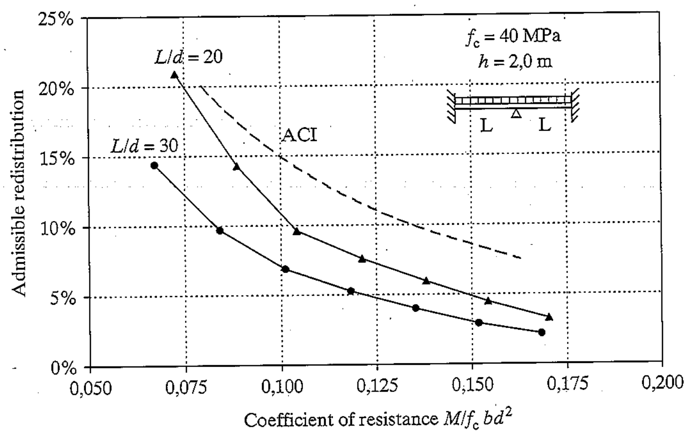

Fig. 8: Predicted admissible redistribution versus coefficient of resistance for different beam slendernesses (solid lines) compared to the ACI prescription (dashed line)

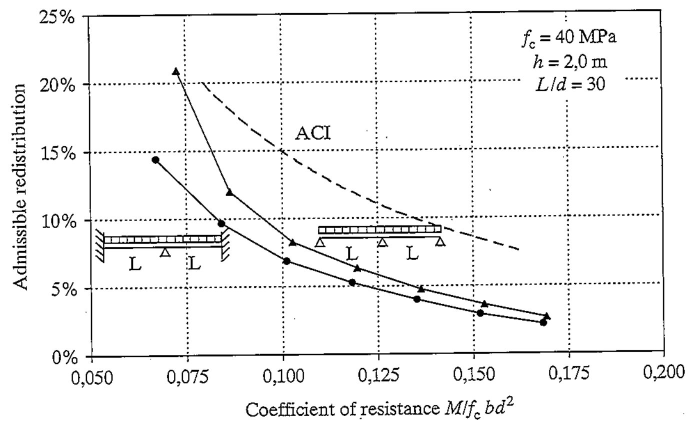

Fig. 9: Predicted admissible redistribution versus coefficient of resistance for different structural systems (solid lines) compared to the ACI prescription (dashed line)

The higher the slenderness, the lower the moment redistribution, as already pointed out in Ref. [6]. For L/d = 20 and L/d = 30, the ACI prescription is unconservative. Finally, the effect of the constraints is investigated in *Fig. 9* for f_c = 40 MPa, h = 2 m and L/d = 30. The redistribution capacity of a fixed-end two-span continuous beam is slightly lower than that of a continuous beam over three supports, as was also found in Ref. [6].

Conclusions

In the present paper, a numerical method able to describe the nonlinear behaviour of RC members during both fracturing and crushing has been presented. With the proposed algorithm, which is based on nonlinear fracture mechanics concepts, it is possible to completely capture the moment versus rotation response under monotonic loadings, taking into account the main nonlinearities. In particular, the behaviour of plastic hinges has been analysed to highlight the limits of the prescriptions of the Eurocode 2 and the ACI Building Code and to provide new easy-to-use design diagrams.

As regards Eurocode 2, the numerical results summarized in *Fig. 6* show that the plastic rotation of RC beams is not dependent only on the neutral axis position. This assumption, in fact, leads to unconservative predictions for deep beams. In order to improve the code provisions, the effect of the structural dimension should be explicitly introduced by considering different design curves as, for instance, those proposed in *Fig. 6*.

The allowable moment redistribution versus coefficient of resistance has also been computed and compared with the ACI Building Code prescriptions. Such prescriptions are, in general, conservative for standard beam depth and slenderness, because of the fact that they involve a safety factor with respect to the effects of slenderness and steel yielding strength.[15] However, the results in *Figs. 7 and 8* show that they become unconservative for very deep and/or slender beams. Also in this case, the effect of the structural dimension should be taken into account by considering different design curves, as those proposed in *Figs. 7–9*.

References

[1] Corley GW. Rotational capacity of reinforced concrete beams. *J. Struct. Div.* 1966; **92**: 121–146.

[2] Macchi G. Limit-states design of statically indeterminate structures composed of linear members. *Costruzioni in Cemento Armato, Studi e Rendiconti* 1969; **6**: 151–191.

[3] Eligehausen R, Langer P. Rotation capacity of plastic hinges and allowable moment redistribution. *CEB Bull. d'Information* 1987; **175**: I7.9–I7.27.

[4] Bigaj AJ, Walraven JC. Size effect on rotational capacity of plastic hinges in reinforced concrete beams. *CEB Bull. d'Information* 1993; **218**: 7–23.

[5] Macchi G. Elastic distribution of moments on continuous beams. In *International Symposium on the Flexural Mechanics of Reinforced Concrete*, ASCE, ACI: Miami, 1964.

[6] Comité Euro-International du Béton. *Bull. d'Information* 1998; **242**.

[7] Comité Euro-International du Béton. *Bull. d'Information* 1961; **30**.

[8] Baker ALL, Amakarone AMN. Inelastic hyperstatic frame analysis. *Proceedings of Conference on Flexural Mechanics of Reinforced Concrete*, Special Publication SP12. American Concrete Institute, FARMINGTON HILLS, MI, 1967; 85-142.

[9] Siviero E. Rotation capacity of monodimensional members in structural concrete. *CEB Bull. d'Information* 1974; **105**: 206–222.

[10] Comité Euro-International du Béton. *CEB-FIP Model Code 1990*. Thomas Telford Ltd: Lausanne, Bulletin No. 213/214, 1993.

[11] CEN TC/250 2004. *Eurocode 2: Design of Concrete Structures, Part 1-1: General Rules and Rules for Buildings*. Brussels, par. 5.6.

[12] Mattock AH. Rotational capacity of hinging regions in reinforced concrete beams. *Flexural Mechanics of Reinforced Concrete, ASCE–ACI International Symposium, Flexural Mechanics of Reinforced Concrete*, ACI SP-12, 1965; 143–181.

[13] Bosco C, Debernardi PG. Influence of some basic parameters on the plastic rotation of reinforced concrete elements. *CEB Bull. d'Information* 1993; **218**: 25–44.

[14] ACI-318. *Building Code Requirements for Reinforced Concrete*, Detroit, 2005.

[15] Mast RF. Unified design provisions for reinforced and prestressed concrete flexural and compression members. *ACI Struct. J.* 1992; **89**(2): 185–199.

[16] van Mier JGM. *Strain-softening of Concrete under Multiaxial Loading Conditions*. PhD Thesis, Eindhoven, University of Technology, 1984.

[17] Hillerborg A. Fracture mechanics concepts applied to moment capacity and rotational capacity of reinforced concrete beams. *Eng. Fract. Mech.* 1990; **35**: 233–240.

[18] Carpinteri A, Corrado M, Paggi M, Mancini G. Cohesive versus overlapping crack model for a size effect analysis of RC elements in bending. In *Proceedings of the 6th International FraMCoS Conference*, vol. 2. Taylor & Francis: London, 2007; 655–663.

[19] Carpinteri A, Corrado M, Paggi M, Mancini G. New model for the analysis of size-scale effects on the ductility of reinforced concrete elements in bending. *ASCE J. Eng. Mech.* 2009; **135**: 221–229.

[20] Carpinteri A, Corrado M, Paggi M. An analytical model based on strain localization for the study of size-scale and slenderness effects in uniaxial compression tests. *Strain*, doi: 10.1111/j.1475-1305.2009.00715.x

[21] Hillerborg A, Modeer M, Petersson PE. Analysis of crack formation and crack growth in concrete by means of fracture mechanics and finite elements. *Cement Concrete Res.* 1976; **6**: 773–782.

[22] Carpinteri A. Interpretation of the Griffith instability as a bifurcation of the global equilibrium. In *Application of Fracture Mechanics to Cementitious Composites*, Martinus Nijhoff Publishers: Dordrecht, 1985; 287–316.

[23] Carpinteri A. Size effects on strength, toughness, and ductility. *ASCE J. Eng. Mech.* 1989; **115**(7): 1375–1392.

[24] Jansen DC, Shah SP. Effect of length on compressive strain softening of concrete. *ASCE J. Eng. Mech.* 1997; **123**: 25–35.

[25] Suzuki M, Akiyama M, Matsuzaki H, Dang TH. Concentric loading test of RC columns with normal- and high-strength materials and averaged stress–strain model for confined concrete considering compressive fracture energy. In *Proceedings of the 2nd Fib Congress, Naples, Italy, June 5–8*, 2006 (CD-ROM).

[26] Ruiz G, Elices M, Planas J. Size effects and bond-slip dependence of lightly reinforced concrete beams. In *Minimum Reinforcement in Concrete Members*, Carpinteri A (ed.). Elsevier Science Ltd.: Oxford, UK, 1999; 127–180.

[27] Carpinteri A, Corrado M, Paggi M. An integrated cohesive/overlapping crack model for the analysis of the nonlinear behaviour of RC beams in bending. *Int. J. Fract.* 2010; **161**: 161–173.

[28] Pecce M. Experimental evaluation of rotation capacity of HPC beams. *CEB Bull. d'Information* 1997; **242**: 197–210.

[29] Carpinteri A, Corrado M, Paggi M, Mancini G. A numerical approach to modelling size effects on the flexural ductility of RC beams. *RILEM Mater. Struct.* 2009; **42**: 1353–1367.

[30] Eligehausen R, Fabritius E, Li L, Zhao R. An analysis of rotation capacity tests. *CEB Bull. d'Information* 1993; **218**: 251–273.

[31] Bachmann H. Influence of shear and bond on rotational capacity of reinforced concrete beams. *IABSE, Zurich* 1970; 11–28.

[32] Dilger W. Veränderlichkeit der Biege- und Schubtragfähigkeit bei Stahlbetontrgwerken und ihr Einfluß auf Schnittkraftverteilung und Traglast bei statisch unbestimmter Lagerung. *Deutscher Ausschuss fuer Stahlbeton* 1966; **179**: 101.

[33] Cosenza E, Greco C, Pecce M. Nonlinear design of reinforced concrete continuous beams. *Struct. Eng. Int.* 1991; **1/91**: 19–27.