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A Mathematical Analogy and a Unified Asymptotic Formulation for Singular Elastic and Electromagnetic Fields at Multimaterial Wedges

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Abstract In the present contribution, the mathematical analogy existing between the singular stress field in elasticity due to antiplane loading and the singular electromagnetic fields in electromagnetism is derived with reference to the problem of isotropic multimaterial wedges. These configurations, where dissimilar sectors converge to the same vertex, are commonly observed in composite materials and may lead to singularities. The proposed analogy permits to extend several elastic solutions for the power of the stress-singularity already available in the elasticity literature to the analogous electromagnetic problems and viceversa. Finally, electromagnetic structures that cannot be treated according to the proposed analogy, such as those related to bi-isotropic multimaterial wedges, are discussed.

Keywords Singularities · Re-entrant corners · Antiplane loading · Linear Elasticity · Electromagnetic fields

Mathematics Subject Classification (2000) 74B05 · 78A25

1 Introduction

Interfaces between two materials are defined as bounding surfaces where a discontinuity of some kind occurs. In general, the interface is a surface through which material characteristics, such as concentration of an element, crystal structure, elastic properties, density, as well as dielectric permittivity and magnetic permeability, change abruptly from one side to

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another. This mismatch in the material properties is the reason for the occurrence of singularities.

Singular stress states can exist in several boundary value problems of linear elasticity where different materials are present (see [1–4] for a wide overview). In this context, the problems of multimaterial wedges or junctions have received a great attention from the scientific community, since they are very commonly observed in composite materials. From the terminology point of view, multimaterial wedges correspond to the situation where two or more different elastic wedges are joined together with a total wedge angle less than 2π. On the contrary, multimaterial junctions imply that the total wedge angle formed by the material regions equals 2π, i.e., the whole plane is occupied by the materials without any voids.

In linear elasticity, the problem of bi-material wedges subjected to in-plane loading was firstly analyzed by Bogy [5] and by Hein and Erdogan [6] in 1971. Bi-material junctions were addressed by Bogy and Wang [7] in the same year and the general mathematical treatment of multimaterial junctions was proposed by Theocaris [8] in 1974. Pageau et al. [9] and Carpinteri and Paggi [10] analyzed several configurations involving tri-material junctions with perfectly bonded or debonded interfaces, whereas Inoue and Koguchi [11] proposed a detailed study on tri-material wedges. In these contributions, three different mathematical techniques were used for the characterization of the singular stress field and a demonstration of their equivalence has been recently provided by Paggi and Carpinteri [4].

Most of the efforts, including those previously mentioned, have been directed to the characterization of stress-singularities for in-plane loading, where the problem is governed by a biharmonic equation. The out-of-plane loading, also referred to as antiplane shear problem, is governed by a simpler harmonic equation. In spite of that, it has received a minor attention as compared to the in-plane problem. From the historical point of view, stress-singularities due to antiplane loading were firstly addressed by Rao [12] in 1971. In his study, a general procedure for the identification of stress-singularities at the intersection of two or more interfaces in domains governed by harmonic equations was presented. Afterwards, Fenner [13] examined the Mode III loading problem of a crack meeting a bi-material interface using the eigenfunction expansion method proposed by Williams [14]. More recently, Ma and Hour [15, 16] analyzed bi-material wedges using the Mellin transform technique and Pageau et al. [17] investigated the singular stress field of bonded and debonded tri-material junctions according to the eigenfunction expansion method.

The mathematical problems characterized by biharmonic or harmonic type of equations where the stress-field is singular present several analogies with other engineering problems. Regarding singular biharmonic problems, the analysis of the stress-singularities at the vertex of a multimaterial wedge or junction has its analogous counterpart in the analysis of the Stokes flow of dissimilar immiscible fluids, as recently pointed out by Paggi and Carpinteri [4]. As far as the harmonic problems are concerned, the mathematical analogy between the steady-state heat transfer and the antiplane loading of composite regions was firstly recognized by Sinclair [18] in 1980. Very recently, Paggi and Carpinteri [4] put into evidence the analogy between antiplane loading and the St. Venant torsion of composite bars. To the knowledge of the present authors, it seems that the analogy between elasticity and electromagnetism has been overlooked. In the solution of diffraction problems, in fact, Bouwkamp [19] and Meixner [20, 21] found that the electromagnetic field vectors may become infinite at the sharp edges of a diffracting obstacle. This is the frequent case when an incident wave meets an obstacle with sharp edges like an antenna. The boundaries of the wedge are often curved but, at a distance much smaller than the radius of curvature of the edge, the latter may be considered as straight [22–24]. Here, the singular behaviour follows from the requirement that the energy density near the edge remains integrable over any finite domain, even if this domain contains singularities, as pointed out by Meixner [21] and then examined by several researchers (see, e.g., [25–28] among others). Multimaterial wedges are also encountered in electric machines and in micro-electro-mechanical-systems. They result from joining different material sectors together and the singularity comes from the mismatch of the electromagnetic properties. Several applications can be found in [24] and regard doubly salient rotor-stator configurations, thin metal microstrips printed on a substrate, as well as cylindrical antennas joined to a metallic half-plane through an insulating ring.

As it will be shown in the sequel, this mathematical problem is governed by the Helmholtz equation (see also [24] for a detailed overview). This partial differential equation admits a separable variable form solution, as for the antiplane problem in elasticity governed by the Laplace equation. Moreover, as far as the asymptotic analysis of the singular electromagnetic fields is concerned, the eigenfunction expansion method can be used as proposed by Meixner [21], in close analogy with the well-known method proposed by Williams [14] in elasticity. The order of the singularity is then determined by the imposition of the boundary conditions (BCs) near the singular point. This leads to two independent eigenproblems, one for the transverse electric (TE) field and another for the transverse magnetic (TM) one, that are mathematically analogous to the corresponding eigenproblem in antiplane elasticity, provided that stress-free or clamped boundary conditions are considered along the limiting edges of the wedge. To achieve this goal, the equations considered by Meixner [14] and widely used in the electromagnetic literature are suitably rewritten in matrix form in order to derive a formulation fully consistent with that used in elasticity.

The advantages stemming from this analogy are manifold. In particular, it possible to extend the results of the parametric analyses in elasticity available in [12, 13, 15, 16, 18] to the analogous electromagnetic problems. In fact, useful diagrams showing the variation of the power of the stress-singularity with respect to the geometrical and mechanical parameters of the joint are provided in those papers. On the other hand, the eigenvalues of the singular TE fields available in the electromagnetic literature can be profitably used in elasticity, where investigations in the case of clamped edges are quite scarce as compared to the more common stress-free boundary conditions. Moreover, the established results on singular-free configurations in electromagnetism [27] and on situations leading to logarithmic singularities in elasticity [18] can now be applied to their respective analogous problems. All of these information resulting from the asymptotic analysis are also important for numerical methods, where generalized singular finite elements are used both in elasticity [29] and in electromagnetism [30] to improve the accuracy of the solution near singular points. Finally, electromagnetic structures that cannot be treated according to the proposed analogy, such as those related to bi-isotropic multimaterial wedges [31], are deeply discussed and complete the paper.

2 Stress-Singularities in Elasticity due to Antiplane Loading

The geometry of a plane elastostatic problem consisting of $n-1$ dissimilar isotropic, homogeneous sectors of arbitrary angles perfectly bonded along their interfaces converging to the same vertex O is shown in Fig. 1. Each of the material regions is denoted by Ω_i with $i = 1, \dots, n-1$, and it is comprised between the interfaces Γ_i and Γ_{i+1}.

Out-of-plane loading due to antiplane shear (Mode III) on composite wedges can lead to stresses that can be unbounded at the junction vertex O. When out-of-plane deformations

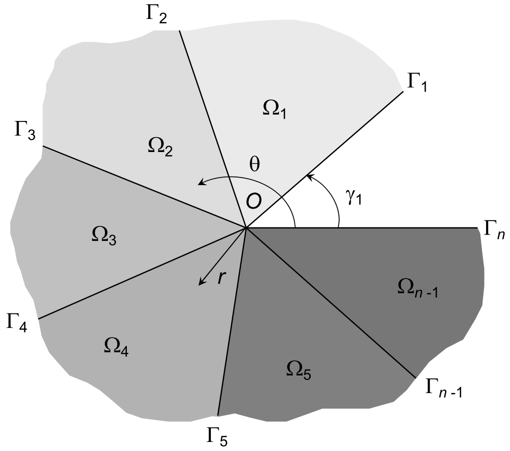


Fig. 1 Scheme of a multimaterial wedge

only exist, the following displacements in cylindrical coordinates can be considered with the origin at the vertex O:

$$u_r = 0, \tag{1a}$$

$$u_\theta = 0, \tag{1b}$$

$$u_z = u_z(r, \theta), \tag{1c}$$

where u_z is the out-of-plane displacement, which depends on r and θ. For such a system of displacements, the strain field components become

$$\varepsilon_r = \varepsilon_\theta = \varepsilon_z = \gamma_{r\theta} = 0, \tag{2a}$$

$$\gamma_{rz} = \frac{\partial u_z}{\partial r}, \tag{2b}$$

$$\gamma_{\theta z} = \frac{1}{r}\frac{\partial u_z}{\partial \theta}. \tag{2c}$$

From the application of the Hooke's law, the stress field components are given by:

$$\sigma_r^i = \sigma_\theta^i = \sigma_z^i = \tau_{r\theta}^i = 0, \tag{3a}$$

$$\tau_{rz}^i = G_i \gamma_{rz}^i = G_i \frac{\partial u_z^i}{\partial r}, \tag{3b}$$

$$\tau_{\theta z}^i = G_i \gamma_{\theta z}^i = \frac{G_i}{r}\frac{\partial u_z^i}{\partial \theta}, \tag{3c}$$

where G_i is the shear modulus of the i-th material region. The equilibrium equations in absence of body forces reduce to a single relationship between the tangential stresses:

$$\frac{\partial \tau_{rz}^i}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}^i}{\partial \theta} + \frac{1}{r}\tau_{rz}^i = 0, \quad \forall (r, \theta) \in \Omega_i. \tag{4}$$

Introducing (3) into (4), the harmonic condition upon u_z^i is derived:

$$\frac{\partial^2 u_z^i}{\partial r^2} + \frac{1}{r}\frac{\partial u_z^i}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_z^i}{\partial \theta^2} = \nabla^2 u_z^i = 0, \quad \forall (r,\theta) \in \Omega_i. \tag{5}$$

In the framework of the eigenfunction expansion method [14], the following separable variable form for the longitudinal displacement u_z^i can be adopted ($\forall (r,\theta) \in \Omega_i$):

$$u_z^i(r,\theta) = \sum_j r^{\lambda_j} f_{i,j}(\theta,\lambda_j) + r^{\lambda_j+1} h_{i,j}(\theta,\lambda_j) + r^{\lambda_j+2} l_{i,j}(\theta,\lambda_j) + \cdots, \tag{6}$$

where λ_j are the eigenvalues of the problem, whereas $f_{i,j}$, $h_{i,j}$ and $l_{i,j}$ are the eigenfunctions. The summation with respect to the subscript j is introduced in (6), since it is possible to have more than one eigenvalue and the Superposition Principle can be applied.

Introducing (6) into (5), we find the following relationship holding for each eigenvalue λ_j:

$$\begin{aligned} &r^{\lambda_j-2}\left(\frac{\mathrm{d}^2 f_{i,j}}{\mathrm{d}\theta^2} + \lambda_j^2 f_{i,j}\right) + r^{\lambda_j-1}\left(\frac{\mathrm{d}^2 h_{i,j}}{\mathrm{d}\theta^2} + (\lambda_j+1)^2 h_{i,j}\right) \\ &\quad + r^{\lambda_j}\left(\frac{\mathrm{d}^2 l_{i,j}}{\mathrm{d}\theta^2} + (\lambda_j+2)^2 l_{i,j}\right) + \cdots = 0. \end{aligned} \tag{7}$$

Hence, the coefficients of the term in r^{λ_j-2} must vanish, implying that the eigenfunctions $f_{i,j}$ are a linear combination of trigonometric functions:

$$f_{i,j}(\theta,\lambda_j) = A_{i,j}\sin(\lambda_j\theta) + B_{i,j}\cos(\lambda_j\theta). \tag{8}$$

The eigenfunctions $f_{i,j}$ are particularly important, since they enter the first term of the series expansion (6), which is responsible for the singular behaviour of the stress field components for $r \to 0$. In fact, if we truncate the series expansion (6) to this first term and we introduce it into (3), the longitudinal displacement and the tangential stresses can be expressed in terms of the eigenfunction and its first derivative:

$$u_z^i = r^{\lambda_j} f_{i,j} = r^{\lambda_j}\left[A_{i,j}\sin(\lambda_j\theta) + B_{i,j}\cos(\lambda_j\theta)\right], \tag{9a}$$

$$\tau_{rz}^i = G_i \lambda_j r^{\lambda_j-1} f_{i,j} = G_i \lambda_j r^{\lambda_j-1}\left[A_{i,j}\sin(\lambda_j\theta) + B_{i,j}\cos(\lambda_j\theta)\right], \tag{9b}$$

$$\tau_{\theta z}^i = G_i r^{\lambda_j-1} f'_{i,j} = G_i \lambda_j r^{\lambda_j-1}\left[A_{i,j}\cos(\lambda_j\theta) - B_{i,j}\sin(\lambda_j\theta)\right]. \tag{9c}$$

The determination of the power of the stress-singularity, $\lambda_j - 1$, can be performed by imposing the boundary conditions (BCs) along the edges Γ_1 and Γ_n and at the bi-material interfaces Γ_i, with $i = 2, \ldots, n-1$. Along the edges Γ_1 and Γ_n, defined by the angles γ_1 and γ_n, we consider two possibilities: one corresponding to unrestrained stress-free edges:

$$\tau_{\theta z}^1(r,\gamma_1) = 0, \tag{10a}$$

$$\tau_{\theta z}^{n-1}(r,\gamma_n) = 0, \tag{10b}$$

and the other for fully restrained (clamped) edges:

$$u_z^1(r,\gamma_1) = 0, \tag{11a}$$

$$u_z^{n-1}(r,\gamma_n)=0. \tag{11b}$$

At the interfaces, the following continuity conditions of displacements and stresses have to be imposed ($i=1,\ldots,n-2$):

$$u_z^i(r,\gamma_{i+1})=u_z^{i+1}(r,\gamma_{i+1}), \tag{12a}$$

$$\tau_{\theta z}^i(r,\gamma_{i+1})=\tau_{\theta z}^{i+1}(r,\gamma_{i+1}). \tag{12b}$$

In this way, a set of $2n-2$ homogeneous equations in the $2n-1$ unknowns $A_{i,j}$, $B_{i,j}$, and λ_j can be symbolically written as:

$$\mathbf{\Lambda}\,\mathbf{v}=\mathbf{0}, \tag{13}$$

where $\mathbf{\Lambda}$ denotes the coefficient matrix which depends on the eigenvalue, and $\mathbf{v}$ represents the vector that collects the unknowns $A_{i,j}$ and $B_{i,j}$. More specifically, the coefficient matrix entering (13) is characterized by a sparse structure:

$$\mathbf{\Lambda}=\begin{bmatrix} \mathbf{N}_{\gamma_1}^1 & & & & & & & \\ \mathbf{M}_{\gamma_2}^1 & -\mathbf{M}_{\gamma_2}^2 & & & & & & \\ & \mathbf{M}_{\gamma_3}^2 & -\mathbf{M}_{\gamma_3}^3 & & & & & \\ & & \cdots & \cdots & & & & \\ & & & \mathbf{M}_{\gamma_i}^{i-1} & -\mathbf{M}_{\gamma_i}^i & & & \\ & & & & \cdots & \cdots & & \\ & & & & & \mathbf{M}_{\gamma_{n-1}}^{n-2} & -\mathbf{M}_{\gamma_{n-1}}^{n-1} \\ & & & & & & \mathbf{N}_{\gamma_n}^{n-1} \end{bmatrix} \tag{14}$$

where the non null elementary matrix $\mathbf{M}_\theta^i$ related to the interface BCs is given by:

$$\mathbf{M}_\theta^i=\begin{bmatrix} \sin(\lambda_j\theta) & \cos(\lambda_j\theta) \\ G_i\cos(\lambda_j\theta) & -G_i\sin(\lambda_j\theta) \end{bmatrix} \tag{15}$$

and the components of the vector $\mathbf{v}$ are:

$$\mathbf{v}=\{\mathbf{v}^1,\mathbf{v}^2,\ldots,\mathbf{v}^i,\ldots,\mathbf{v}^{n-2},\mathbf{v}^{n-1}\}, \tag{16}$$

with $\mathbf{v}^i=\{A_{i,j},B_{i,j}\}^T$. The term $\mathbf{N}_\theta^i$ depends on the BCs along the edges Γ_1 and Γ_n. For stress-free edges we have:

$$\mathbf{N}_\theta^i=\{\cos(\lambda_j\theta),-\sin(\lambda_j\theta)\}, \tag{17}$$

whereas for clamped edges it is given by

$$\mathbf{N}_\theta^i=\{\sin(\lambda_j\theta),\cos(\lambda_j\theta)\}. \tag{18}$$

A nontrivial solution of the equation system (13) exists if and only if the determinant of the coefficient matrix vanishes. This condition yields an eigenequation which has to be solved for the eigenvalues λ_j that, in the most general case, do depend on the elastic properties of the materials. Complex eigenvalues leading to logarithmic singularities can also be found and are discussed in [18].

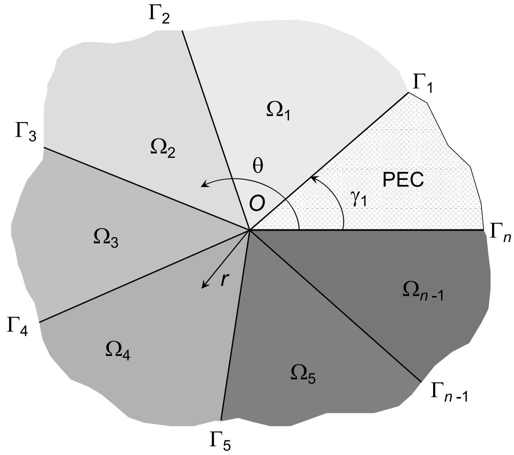


Fig. 2 Scheme of a multimaterial wedge with a PEC material

3 Singularities in the Electro-magnetic Fields

In the solution of diffraction problems, where an incident wave meets an obstacle with sharp edges like an antenna, the electromagnetic field vectors may diverge. Multimaterial wedges are also encountered in electric machines and micro-electro-mechanical-systems. They result from joining different material sectors together and the singularity comes from the mismatch of the electromagnetic properties. Several applications can be found in [24] and regard doubly salient rotor-stator configurations, thin metal microstrips printed on a substrate, as well as cylindrical antennas joined to a metallic half-plane through an insulating ring.

For the sake of generality, let us consider the multimaterial wedge shown in Fig. 2, consisting in perfectly joined material sectors. Each material is isotropic and has a dielectric permittivity ϵ_i and a magnetic permeability μ_i. The permittivity is a physical quantity that describes how an electric field affects a dielectric medium, and is related to the ability of a material to polarize in response to the field. Similarly, the magnetic permeability is related to the degree of magnetization of a material that responds linearly to an applied magnetic field (for more details on these fundamental aspects, see [32], Chap. 1).

We also admit the presence of a Perfect Electric Conductor (PEC) in the region 1 defined by the interfaces Γ_1 and Γ_n. From the physical point of view, a PEC corresponds to a material with no resistivity or, equivalently, with infinite conductivity. On its surface the tangential component of the electric field and the normal component of the magnetic field are zero. Metals such as copper or silver have such a high conductivity that are often modelled as PEC.

For periodic fields with circular frequency ω, the Maxwell's equations for each homogeneous angular domain read [21]:

$$\mathrm{j}\omega\epsilon_i \mathbf{E}^i = \nabla \times \mathbf{H}^i, \tag{19a}$$

$$-\mathrm{j}\omega\mu_i \mathbf{H}^i = \nabla \times \mathbf{E}^i, \tag{19b}$$

where $\mathbf{E}^i$ and $\mathbf{H}^i$ are, respectively, the electric and magnetic fields, and the symbol j stands for the imaginary unit.

In cylindrical coordinates r, θ, z, with the z axis perpendicular to the plane of the wedge, and considering electromagnetic fields independent of z, the Maxwell's equations reduce to the following conditions upon the components of the electric and magnetic fields:

$$\mathrm{j}\omega\epsilon_i E_r^i = \frac{1}{r}\frac{\partial H_z^i}{\partial \theta}, \tag{20a}$$

$$\mathrm{j}\omega\epsilon_i E_\theta^i = -\frac{\partial H_z^i}{\partial r}, \tag{20b}$$

$$\mathrm{j}\omega\epsilon_i E_z^i = \frac{1}{r}\frac{\partial}{\partial r}(r H_\theta^i) - \frac{1}{r}\frac{\partial H_r^i}{\partial \theta}, \tag{20c}$$

$$-\mathrm{j}\omega\mu_i H_r^i = \frac{1}{r}\frac{\partial E_z^i}{\partial \theta}, \tag{20d}$$

$$-\mathrm{j}\omega\mu_i H_\theta^i = -\frac{\partial E_z^i}{\partial r}, \tag{20e}$$

$$-\mathrm{j}\omega\mu_i H_z^i = \frac{1}{r}\frac{\partial}{\partial r}(r E_\theta^i) - \frac{1}{r}\frac{\partial E_r^i}{\partial \theta}. \tag{20f}$$

It is easy to verify that the E_z^i and H_z^i components satisfy the Helmholtz equation:

$$\frac{\partial^2 E_z^i}{\partial r^2} + \frac{1}{r}\frac{\partial E_z^i}{\partial r} + \frac{1}{r^2}\frac{\partial^2 E_z^i}{\partial \theta^2} + k_i^2 E_z^i = \nabla^2 E_z^i + k_i^2 E_z^i = 0, \tag{21a}$$

$$\frac{\partial^2 H_z^i}{\partial r^2} + \frac{1}{r}\frac{\partial H_z^i}{\partial r} + \frac{1}{r^2}\frac{\partial^2 H_z^i}{\partial \theta^2} + k_i^2 H_z^i = \nabla^2 H_z^i + k_i^2 H_z^i = 0, \tag{21b}$$

where $k_i = \omega^2\epsilon_i\mu_i$.

In close analogy with the antiplane problem in linear elasticity, the following separable form for E_z^i and H_z^i can be postulated ($\forall(r,\theta) \in \Omega_i$) [20, 21]:

$$E_z^i(r,\theta) = \sum_j r^{\lambda_j} f_{i,j}(\theta,\lambda_j) + r^{\lambda_j+1} h_{i,j}(\theta,\lambda_j) + r^{\lambda_j+2} l_{i,j}(\theta,\lambda_j) + \cdots, \tag{22a}$$

$$H_z^i(r,\theta) = \sum_j r^{\lambda_j} F_{i,j}(\theta,\lambda_j) + r^{\lambda_j+1} H_{i,j}(\theta,\lambda_j) + r^{\lambda_j+2} L_{i,j}(\theta,\lambda_j) + \cdots, \tag{22b}$$

where λ_j are the eigenvalues, whereas $f_{i,j}$, $h_{i,j}$, $l_{i,j}$, $F_{i,j}$, $H_{i,j}$ and $L_{i,j}$ are the eigenfunctions.

We can introduce (22) into (21), obtaining the following equalities:

$$\begin{aligned} r^{\lambda_j-2}\left(\frac{\mathrm{d}^2 f_{i,j}}{\mathrm{d}\theta^2} + \lambda_j^2 f_{i,j}\right) + r^{\lambda_j-1}\left(\frac{\mathrm{d}^2 h_{i,j}}{\mathrm{d}\theta^2} + (\lambda_j+1)^2 h_{i,j}\right) \\ + r^{\lambda_j}\left(\frac{\mathrm{d}^2 l_{i,j}}{\mathrm{d}\theta^2} + (\lambda_j+2)^2 l_{i,j} + k_i^2 f_{i,j}\right) + \cdots = 0, \end{aligned} \tag{23a}$$

$$\begin{aligned} r^{\lambda_j-2}\left(\frac{\mathrm{d}^2 F_{i,j}}{\mathrm{d}\theta^2} + \lambda_j^2 F_{i,j}\right) + r^{\lambda_j-1}\left(\frac{\mathrm{d}^2 H_{i,j}}{\mathrm{d}\theta^2} + (\lambda_j+1)^2 H_{i,j}\right) \\ + r^{\lambda_j}\left(\frac{\mathrm{d}^2 L_{i,j}}{\mathrm{d}\theta^2} + (\lambda_j+2)^2 L_{i,j} + k_i^2 F_{i,j}\right) + \cdots = 0. \end{aligned} \tag{23b}$$

Note that the only difference with respect to (7) derived in elasticity is represented by the coefficients multiplying the terms r^{λ_j}. These terms lead to two ordinary differential equations for the eigenfunctions $l_{i,j}$ and $L_{i,j}$ which solely contribute to the higher-order terms of the eigenfunction expansion. Hence, focusing on the leading term r^{λ_j-2}, we find that the eigenfunctions $f_{i,j}$ and $F_{i,j}$ are linear combinations of trigonometric functions, in perfect analogy with the eigenfunction $f_{i,j}$ in antiplane elasticity (see (8)):

$$f_{i,j}(\theta,\lambda_j) = A_i \sin(\lambda_j\theta) + B_i \cos(\lambda_j\theta), \tag{24a}$$

$$F_{i,j}(\theta,\lambda_j) = C_i \sin(\lambda_j\theta) + D_i \cos(\lambda_j\theta). \tag{24b}$$

These eigenfunctions are particularly important, since they enter the first terms of the series expansions (22a) and (22b) and are responsible for the singular behaviour of the components E_r^i, E_θ^i, H_r^i and H_θ^i of the electric and magnetic fields near the wedge apex. In particular, from (20), we observe that:

$$E_r^i = \frac{1}{r\mathrm{j}\omega\epsilon_i}\frac{\partial H_z^i}{\partial\theta} = \frac{1}{\mathrm{j}\omega\epsilon_i}\sum_j \lambda_j r^{\lambda_j-1}F'_{i,j} + \cdots \sim O(r^{\lambda_j-1}), \tag{25a}$$

$$E_\theta^i = -\frac{1}{\mathrm{j}\omega\epsilon_i}\frac{\partial H_z^i}{\partial r} = -\frac{1}{\mathrm{j}\omega\epsilon_i}\sum_j \lambda_j r^{\lambda_j-1}F_{i,j} + \cdots \sim O(r^{\lambda_j-1}), \tag{25b}$$

$$H_r^i = -\frac{1}{r\mathrm{j}\omega\mu_i}\frac{\partial E_z^i}{\partial\theta} = -\frac{1}{\mathrm{j}\omega\mu_i}\sum_j \lambda_j r^{\lambda_j-1}f'_{i,j} + \cdots \sim O(r^{\lambda_j-1}), \tag{25c}$$

$$H_\theta^i = \frac{1}{\mathrm{j}\omega\mu_i}\frac{\partial E_z^i}{\partial r} = \frac{1}{\mathrm{j}\omega\mu_i}\sum_j \lambda_j r^{\lambda_j-1}f'_{i,j} + \cdots \sim O(r^{\lambda_j-1}). \tag{25d}$$

Hence, $E_z^i \sim O(r^{\lambda_j})$ and $H_z^i \sim O(r^{\lambda_j})$ are the analogous counterparts of u_z^i and remain finite for $r \to 0$. Moreover, the radial components of the electric and magnetic fields, E_r^i and H_r^i, are analogous to $\tau_{\theta z}^i$ and the circumferential components, E_θ^i and H_θ^i, are analogous to τ_{rz}^i. More specifically, we have $E_r^i = \tau_{\theta z}^i/(\mathrm{j}\omega\epsilon_i G_i)$, $H_r^i = -\tau_{\theta z}^i/(\mathrm{j}\omega\mu_i G_i)$, $E_\theta^i = -\tau_{rz}^i/(\mathrm{j}\omega\epsilon_i G_i)$ and $H_\theta^i = \tau_{rz}^i/(\mathrm{j}\omega\mu_i G_i)$. All of these components diverge when $r \to 0$ with a power-law singularity of order $-1 < (\lambda_j - 1) < 0$.

As far as the BCs are concerned, a schematic representation is shown in Fig. 3 ($n = 4$). More specifically, along the PEC edges Γ_1 and Γ_n the tangential components of the electric field vanish, hence

$$E_z^1(r,\gamma_1) = 0, \tag{26a}$$

$$E_z^{n-1}(r,\gamma_n) = 0, \tag{26b}$$

$$E_r^1(r,\gamma_1) = 0, \tag{26c}$$

$$E_r^{n-1}(r,\gamma_n) = 0. \tag{26d}$$

On the PEC surface also $H_\theta = 0$, but this condition needs not be enforced, since it is a consequence of the previous ones. Along each bi-material interface ($i = 1, \ldots, n-2$), the tangential components of the electric and magnetic fields are continuous, i.e.

$$E_z^i(r,\gamma_{i+1}) = E_z^{i+1}(r,\gamma_{i+1}), \tag{27a}$$

$$E_r^i(r,\gamma_{i+1}) = E_r^{i+1}(r,\gamma_{i+1}), \tag{27b}$$

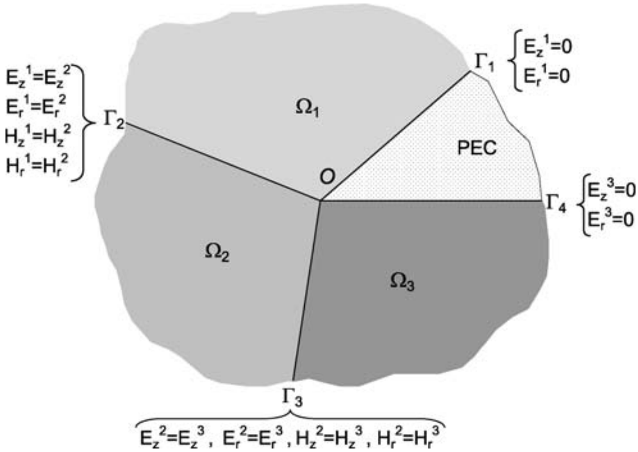


Fig. 3 Boundary conditions for a multimaterial wedge with a PEC material

$$H_z^i(r,\gamma_{i+1}) = H_z^{i+1}(r,\gamma_{i+1}), \tag{27c}$$

$$H_r^i(r,\gamma_{i+1}) = H_r^{i+1}(r,\gamma_{i+1}). \tag{27d}$$

On the contrary the field normal components are discontinuous. Indeed, (20b) and (20e) show that the continuity of E_z^i and H_z^i causes the continuity of $\epsilon_i E_\theta^i$ and $\mu_i H_\theta^i$. Therefore, the discontinuity of the material properties leads to discontinuous E_θ^i and H_θ^i. Note that the continuity of E_z and H_r implies the continuity of the eigenfunction f and of its first derivative. Similarly, the continuity of H_z and E_r leads to the continuity of the eigenfunction F and of its first derivative.

Using (25), the BCs (26) become:

$$E_z^1(r,\gamma_1) = 0, \tag{28a}$$

$$E_z^{n-1}(r,\gamma_n) = 0, \tag{28b}$$

$$\frac{\partial H_z^1}{\partial \theta}(r,\gamma_1) = 0, \tag{28c}$$

$$\frac{\partial H_z^{n-1}}{\partial \theta}(r,\gamma_n) = 0, \tag{28d}$$

whereas those defined by (27) become ($i = 1, \ldots, n-2$):

$$E_z^i(r,\gamma_{i+1}) = E_z^{i+1}(r,\gamma_{i+1}), \tag{29a}$$

$$\frac{1}{\epsilon_i}\frac{\partial H_z^i}{\partial \theta}(r,\gamma_{i+1}) = \frac{1}{\epsilon_{i+1}}\frac{\partial H_z^{i+1}}{\partial \theta}(r,\gamma_{i+1}), \tag{29b}$$

$$H_z^i(r,\gamma_{i+1}) = H_z^{i+1}(r,\gamma_{i+1}), \tag{29c}$$

$$\frac{1}{\mu_i}\frac{\partial E_z^i}{\partial \theta}(r,\gamma_{i+1}) = \frac{1}{\mu_{i+1}}\frac{\partial E_z^{i+1}}{\partial \theta}(r,\gamma_{i+1}). \tag{29d}$$

It is interesting to note that (21), (28) and (29) can be separated into two independent sets of equations, one involving only H_z and another involving only E_z (see Figs. 4(a) and (b) for the case $n = 4$). Hence, the electromagnetic field for this problem can be decomposed

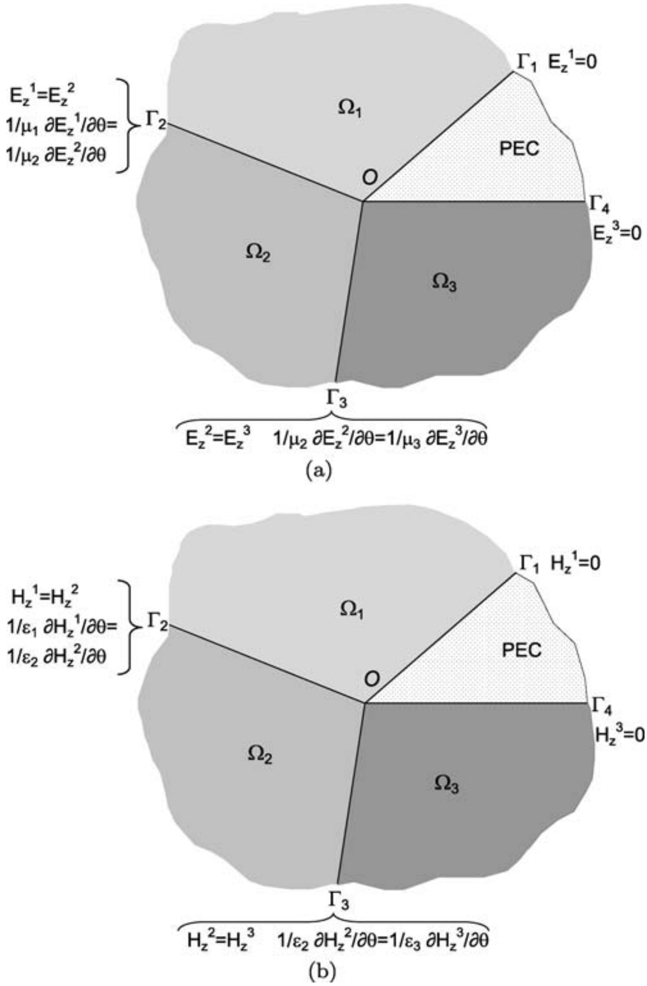


Fig. 4 Boundary conditions for a multimaterial wedge with a PEC material ($n = 4$): (**a**) TM case, BCs for E_z and (**b**) TE case, BCs for H_z

into two distinct independently evolving fields, the so-called Transverse Electric (TE) and Transverse Magnetic (TM) fields, respectively. In particular, the TE (resp. TM) field has vanishing electric (resp. magnetic) but nonzero magnetic (resp. electric) field parallel to the cylinder axis z (see [32], Chap. 5, for fundamentals on TE and TM fields).

Considering the series expansion for E_z and H_z truncated at the first term, along with the expressions for the eigenfunctions $f_{i,j}$ and $F_{i,j}$, the boundary value problem consists of two sets of $2n-2$ equations in $2n-1$ unknowns, one for E_z and another for H_z. The former equation set (TM case) involves the coefficients $A_{i,j}$, $B_{i,j}$ and λ_j and can be symbolically written as:

$$\mathbf{\Lambda}\,\mathbf{v} = \mathbf{0}, \tag{30}$$

where $\mathbf{\Lambda}$ denotes the coefficient matrix which depends on the eigenvalue and $\mathbf{v}$ represents the vector which collects the unknowns $A_{i,j}$ and $B_{i,j}$. The coefficient matrix in (30) has exactly the same structure as that for the elasticity problem in (13), provided that we consider $\mathbf{N}_\theta^i = \{\sin(\lambda_j\theta), \cos(\lambda_j\theta)\}$ and we set $G_i = 1/\mu_i$.

The latter equation set (TE case) involves the coefficients $C_{i,j}$, $D_{i,j}$ and λ_j and can be symbolically written as:

$$\boldsymbol{\Lambda}\,\mathbf{w} = \mathbf{0}, \tag{31}$$

where $\boldsymbol{\Lambda}$ is the coefficient matrix which depends on the eigenvalue and $\mathbf{w}$ represents the vector which collects the unknowns $C_{i,j}$ and $D_{i,j}$. Again, the coefficient matrix in (31) has exactly the same structure as that for the elasticity problem in (13), provided that we consider $\mathbf{N}_\theta^i = \{\cos(\lambda_j\theta), -\sin(\lambda_j\theta)\}$ and we set $G_i = 1/\epsilon_i$.

For the existence of nontrivial solutions, the determinants of the coefficient matrices must vanish, yielding two eigenequations that, for given values of ϵ_i and μ_i, determine the eigenvalues λ_j^{TE} and λ_j^{TM}. Hence, this proves that the analysis of the singularities of the electro-magnetic field is mathematically analogous to that for the elastic field due to antiplane loading. For a given multimaterial wedge problem, in elasticity we can distinguish between singularities due to either stress-free or clamped edges, depending on the BCs specified along the edges defined by the interfaces Γ_1 and Γ_n. In electromagnetism, these BCs are both present when a multimaterial wedge includes a PEC material. In this instance, the singularities related to the homogeneous equation system (30) (TM) correspond to those obtained from the analogous elastic problem with $G_i = 1/\mu_i$ and with clamped edges Γ_1 and Γ_n. On the other hand, the singularities related to the homogeneous equation system (31) (TE) correspond to those obtained from the analogous elastic problem with $G_i = 1/\epsilon_i$ and with stress-free edges Γ_1 and Γ_n.

A notable limit case is represented by a PEC embedded into a single homogeneous material, say Ω_1 (see Fig. 1). In this case, the TE and TM fields have the same singularity, whose power is independent of the material properties of Ω_1:

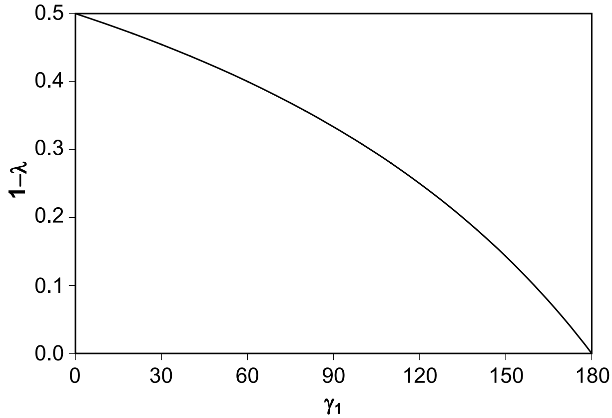
$$\lambda^{TE} = \lambda^{TM} = m\frac{\pi}{2\pi - \gamma_1}, \tag{32}$$

where m is a natural number. Considering the minimum eigenvalues ($m = 1$), the power of the electromagnetic-singularity, $(1 - \lambda)$, is shown in Fig. 5 as a function of the angle γ_1. The minimum eigenvalue is equal to $1/2$ for $\gamma_1 = 0$. In elasticity, this situation corresponds to a crack (when stress-free BCs are imposed) or to a rigid line inclusion or anti-crack (when clamped BCs are imposed). For higher values of γ_1, $(1 - \lambda)$ diminishes and vanishes for a half-plane ($\gamma_1 = \pi$). For $\gamma_1 > \pi$, the electromagnetic fields are no longer singular.

4 Electromagnetic Structures that Cannot be Treated According to the Proposed Analogy

In physics and engineering, bi-isotropic media constitute a class of materials having special optical properties that can twist the polarization of light in either refraction or transmission. Such a twist effect is based on the chirality and non-reciprocity of the structure of the material and is responsible for an unusual interaction between the electric and magnetic fields of an incident electromagnetic wave. Such advanced materials were formalized for the first time in 1948 by Tellegen [33] and then received attention from the scientific community since they offer novel promising applications in microwave technology and radio engineering.

Fig. 5 The power of the electromagnetic singularity as a function of the wedge angle γ_1 of the PEC



Bi-isotropic materials are characterized by the following constitutive relations:

$$\mathbf{D}^i = \epsilon_i \mathbf{E}^i + \xi_i \mathbf{H}^i, \tag{33a}$$

$$\mathbf{B}^i = \zeta_i \mathbf{E}^i + \mu_i \mathbf{H}^i, \tag{33b}$$

where $\mathbf{D}^i$ is the electric displacement field, $\mathbf{B}^i$ is the magnetic induction, $\mathbf{E}^i$ is the electric field and $\mathbf{H}^i$ is the magnetic field. The constants ϵ_i and μ_i are, respectively, the classical dielectric permittivity and magnetic permeability already introduced in the previous section. The novelty here is represented by the presence of the coupling constants ξ_i and ζ_i. Particular cases of bi-isotropicity are represented by the Pasteur and the Tellegen media. In the former case, $\zeta_i = -\xi_i$, whereas in the latter we have $\zeta_i = \xi_i$.

For bi-isotropic materials, the Maxwell's equations (19) become:

$$\mathrm{j}\omega\left(\epsilon_i \mathbf{E}^i + \xi_i \mathbf{H}^i\right) = \nabla \times \mathbf{H}^i, \tag{34a}$$

$$-\mathrm{j}\omega\left(\mu_i \mathbf{H}^i + \zeta_i \mathbf{E}^i\right) = \nabla \times \mathbf{E}^i. \tag{34b}$$

Hence, (20a)–(20f) become:

$$\mathrm{j}\omega\left(\epsilon_i E_r^i + \xi_i H_r^i\right) = \frac{1}{r}\frac{\partial H_z^i}{\partial \theta}, \tag{35a}$$

$$\mathrm{j}\omega\left(\epsilon_i E_\theta^i + \xi_i H_\theta^i\right) = -\frac{\partial H_z^i}{\partial r}, \tag{35b}$$

$$\mathrm{j}\omega\left(\epsilon_i E_z^i + \xi_i H_z^i\right) = \frac{1}{r}\frac{\partial}{\partial r}(r H_\theta^i) - \frac{1}{r}\frac{\partial H_r^i}{\partial \theta}, \tag{35c}$$

$$-\mathrm{j}\omega\left(\zeta_i E_r^i + \mu_i H_r^i\right) = \frac{1}{r}\frac{\partial E_z^i}{\partial \theta}, \tag{35d}$$

$$-\mathrm{j}\omega\left(\zeta_i E_\theta^i + \mu_i H_\theta^i\right) = -\frac{\partial E_z^i}{\partial r}, \tag{35e}$$

$$-\mathrm{j}\omega\left(\zeta_i E_z^i + \mu_i H_z^i\right) = \frac{1}{r}\frac{\partial}{\partial r}(r E_\theta^i) - \frac{1}{r}\frac{\partial E_r^i}{\partial \theta}. \tag{35f}$$

In this case, after some manipulation of (35), we find that E_z^i and H_z^i do not satisfy the Helmholtz equation, but we have:

$$\nabla^2 E_z^i + \omega^2 \left[(\epsilon_i \mu_i - \zeta_i^2) E_z^i + \mu_i (\xi_i - \zeta_i) H_z^i \right] = 0, \tag{36a}$$

$$\nabla^2 H_z^i + \omega^2 \left[(\epsilon_i \mu_i - \xi_i^2) H_z^i + \epsilon_i (\zeta_i - \xi_i) E_z^i \right] = 0. \tag{36b}$$

Hence, only in the case of a Tellegen medium ($\xi_i = \zeta_i$) do the partial differential equations for E_z^i and H_z^i become uncoupled and reduce to the Helmholtz equations. In the most general situation, the following fourth order uncoupled partial differential equations should be considered instead of (36):

$$\begin{aligned} &\nabla^4 E_z^i + \omega^2 \left(2\epsilon_i \mu_i - \zeta_i^2 - \xi_i^2\right) \nabla^2 E_z^i \\ &\quad + \omega^4 \left[\epsilon_i^2 \mu_i^2 - \epsilon_i \mu_i (\xi_i^2 + \zeta_i^2) + \zeta_i^2 \xi_i^2 - (\zeta_i - \xi_i)^2 \right] E_z^i = 0, \end{aligned} \tag{37a}$$

$$\begin{aligned} &\nabla^4 H_z^i + \omega^2 \left(2\epsilon_i \mu_i - \zeta_i^2 - \xi_i^2\right) \nabla^2 H_z^i \\ &\quad + \omega^4 \left[\epsilon_i^2 \mu_i^2 - \epsilon_i \mu_i (\xi_i^2 + \zeta_i^2) + \zeta_i^2 \xi_i^2 + (\zeta_i - \xi_i)^2 \right] E_z^i = 0. \end{aligned} \tag{37b}$$

Hence, the problem of bi-isotropic materials presents some analogies with the in-plane elastic problem, where the Airy stress function is governed by a biharmonic type of equation. In spite of these differences with respect to the isotropic case, if we consider again the eigenfunction expansion approximation (22) for E_z^i and H_z^i, as also proposed in [31], and we introduce them into (36), then we find that the ordinary differential equations (23) holding for $f_{i,j}$ and $F_{i,j}$ remain unchanged. In fact, it easy to verify that the bi-isotropicity affects only the terms multiplying r^{λ_j} that are not responsible for the singular behaviour of E_z^i and H_z^i for $r \to 0$. As a consequence, the asymptotic analysis can be carried out as for the isotropic case, considering the trigonometric expressions (24) for $f_{i,j}$ and $F_{i,j}$.

However, when the BCs (26) and (27) are imposed, we have the following conditions instead of (28) and (29):

$$E_z^1(r, \gamma_1) = 0, \tag{38a}$$

$$E_z^{n-1}(r, \gamma_n) = 0, \tag{38b}$$

$$\frac{1}{\epsilon_1} \frac{\partial H_z^1}{\partial \theta} - \frac{\xi_1}{(\zeta_1 \xi_1 - \mu_1 \epsilon_1)} \left(\frac{\partial E_z^1}{\partial \theta} + \frac{\zeta_1}{\epsilon_1} \frac{\partial H_z^1}{\partial \theta} \right) = 0, \tag{38c}$$

$$\frac{1}{\epsilon_1} \frac{\partial H_z^{n-1}}{\partial \theta} - \frac{\xi_{n-1}}{(\zeta_{n-1} \xi_{n-1} - \mu_{n-1} \epsilon_{n-1})} \left(\frac{\partial E_z^{n-1}}{\partial \theta} + \frac{\zeta_{n-1}}{\epsilon_{n-1}} \frac{\partial H_z^{n-1}}{\partial \theta} \right) = 0, \tag{38d}$$

$$E_z^i(r, \gamma_{i+1}) = E_z^{i+1}(r, \gamma_{i+1}), \tag{39a}$$

$$\begin{aligned} &\frac{1}{\epsilon_i} \frac{\partial H_z^i}{\partial \theta} - \frac{\xi_i}{(\zeta_i \xi_i - \mu_i \epsilon_i)} \left(\frac{\partial E_z^i}{\partial \theta} + \frac{\zeta_i}{\epsilon_i} \frac{\partial H_z^i}{\partial \theta} \right) \\ &\quad = \frac{1}{\epsilon_{i+1}} \frac{\partial H_z^{i+1}}{\partial \theta} - \frac{\xi_{i+1}}{(\zeta_{i+1} \xi_{i+1} - \mu_{i+1} \epsilon_{i+1})} \left(\frac{\partial E_z^{i+1}}{\partial \theta} + \frac{\zeta_{i+1}}{\epsilon_{i+1}} \frac{\partial H_z^{i+1}}{\partial \theta} \right), \end{aligned} \tag{39b}$$

$$H_z^i(r, \gamma_{i+1}) = H_z^{i+1}(r, \gamma_{i+1}), \tag{39c}$$

$$\begin{aligned} &\frac{\epsilon_i}{\xi_i \zeta_i - \mu_i \epsilon_i} \left(\frac{\partial E_z^i}{\partial \theta} + \frac{\zeta_i}{\epsilon_i} \frac{\partial H_z^i}{\partial \theta} \right) \\ &\quad = \frac{\epsilon_{i+1}}{\xi_{i+1} \zeta_{i+1} - \mu_{i+1} \epsilon_{i+1}} \left(\frac{\partial E_z^{i+1}}{\partial \theta} + \frac{\zeta_{i+1}}{\epsilon_{i+1}} \frac{\partial H_z^{i+1}}{\partial \theta} \right). \end{aligned} \tag{39d}$$

Hence, in the case of bi-isotropicity, it is no longer possible to decouple the homogeneous equation set resulting from the imposition of the BCs into two equation sets, one for E_z^i and another for H_z^i. Therefore, the resulting eigenequation cannot be derived from the antiplane shear problems and the mathematical analogy does not apply in this case.

5 Conclusions

In the present paper, we have demonstrated that the asymptotic analysis of the stress-singularities at the vertex of multimaterial wedges and junctions in antiplane elasticity is analogous to the corresponding problem in electromagnetism. In particular, when an isotropic multimaterial wedge with PEC boundaries is considered, we have shown that two independent problems can be defined, one for TE fields, associated to an eigenequation for H_z^i, and one for TM fields, associated to an eigenequation for E_z^i. The eigenequation for E_z corresponds exactly to that obtained for the same geometrical configuration in antiplane elasticity by setting $G_i = 1/\mu_i$ and replacing the PEC region with an infinitely stiff material leading to clamped edge BCs along Γ_1 and Γ_n. Similarly, the other eigenequation for H_z can be obtained in antiplane elasticity for the same geometrical configuration by setting $G_i = 1/\epsilon_i$ and replacing the PEC region with an infinitely soft material leading to stress-free BCs along Γ_1 and Γ_n.

We have also shown that the case of bi-isotropic wedges can still be analyzed using the eigenfunction expansion method as for the isotropic case and that the determination of the eigenvalues can still be performed in a way similar to that for isotropic materials. However, in this case, due to the coupling between E_z^i and H_z^i, the resulting homogeneous equation set coming from the imposition of the boundary conditions is no longer analogous to that in antiplane elasticity.

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