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A unified fractal approach for the interpretation of the anomalous scaling laws in fatigue and comparison with existing models

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Abstract In this paper, a critical reexamination of the fractal models for the analysis of crack-size effects in fatigue is proposed. The enhanced ability to detect and measure very short cracks has in fact pointed out the so-called anomalous behavior of short cracks with respect to their longer counterparts. The crack-size dependencies of both the fatigue threshold and the Paris' constant C are only two notable examples of these anomalous scaling laws. In this context, a unified theoretical model seems to be missing and the behavior of short cracks can still be considered as an open problem. A new generalized theory based on fractal geometry is herein proposed, which permits to consistently interpret the short crack-related anomalous scaling laws within a unified theoretical framework. The proposed model is used to interpret relevant experimental data related to the crack-size dependence of the fatigue threshold in metals. As a main result, the model gives an explanation to the experimentally observed variability in the slope of the asymptote of the scaling law for the fatigue threshold in the short crack regime.

Keywords Fatigue crack growth · Short cracks · Fatigue threshold · Fractals · Scaling laws

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1 Introduction

During the last decades, the enhanced ability to detect and measure very short cracks, along with a great interest in applying fracture mechanics formulae to smaller and smaller crack sizes, has pointed out the so-called anomalous behavior of short cracks with respect to their longer counterparts (see e.g. Taylor 1981; Miller 1982, Suresh and Ritchie 1984 for a review). Pearson (1975) firstly reported the observation that such cracks are characterized by a crack growth rate, $\mathrm{d}a/\mathrm{d}N$, higher than what would be predicted by the Paris' law for a given stress–intensity factor range, ΔK. Moreover, experimental results by Lankford (1982) for a peak-aged 7075 aluminium alloy firstly showed a decrease in $\mathrm{d}a/\mathrm{d}N$ with increasing ΔK, as well as the occurrence of crack growth at ΔK values lower than the long-crack threshold.

On the other hand, most of the fatigue crack growth parameters are experimentally found to be crack-size dependent. Among them, it is worth mentioning the fatigue threshold stress–intensity range, ΔK_{th}, which was introduced by McClintock (1963) in the framework of the Paris' law (Paris et al. 1961; Paris and Erdogan 1963). By definition, this parameter represents a threshold condition of nonpropagation of a pre-existing crack or notch subjected to cyclic loading. Therefore, in close analogy with fracture toughness defining the limit of unstable crack propagation, the threshold ΔK_{th} defines the limit of stable fatigue crack growth.

As the fatigue threshold represents a very important parameter in design and failure analysis, much research effort has been devoted during the last decades to reveal underlying principles. Experimentally, the threshold stress–intensity range is calculated in correspondence to a conventional very low propagation rate, $\mathrm{d}a/\mathrm{d}N$, (usually 10^{-11} m/cycle, see e.g. Radhakrishnan 1980) and is typically determined for long cracks. For such crack sizes, the threshold stress–intensity range, $\Delta K_{\mathrm{th}}^{\infty}$, is found to be a material property independent of the crack length.

Frost (1966) questioned the validity of LEFM-based threshold stress intensity range in the region of short cracks. He found a crack-size dependence of ΔK_{th} in mild steel, aluminium and copper, where ΔK_{th} becomes a decreasing function of the crack length. A similar trend was also reported for several steels by Usami and Shida (1979), and for a high-strength steel by Kitagawa and Takahashi (1976). The same Authors (Kitagawa and Takahashi 1979) later found that there exists a transition crack length below which ΔK_{th} is smaller than that for long cracks, and that such a length is dependent on the material microstructure. Thus, the short crack problem is essentially an outcome of the inapplicability of the fracture mechanics parameters to uniquely characterize the growth of fatigue cracks independently of their size.

In the last two decades, the properties of *fractality* and *multifractality* of fracture surfaces have been widely recognized in the case of both quasi-brittle (Carpinteri 1994a,b,c; Carpinteri and Ferro 1994; Carpinteri and Chiaia 1995) and ductile materials (Mandelbrot et al. 1984; Williford 1988; Underwood and Banerjee 1987). The former concept is related to *self-similar* domains characterized by a constant fractal dimension, whereas the latter permits to analyze *self-affine* domains where their fractal dimension depends on the scale of observation. According to this second approach, Carpinteri (1994b) firstly proposed two geometrical multifractal scaling laws for strength and toughness of disordered materials.

Starting from these theories, successful applications of fractal geometry to size effect-related fatigue problems have recently been proposed by Carpinteri et al. (2002b) and by Carpinteri and Spagnoli (2004). A self-similar invasive fractal set has been exploited in Carpinteri and Spagnoli (2004) to model fracture surfaces, and a size-dependent crack propagation law has been proposed. Recent applications of this model have concerned the analysis of the crack-size dependence of the constant C entering the Paris' law (Spagnoli 2005). On the other hand, self-affinity of fracture surfaces was also postulated by Spagnoli (2004) to reinterpret the anomalous short-crack behavior of the fatigue threshold according to a multifractal scaling law. In this case, however, an ad hoc assumption on the asymptotic value of the fractal dimension for small cracks is put forward, which lies outside the typical experimental range of variation of this parameter. Moreover, since both the Paris' constant C and the fatigue threshold ΔK_{th} are experimentally found to be crack-size dependent, it is reasonable expecting to interpret these anomalous behaviors within the same theoretical framework.

From this brief overview, we note that fractal geometry emerges as a powerful tool for the explanation of the anomalous scaling laws in fatigue that are usually interpreted according to empirical laws without a theoretical ground. On the other hand, as a criticism, we observe that none of the aforementioned models is able to consistently interpret all the anomalous scaling laws due to short cracks without introducing ad hoc assumptions.

For all of these reasons, in this paper we propose a reexamination of the fractal models for the analysis of crack-size effects in fatigue. Whereas in a previous paper (Carpinteri and Paggi 2007) we focused the attention on the asymptote of the Paris' curve for $K_{\mathrm{I}} \to K_{\mathrm{IC}}$ to derive a correlation between the Paris' constants, in this study we analyze the asymptote for $\Delta K \to \Delta K_{\mathrm{th}}$. The limitations of the existing models are put into evidence and removed. At the end, a unified theory based on fractal geometry is proposed, which permits to consistently interpret the short crack-related anomalous scaling laws within the same theoretical formulation.

2 Fractal and multifractal scaling laws

Experimental observations have shown that, within a certain range of scales, the fracture surfaces exhibit self-similar characteristics, i.e., they look the same at different magnification levels. Self-similarity implies that a (statistically) similar morphology appears in a wide range of magnifications of the fracture surface. The fractal dimension of these surfaces is a way to quantify this "order behind chaos", i.e., a measure of the correlation in the surface topology. Experimental evidence of self-similarity (Mandelbrot 1982) over a broad range of scales has been reported frequently in the literature for a wide range of materials: fractured surfaces of steel (Mandelbrot et al. 1984), Molybdenum (Matsuoka et al. 1992), natural rocks (Brown and Scholz 1985) and also concrete (Carpinteri et al. 1999).

Formidable advances have been made in the last few years in the study of the fractal aspects of crack morphology and energy dissipation over fractal domains. Carpinteri (1994a) firstly proposed to model concrete damage by assuming that the rarefied resisting cross-sections in correspondence to the peak load can be represented by stochastic lacunar fractal sets with dimension $2 - d_\sigma$ ($d_\sigma \geq 0$). According to this approach, a straightforward application of the Renormalization Group procedure (Wilson 1971) permits to derive the following scaling law for the nominal tensile strength (see also Carpinteri et al. 2006 for a detailed review):

$$\sigma_u = \sigma_u^* b^{-d_\sigma}, \tag{1}$$

where σ_u^* presents the anomalous physical dimensions $[F][L]^{(2-d_\sigma)}$ and is a scale-invariant mechanical property.

To highlight the scaling on fracture energy, Carpinteri (1994b) analyzed the work W necessary to break a specimen of nominal cross-section b^2. In this case, a straightforward application of the Renormalization Group procedure permits to derive the following scaling law for the nominal fracture energy:

$$\mathcal{G}_\mathrm{F} = \mathcal{G}_\mathrm{F}^* b^{d_\mathcal{G}}, \tag{2}$$

where $\mathcal{G}_\mathrm{F}^*$ is the renormalized fracture energy whose anomalous physical dimensions $[F][L]^{-(1+d_\mathcal{G})}$ imply that the energy dissipation is intermediate between a purely surface dissipation and a bulk dissipation. Here, the crack surface is modeled as an invasive fractal whose topological dimension is equal to $2 + d_\mathcal{G}$ ($d_\mathcal{G} \geq 0$).

Considering the well-known Griffith's energetic approach to the problem of an infinite plate of unit thickness, containing a crack of Euclidean length $2a$ and subjected to a remote tensile stress σ, Carpinteri (1994b) also determined a renormalized expression for the stress–intensity factor within the context of fractal cracks:

$$K_\mathrm{I} = K_\mathrm{I}^* a^{d_\mathcal{G}/2}, \tag{3}$$

where the renormalized quantity K_I^* has the following physical dimensions: $[F][L]^{-(3+d_\mathcal{G})/2}$. It has to be remarked that the exponents d_σ and $d_\mathcal{G}$ of the aforementioned scaling laws are not uncorrelated, as demonstrated by Carpinteri et al. (2002a). At the smaller scales, we have $d_\sigma + d_\mathcal{G} = 1$.

The scaling variations described by Eqs.(1) and (2) are represented by a constant scaling exponent and therefore can be called *monofractal scaling laws*. If specimens of different sizes, made of the same material, are tested in uniaxial tension, experiments show that the monofractal scaling of σ_u and $\mathcal{G}_\mathrm{F}$ is strictly valid only within a limited scale range, where the fractal dimensions of the supporting domains can be considered as constants. As the size increases, in fact, the concept of *geometrical multifractality*, strictly connected to the characteristics of self-affine fractals (Carpinteri 1994b), implies the progressive vanishing of fractality ($d_\sigma \to 0$, $d_\mathcal{G} \to 0$) with a corresponding homogenization of the domains. Intuitively, since the microstructure of a disordered material is the same, independently of the macroscopic specimen size, the influence of disorder on the mechanical properties essentially depends on the ratio between a characteristic material length, l_ch, and the external size, b, of the specimen. Therefore, the effect of microstructural disorder on the mechanical behavior of the material becomes progressively less important at the larger scales. On the other hand, Carpinteri (1994b) observed that a Brownian disorder is the highest possible at the smaller scales, yielding to fractal scaling exponents equal to $d_\sigma = d_\mathcal{G} = 1/2$ for both invasive and lacunar morphologies.

This transition from a disordered (fractal) regime to an ordered (homogeneous) one can therefore be emphasized in the scaling behavior of any mechanical quantity. The analytical expressions of the *Multifractal Scaling Laws* (MFSL) for tensile strength and fracture energy (Carpinteri 1994b; Carpinteri and Chiaia 1995, 1996), which are shown in Fig. 1, are the following:

$$\sigma_u(b) = f_t \left(1 + \frac{l_\mathrm{ch}}{b}\right)^{1/2}, \tag{4a}$$

$$\mathcal{G}_\mathrm{F} = \mathcal{G}_\mathrm{F}^\infty \left(1 + \frac{l_\mathrm{ch}}{b}\right)^{-1/2}. \tag{4b}$$

These scaling laws are both two-parameters models, where the asymptotic value of the nominal quantity ($\mathcal{G}_\mathrm{F}^\infty$ or f_t), corresponding, respectively, to the highest nominal fracture energy and to the lowest nominal tensile strength, is reached only in the limit case of infinite sizes. The nondimensional term into round brackets, which is controlled by the characteristic length

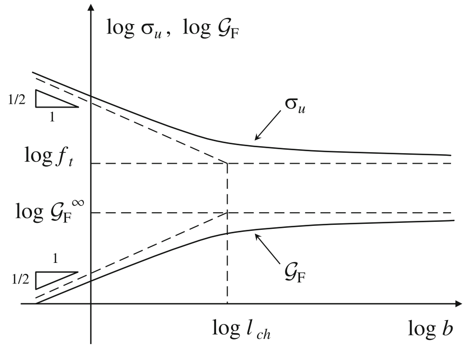


Fig. 1 Multifractal scaling laws for tensile strength and fracture energy, as proposed in Carpinteri (1994b)

l_{ch}, represents the variable influence of disorder on the mechanical behavior. In the bilogarithmic diagram, shown in Fig. 1, the transition from the fractal scaling regime to the Euclidean one is evident, the transition scale being represented by the point of abscissa $\log l_{ch}$.

3 Fractal approaches to fatigue

The well-known Paris' law (Paris and Erdogan 1963) describes the kinetics of crack propagation in the intermediate range of ΔK_I:

$$\frac{da}{dN} = C\,(\Delta K_I)^m\,, \tag{5}$$

where C and m are the Paris' law parameters, N is the number of fatigue cycles, da/dN is the crack propagation rate and ΔK_I is the stress–intensity factor range.

An early application to fatigue of the innovative concepts of fractals and multifractal measures introduced by Mandelbrot (1982) can be traced back to the work by Williford (1988, 1990). He modeled the fracture surfaces near the crack tip as an invasive fractal and proposed a power-law relationship between the J-integral and the crack length increment, Δa, with a non-integer exponent which coincides with that in Eq. (2):

$$J = J^*(\Delta a)^{d_{\mathcal{G}}}. \tag{6}$$

Parameter J^* was considered as a material constant, although its anomalous physical dimensions were not pointed out. On the basis of this scaling law, Williford (1990) proposed a modified Paris' law where both the Paris' parameters are functions of the surface fractal dimension. Hence, for the usual range of variation of the exponent m of metals, i.e. $2 \leq m \leq 6$, he suggested a theoretical variability of $d_{\mathcal{G}}$ in the range $0.6 \geq d_{\mathcal{G}} \geq 0.2$. For AISI 4340 steel tested under monotonic loading (Yokobori 1979), measured values of the fractal dimension using fractographic images led to $d_{\mathcal{G}} = 0.51$, which was also considered by Williford (1990) as the representative value for fatigue conditions.

In the 1990s, experimental evidences by Bažant and Xu (1991) and Bažant and Shell (1993) pointed out a dependence of the crack growth rate on the specimen size, i.e., a size effect on fatigue crack growth. Thus, exploiting the renormalized quantities related to fractal cracks (whose surfaces can be modelled as invasive fractals according to the results achieved by Carpinteri and summarized in the previous section), Carpinteri and Spagnoli (2004) proposed the following size-independent fatigue crack growth law:

$$\frac{da^*}{dN} = C\left(\Delta K_I^*\right)^m\,, \tag{7}$$

where $a^* = a^{1+d_{\mathcal{G}}}$ and ΔK_I^* is given by Eq. (3). As shown by several authors, including Pippan et al. (1994), the crack path often shows deflections from the macroscopic propagation direction due to Mixed Mode effects, leading to a zig-zag shape. In the present macroscopic model, the problem is simplified as a Mode I crack with a fractal profile.

A scaling law was obtained by Carpinteri and Spagnoli (2004) from Eq. (7), by rewriting such a relationship in terms of the nominal crack propagation rate, da/dN, and the nominal stress–intensity factor range, ΔK_I. Using a derivation chain rule to calculate the crack propagation rate da^*/dN of the fractal crack, they obtained:

$$\frac{da}{dN} = \frac{C}{1+d_{\mathcal{G}}} a^\beta \Delta K_I^m = C_F(a)\Delta K_I^m,\qquad \beta = -d_{\mathcal{G}}\left(1+\frac{m}{2}\right). \tag{8}$$

For geometrically similar cracked bodies, a is proportional to b. Consequently, Eq. (8) can be regarded either as a structural-size dependent Paris' law (Carpinteri and Spagnoli 2004), or as a crack-size dependent fatigue crack growth law (Spagnoli 2005). Note that such an equation is formally identical to the classical Paris' law in Eq. (5), but the new coefficient of proportionality, C_F, is no longer a material constant. Since $\beta < 0$, Eq. (8) would predict that the crack growth rate da/dN is a decreasing function of the structural dimension or the crack length. In the case of large structural sizes where the transition from disorder to order takes place, $d_{\mathcal{G}} \rightarrow 0$, and no size effects occur.

This model was referred to as *monofractal approach* to size effect on fatigue crack growth in (Carpinteri and Spagnoli 2004; Spagnoli 2005). It was also noticed that it can be applied within a limited scale range. The use of a *multifractal approach* was also suggested in (Carpinteri and Spagnoli 2004; Spagnoli 2005) to model the propagation of both short and long fatigue cracks, although this possibility remained unexplored.

On the other hand, Spagnoli (2004) proposed an original interpretation of crack-size effects on the fatigue threshold according to geometric multifractality. More specifically, the well-known Kitagawa diagram (Kitagawa and Takahashi 1976) describes the variation of the threshold stress–intensity range as a function of the crack length, showing the existence of a transition length beyond which the threshold of fatigue crack growth becomes a material constant, $\Delta K_{\mathrm{th}}^{\infty}$. For shorter crack lengths, ΔK_{th} is progressively reduced, as schematically shown in Fig. 2.

To interpret this anomalous trend in the context of fractal geometry, Spagnoli (2004) supposed that the monofractal scaling law in Eq. (3) would apply not only to the generalized stress–intensity factor, but also to the threshold stress–intensity range. According to this equation, ΔK_{th} is expected to be a function of a raised to $d_{\mathcal{G}}/2$ within a limited scale range. Then, he treated fracture surfaces as invasive self-affine fractal sets and postulated the existence of the following multifractal scaling law for ΔK_{th} to bridge the two experimentally observed asymptotical tendencies of $\Delta K_{\mathrm{th}} \propto a^{1/2}$ for $a \rightarrow 0$ and $\Delta K_{\mathrm{th}} = \text{constant}$ for $a \rightarrow \infty$:

$$\Delta K_{\mathrm{th}} = \Delta K_{\mathrm{th}}^{\infty} \left(1 + \frac{a_0}{a}\right)^{-1/2}, \tag{9}$$

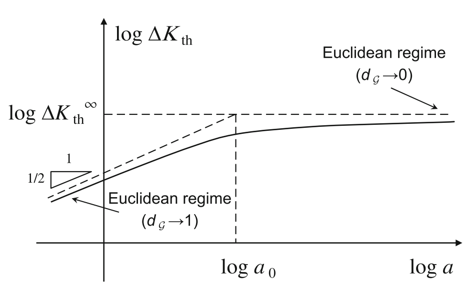


Fig. 2 Multifractal scaling law for the threshold stress–intensity range, as proposed in Spagnoli (2004)

where a_0 denotes a characteristic crack length. This equation, which is identical to the empirical relationship by El Haddad et al. (1979), implies that $d_{\mathcal{G}} \rightarrow 0$ for long cracks, leading to $\Delta K_{\mathrm{th}} = \Delta K_{\mathrm{th}}^{\infty}$ and the disappearance of size-effects. On the other hand, for short cracks ($a \rightarrow 0$), the slope $1/2$ of the asymptote can only be predicted in this model by setting $d_{\mathcal{G}} \rightarrow 1$ (see Eq. (3) and the diagram in Fig. 2).

As previously pointed out, $d_{\mathcal{G}}$ has to be lower than or equal to $1/2$, and therefore the assumption in (Spagnoli 2004) seems to be in contrast with the common range of variation of this exponent. On the other hand, from the geometrical point of view, fractal dimensions larger than 2.5 would imply overhangs along the surface, which clearly are not admissible in the kinematics of a fracture process.

More importantly, the application of the scaling law (3) to the fatigue threshold is questionable. In fact, ΔK_{th} is experimentally computed in correspondence of a conventional very low crack propagation rate, and thus it should depend on the parameters defining the kinetics of the fatigue crack propagation phenomenon, such as m and R, as experimentally found in (Radhakrishnan 1980).

In this sense, since both the Paris' constant C and the fatigue threshold ΔK_{th} are experimentally found to be crack-size dependent, it should be reasonable expecting to interpret these anomalous behaviors within the same theoretical framework.

4 A unified fractal model for the interpretation of the crack-size dependencies in fatigue

In this section, we propose a unified fractal approach encompassing all the aforementioned size-effects related to the anomalous behavior of short cracks observed in fatigue. To this aim, let us consider the crack-size dependent monofractal Paris' law in Eq. (8). Since the threshold stress–intensity factor range is experimentally calculated for a conventional very low propagation rate, $v_{\mathrm{th}} \cong 10^{-11}$ m/cycle (Radhakrishnan 1980), we can compute the fatigue threshold range by inverting Eq. (8) in correspondence of $\mathrm{d}a/\mathrm{d}N = v_{\mathrm{th}}$:

$$\Delta K_{\mathrm{th}} = \left(\frac{v_{\mathrm{th}}}{C_{\mathrm{F}}}\right)^{1/m} = v_{\mathrm{th}}^{1/m} \left[\frac{1 + d_{\mathcal{G}}}{C} a^{d_{\mathcal{G}}\left(1+\frac{m}{2}\right)}\right]^{1/m}. \tag{10}$$

For long cracks, a transition from disorder to the Euclidean order is expected, so that $d_{\mathcal{G}} \rightarrow 0$. As a consequence, the crack-size dependence disappears and we obtain the asymptotic value of the fatigue threshold for long cracks, $\Delta K_{\mathrm{th}}^{\infty}$:

$$\Delta K_{\mathrm{th}}^{\infty} = \left(\frac{v_{\mathrm{th}}}{C}\right)^{1/m}. \tag{11}$$

In this case, recalling the correlation between the Paris' law parameters established in (Carpinteri and Paggi 2007), we have:

$$\Delta K_{\mathrm{th}}^{\infty} = \left(\frac{v_{\mathrm{th}}}{v_{\mathrm{cr}}}\right)^{1/m} K_{\mathrm{IC}}(1-R), \tag{12}$$

where v_{cr} and K_{IC} denote the coordinates of the point of the Paris' curve corresponding to the onset of crack growth instability, whereas $R = K_{\min}/K_{\max}$ is the loading ratio. Equation (12) shows that the physical dimensions of the fatigue threshold for long cracks coincide with those of fracture toughness, i.e., $[F][L]^{-3/2}$. For long cracks, Eq. (12) can also be used to establish a useful relationship between the parameter m and the coordinates of two special points of the Paris' curve: the point corresponding to the onset of crack growth instability, and the point defining the threshold condition. Therefore, it should be possible, in principle, to characterize the process of fatigue crack growth in the intermediate regime (Region II) using the coordinates of the aforementioned special points instead of the parameters C and m.

According to Eq. (11), Eq. (10) can now be rewritten in this synthetic form:

$$\Delta K_{\mathrm{th}} = \Delta K_{\mathrm{th}}^{\infty}\left[(1+d_{\mathcal{G}})a^{d_{\mathcal{G}}(1+\frac{m}{2})}\right]^{1/m}. \tag{13}$$

For metals, where $2 \le m \le 6$, this equation would predict a scaling law for the fatigue threshold of the type $\Delta K_{\mathrm{th}} \propto a^{\gamma}$, with an exponent γ ranging from $2/3\, d_{\mathcal{G}}$ for $m = 6$, up to $d_{\mathcal{G}}$ for $m = 2$. Note, incidentally, that this approach leads to $\Delta K_{\mathrm{th}} \propto \sqrt{a}$ for short cracks ($d_{\mathcal{G}} \to 1/2$) in the case of $m = 2$, which is far commonly reported for steels. In this case, ΔK_{th} assumes the physical dimensions of stress, i.e. $[F][L]^{-2}$. This is in agreement with the common interpretation of the crack-size effects on the fatigue threshold at the small scales based on the well-known Hall–Petch law (Verhoeven 1975; Masounave and Bailon 1976). In fact, the crack length can be considered as proportional to the average grain size of the material microstructure and we have $\Delta K_{\mathrm{th}} \propto \sqrt{d_{\mathrm{grain}}}$. Therefore, the dimensional transition from $[F][L]^{-2}$ in the case of short cracks to $[F][L]^{-3/2}$ for long cracks can be physically interpreted as a transition from a dislocation-based diffuse damage to a macroscopic fracturing.

Exploiting the concept of self-affinity, we can also consider a multifractal Paris' law, as suggested by Carpinteri and Spagnoli (2004):

$$\frac{\mathrm{d}a}{\mathrm{d}N} = C_{\mathrm{MF}}(a)\Delta K^{m}. \tag{14}$$

The new Paris' law parameter is now given by the following equation:

$$C_{\mathrm{MF}}(a) = C\left(1+\frac{a_0}{a}\right)^{\frac{1}{2}(1+\frac{m}{2})}, \tag{15}$$

where a_0 is a characteristic length as that defined in the multifractal scaling laws for tensile strength and fracture energy. This equation, which is a decreasing function of the crack length, is schematically shown in Fig. 3 and bridges the two asymptotic behaviors for the short and the long crack length regimes. For short cracks, we have the asymptote $\frac{1}{2}(1+m/2)$, which is obtained by setting $d_{\mathcal{G}} \to 1/2$ in Eq. (8). On the other hand, the asymptote for long cracks is characterized by $d_{\mathcal{G}} \to 0$, and therefore the crack-size effect disappears.

Again, inverting Eq. (14) in correspondence of $\mathrm{d}a/\mathrm{d}N = v_{\mathrm{th}}$, we obtain the scaling law for the fatigue

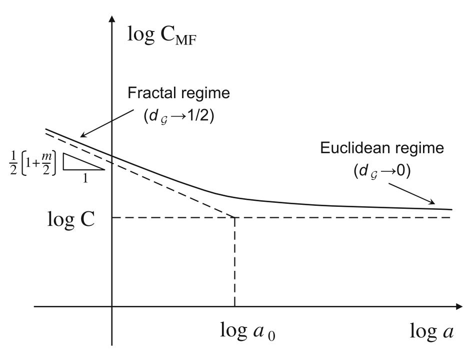


Fig. 3 Multifractal scaling law for the Paris' constant C

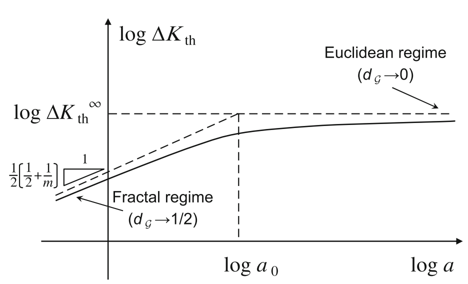


Fig. 4 New multifractal scaling law for the threshold stress–intensity range

threshold stress–intensity range (see Fig. (4) for its graphical representation):

$$\Delta K_{\mathrm{th}} = \left(\frac{v_{\mathrm{th}}}{C_{\mathrm{MF}}}\right)^{1/m} = \Delta K_{\mathrm{th}}^{\infty}\left(1+\frac{a_0}{a}\right)^{-\frac{1}{2}\left(\frac{1}{2}+\frac{1}{m}\right)}. \tag{16}$$

It is worth noting that this equation has been derived as a straightforward consequence of the multifractal scaling law in Eq. (15), without introducing additional ad hoc assumptions or limitations on the value of $d_{\mathcal{G}}$ like in (Spagnoli 2004). The exponent of the proposed scaling law ranges from $1/3$ for $m = 6$, up to $1/2$ for $m = 2$. It is important to note that the variability in the parameter m can also be ascribed to the effect of the loading ratio, $R = K_{\min}/K_{\max}$, as firstly observed by Radhakrishnan (1979), who proposed the following empirical relationship:

$$m = m_0\left(\frac{1}{1-R}\right)^{1/4}, \tag{17}$$

where m_0 denotes the value of m for $R = 0$. The range of variation of the exponent of the multifractal scaling law for the fatigue threshold, which corresponds to the slope of the asymptote of the ΔK_{th} versus a bilogarithmic curve at the small scales, is quantified in Fig. 5. More specifically, we have considered two cases: (i) $R=0$ and m ranging from 2 to 6; (ii) $m_0=3$, R ranging from 0 to 0.8 and m computed according to Eq. (17). The exponent of the multifractal scaling law turns out to be a decreasing function of m and R, implying that, for a given crack length a, the fatigue threshold increases either in materials having high values of m, or for a given material tested under increasing loading ratios,

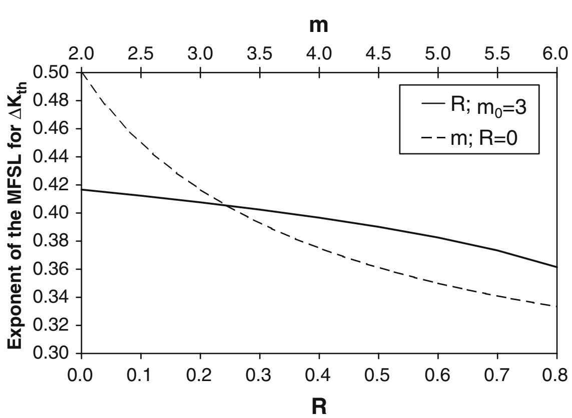


Fig. 5 The effect of m and R on the exponent of the multifractal scaling law for the threshold stress–intensity range

R. This prediction is in close agreement with the empirical correlations proposed in (Radhakrishnan 1980).

Considering the data collected in (Tanaka 2003), an experimental assessment of Eq. (16) is proposed in Fig. 6 for a variety of metals. By performing a nonlinear regression analysis on the experimental data, the value of a_0 and the exponent of the multifractal scaling law are determined. Whereas the parameter a_0 ranges from 1–10 μ m for very high strength steels to 100–1000 μ m for very low strength steels, the exponent of the scaling law ranges from $0.33 \sim 1/3$ to $0.51 \sim 1/2$, in good agreement with the theoretically predicted range of variation for this parameter.

5 Comparison with existing approaches

In this section, a critical analysis of (i) the $\sqrt{Area}$ empirical approach, (ii) the models based on the effect of crack closure, (iii) the generalized Frost and Dugdale crack growth law, and (iv) the approach based on *quantized fracture mechanics* is proposed in view of the results obtained according to fractal geometry.

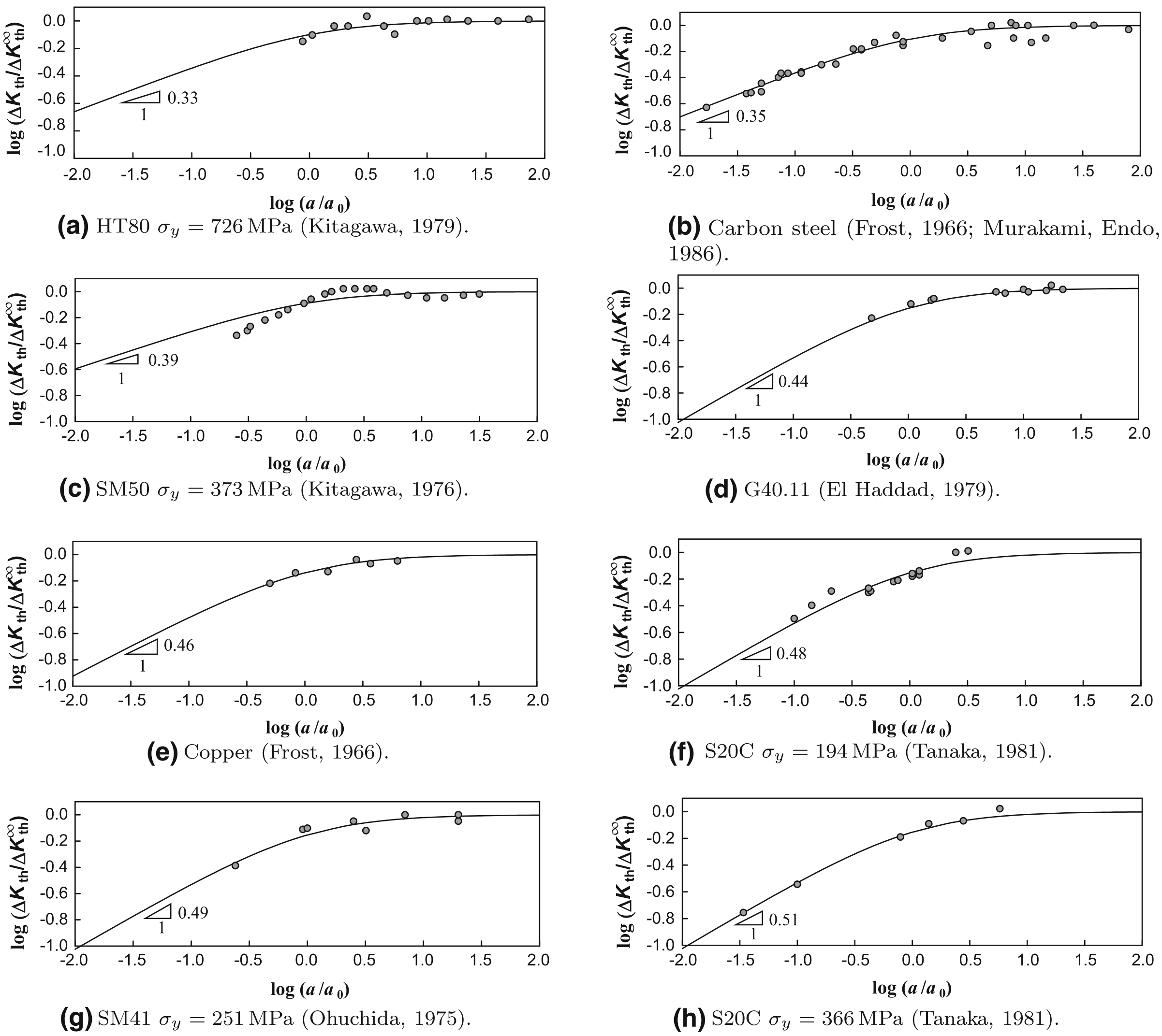
5.1 Comparison with the $\sqrt{Area}$ empirical approach

An empirical relationship for the estimate of the short fatigue crack threshold was proposed by Murakami and Endo (1986):

$$\Delta K_{\mathrm{th}} = 3.3\times 10^{-3}(H_v+120)\left(\sqrt{Area}\right)^{1/3}, \tag{18}$$

where H_v is the Vickers hardness in [kgf/mm^2] and *Area*, measured in [m], is the area of a defect or a crack projected onto the plane normal to the maximum tensile stress. Equation (18) suggests a power-law relationship between the threshold stress–intensity factor range and the crack length. In the case of a penny shape crack, we simply have $\Delta K_{\mathrm{th}} \propto a^{1/3}$, which provides a reasonable approximation to experimental results. However, it is worth noting that the determination of the projected area of a defect entering Eq. (18), well-defined for an ideal smooth crack, is highly questionable when we deal with real rough cracks or microdefects. Considering, for instance, a fractal crack, we have $a^* = a^{1+d_{\mathcal{G}}}$, with $0 \leqslant d_{\mathcal{G}} \leqslant 1/2$. Consequently, Eq. (18) provides the following approximate scaling law:

$$\Delta K_{\mathrm{th}} \propto \left(a^*\right)^{1/3} = a^{(1+d_{\mathcal{G}})/3} \propto \begin{cases} a^{1/3} & \text{if } d_{\mathcal{G}}=0, \\ a^{1/2} & \text{if } d_{\mathcal{G}}=1/2. \end{cases} \tag{19}$$



(a) HT80 $\sigma_y = 726$ MPa (Kitagawa, 1979).

(b) Carbon steel (Frost, 1966; Murakami, Endo, 1986).

(c) SM50 $\sigma_y = 373$ MPa (Kitagawa, 1976).

(d) G40.11 (El Haddad, 1979).

(e) Copper (Frost, 1966).

(f) S20C $\sigma_y = 194$ MPa (Tanaka, 1981).

(g) SM41 $\sigma_y = 251$ MPa (Ohuchida, 1975).

(h) S20C $\sigma_y = 366$ MPa (Tanaka, 1981).

Fig. 6 Experimental assessment of the proposed multifractal scaling law for the threshold stress–intensity range versus crack length (experimental data are taken from the collections reported in Tanaka (2003) and Spagnoli (2004))

Hence, a variability in the exponent characterizing the power-law regime from 1/3 to 1/2 is very likely to occur, as confirmed by the experimental results shown in Fig. 6. This variability is also predicted by the more accurate fractal model proposed in the present paper.

5.2 Comparison with the models based on crack closure

The phenomenon of crack closure has received a great attention in the literature (see Pippan et al. 1994 for a description of the physical mechanism). In order to explain the effects of the loading ratio R on fatigue crack growth, Elber (1970) firstly introduced the concepts of crack closure and effective stress–intensity factor range, $\Delta K_{\mathrm{th,eff}} = K_{\max} - K_{\mathrm{op}}$, where $K_{\max}$ and K_{op} denote, respectively, the stress–intensity factor calculated in correspondence of the maximum load and that corresponding to the crack-opening load. The introduction of the concept of effective stress–intensity factor range has contributed to explain the so call anomalous behaviour of short cracks: the behaviour of a stage II short crack, poorly sensitive to microstructure, has been shown to be essentially related to the variation of crack closure with the crack length. However, according to the recent review paper by Cui (2002) and by the critical examination by Vasudevan et al. (1994), it seems that the physical effects of crack closure have been greatly overestimated in the past. The various forms of closure, e.g. plasticity, roughness and oxide-induced closures, make the problem rather complex and highly debated by the scientific community. For instance, in the long-crack-growth threshold regime, the plasticity-induced closure model is not always able to collapse the threshold ΔK vs. $\mathrm{d}a/\mathrm{d}N$ data onto a unique ΔK_{eff} vs. $\mathrm{d}a/\mathrm{d}N$ relationship as it would be expected (see Newman et al. 1999). A partial crack closure model (Donald and Paris 1999; Kujawski 2001a) was proposed to overcome the shortcomings of the original crack closure model. Later, Kujawski (2001b) found that the loading ratio effect can be explained even better without using the crack closure concept.

The crack closure approach was profitably pursued by McEvily et al. (2003) to provide a rational interpretation to the empirical relationship by Murakami and Endo (1986) for the estimation of the fatigue threshold of short cracks [see Eq. (18)]. The modified expression of the crack growth rate proposed by McEvily et al. (2003), taking into account crack closure and the elasto-plastic behaviour of the material, was able to recover the asymptotic limit of $\Delta K_{\mathrm{th}} \propto a^{1/3}$ for short cracks. The mathematical expression of the modified Paris' law is the following:

$$\frac{\mathrm{d}a}{\mathrm{d}N} = A\left[\left(\sqrt{2\pi r_e F} + Y\sqrt{\pi a F}\right)\Delta\sigma - \left(1 - \mathrm{e}^{-k\lambda}\right)\left(K_{\mathrm{op,max}} - K_{\mathrm{min}}\right) - \Delta K_{\mathrm{th,eff}}\right]^2 \tag{20}$$

where the definition of the symbols is provided in (McEvily et al. 2003) and is not repeated here.

From the mathematical point of view, Eq. (20) is completely different from the generalized Paris' law derived according to fractal geometry [see Eqs. (8) and (14)] and therefore a direct comparison cannot be made. However, in spite of the fact the two approaches focus on two different aspects, one giving prominence to crack closure and the other addressing the effect of roughness of the crack surfaces, the modified Paris' laws determined in the two cases share a common feature. Equations (8) and (14) suggest that the Paris' law parameter C is dependent on the crack length and, as we have demonstrated, this property has a direct consequence on the anomalous scaling of the fatigue threshold. Similarly, Eq. (20) presents a multiplicative coefficient of ΔK_{eff} which is crack-size dependent. In fact, the term $\left(1 - \mathrm{e}^{-k\lambda}\right)$, which governs the rate of increase of crack closure with crack propagation, depends on the increase of crack length from the initial crack size, λ.

5.3 Comparison with the generalized Frost and Dugdale crack growth law

Several researchers have questioned the validity of the similitude hypothesis, which states that "two different sized cracks embedded into two different sized bodies subjected to the same stress–intensity factor range should grow at the same rate". As a support of the theories against the similitude hypothesis, we mention the experimental results by Newman et al. (2004), who observed that "in the threshold regime there is something missing in the (closure) model", and those by Forth et al. (2006), revealing that similitude does not hold in Region I (the near-threshold region) and also in the lower portion of Region II. To solve this problem, Molent et al. (2006) and Jones et al. (2007) have recently proposed a generalized Frost and Dugdale (1958) crack growth law, assuming that the crack growth rate is proportional to the accumulated plastic strain, averaged over a characteristic length ahead of the crack tip:

$$\frac{\mathrm{d}a}{\mathrm{d}N} = C' a^{(1-m'/2)} \Delta K^{m'} \tag{21}$$

where C' and m' are regarded as material constants. This equation states that $\mathrm{d}a/\mathrm{d}N$ is not only a function of the stress–intensity factor range, ΔK, but also of the crack length, a. Such a generalized Frost and Dugdale crack growth equation was then successfully used to predict the growth of near micron sized cracks in both coupon and full scale aircraft fatigue tests. A close comparison between Eq. (21) and Eq. (8) reveals that the multiplicative factor of ΔK, which would correspond to the generalized Paris' law parameter C_F in Eq. (8), depends on the crack length and on the Paris' law exponent m, in agreement with our proposed model.

5.4 Comparison with the approach based on quantized fracture mechanics

Pugno et al. (2006, 2007) have recently proposed a model based on *quantized fracture mechanics* for generalizing the Paris' law for fatigue crack growth. By considering an increased effective crack length as typically assumed in the critical distance theories, they obtained the following expression for the effective Mode I stress–intensity factor in the case of a Griffith crack:

$$K_{\mathrm{I}}^{\mathrm{e}} = \sigma\sqrt{\pi(a + \Delta a/2)}, \tag{22}$$

where σ is the remote applied stress and Δa is the so-called *fracture quantum*. After some manipulation, it is possible to determine the expression between $K_{\mathrm{I}}^{\mathrm{e}}$ and the classical K_{I}:

$$K_{\mathrm{I}}^{\mathrm{e}} = K_{\mathrm{I}}\sqrt{1 + \frac{\Delta a}{2a}}. \tag{23}$$

Considering this expression in the generalized Paris' law defined in (Pugno et al. 2006), we obtain:

$$\begin{aligned}\frac{\mathrm{d}a}{\mathrm{d}N} &= C\left(\Delta K_{\mathrm{I}}^{\mathrm{e}}\right)^m = C\left(1 + \frac{\Delta a}{2a}\right)^{m/2}(\Delta K_{\mathrm{I}})^m \\ &= C'\left(\Delta K_{\mathrm{I}}\right)^m,\end{aligned} \tag{24}$$

where the new Paris' constant C' is given by:

$$C' = C\left(1 + \frac{\Delta a}{2a}\right)^{m/2}. \tag{25}$$

Finally, this approach yields to the following scaling law for the threshold stress–intensity factor range:

$$\Delta K_{\mathrm{th}} = \Delta K_{\mathrm{th}}^{\infty}\left(1 + \frac{\Delta a}{2a}\right)^{-1/2}. \tag{26}$$

These expressions resemble those obtained according to fractal concepts in Eqs. (15) and (16). However, in this case, the parameter Δa is no longer a material constant and it is given by a complicated expression involving the parameters of the Paris' law and of the Wöhler curve. In any case, the expression in Eq. (25) provides another independent confirmation of the dependence of the scaling law for C on the parameter m, as herein demonstrated according to fractal geometry.

6 Conclusions

In the present paper, we have proposed a unified fractal approach for the interpretation of the anomalous scaling laws in fatigue. Early applications of the concepts of fractality and multifractality to the interpretation of size-effects on tensile strength and fracture energy of disordered materials were proposed by Carpinteri since the 1990s. In this framework, the renormalized fracture energy is represented by a dissipation over a surface having a dimension higher than two. The dimensional increment with respect to the Euclidean dimension represents self-similar tortuosity of the fracture surface due to grains and inclusions present in the material microstructure. In physical reality, fracture surfaces after rupture can be considered as multifractals having dimension 2.5 at small scales and dimension two at large scales. This implies that a transition from extreme disorder to extreme order takes place by considering different scales of observation.

Recent developments concerning the analysis of crack-size effects in fatigue have shown that the coefficient C entering the Paris' law is no longer a material constant, but it depends on the initial crack length. Following the idea suggested by Carpinteri and Spagnoli (2004), a multifractal scaling law for the constant C has been proposed. As a consequence of this scaling law, the threshold stress–intensity factor range, which is computed by definition in correspondence of a conventional low crack propagation rate, is found to be crack-size dependent in its turn. This newly proposed scaling law provides an original interpretation of the so-called Kitagawa diagram, where the stress–intensity range threshold is plotted versus the crack length in a bilogarithmic plot. In this diagram, ΔK_{th} is an increasing function of the crack length, approaching a constant value in the long crack regime. For the short crack regime, the slope of the asymptote is found to be dependent on the exponent m of the Paris' law. This dependence sheds a new light on the scatter of the experimental results noticed at the small scales, which cannot be interpreted according to the existing empirical formulae.

Finally, we remark that the critical examination of the approaches available in the literature proposed in Sect. 5 suggests that the anomalous scaling laws of short cracks can be explained according to different models, each one focusing on different physical aspects. Certainly, further studies are required to assess the interaction between different phenomena, such as roughness of the crack surfaces, crack closure, and plasticity.

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