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Asymptotic analysis in Linear Elasticity: From the pioneering studies by Wieghardt and Irwin until today

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ABSTRACT

The asymptotic analysis of the stress distribution around an elastic wedge-shaped domain is one of the most fundamental problems in Linear Elasticity. In occasion of the century anniversary of the pioneering paper by Wieghardt on *splitting and cracking of elastic bodies*, and of the half-a-century anniversary of the Irwin's paper on *the analysis of stresses and strains near the end of a crack*, we propose a review of the most important contributions leading to fundamental advances in this research field. Special focus will be given to the epistemological steps towards a full appreciation of the mathematical and engineering relevance of the stress-singularities. We also provide the reader with a review of the geometrical configurations and mechanical conditions that can relieve or remove the singularities, including: re-entrant corners; power-law hardening constitutive laws; fractal cracks; multi-material junctions and wedges; nonhomogeneous materials.

1. Introduction

In various exact solutions to boundary value problems of Linear Elasticity, the stress field is found to have singularities. They are observed in re-entrant corners of plates and also at the points where discontinuities in the material properties occur. Physically, these singularities correspond to regions of high stress in which plastic flow or even fracture of the material may occur. For all of these reasons, this field has been the subject of extensive research since the end of the 18th century.

Although it cannot be the purpose of this paper to list and review all the valuable studies in this field, it is felt this contribution should be devoted to acquainting the reader briefly with the main efforts and research developments towards a full appreciation of the mathematical and engineering relevance of stress-singularities (see also the review papers [1–4] for a detailed discussion on mathematical and numerical aspects).

We also provide the reader with a review of the geometrical configurations and mechanical conditions that can relieve the power of the singularities with respect to the well-known square-root singularity of Linear Elastic Fracture Mechanics. The discussion includes the effect of the notch angle, the influence of the exponent of the stress–strain relationship in power-law hardening materials, as well as the effect of the roughness of crack surfaces. Special attention is also devoted to the problem of stress-singularities in multi-material junctions and wedges, a situation frequently observed in mechanical and composite engineering. Finally, a section on stress-singularities in nonhomogeneous materials is presented, which is a new research field that can now be explored due to the enormous advances in materials processing achieved during the 20th century.

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Nomenclature

$P; P_{\max}$	applied line loading; maximum or failure load
E	Young's modulus
E^*	tangent Young's modulus of the yielded material
ν	Poisson's ratio
a	crack length
a^*	fractal crack length
l	beam span
h	beam depth
t	beam width
γ	notch angle
u_r	radial displacement
σ_r	radial stress
$W_e; W_s$	strain energy release; dissipated energy
Φ	Airy stress function
λ_j	eigenvalue
$f_j, f_{i,j}$	eigenfunction
σ_p	material tensile strength
σ_f	stress at failure
K_{I}	Mode I stress-intensity factor
K_{I}^*	generalized Mode I stress-intensity factor
K_{IC}	fracture toughness
K_{IC}^*	generalized fracture toughness
$\mathscr{G}_{\mathrm{I}}$	Mode I strain energy release rate
$\mathscr{G}_{\mathrm{IC}}$	fracture energy
$\mathscr{G}_{\mathrm{F}}^*$	renormalized fracture energy
s	brittleness number
n	Ramberg–Osgood stress–strain exponent
$1 + d_{\mathscr{G}}$	fractal dimension of the crack profile

2. Pioneering investigations and recent developments on the elastic singularities

As a first example of singularity in Linear Elasticity, we present the problem of *a single normal force acting on a straight edge*, which is also referred to as *Flamant's problem* [5]. In his note, transmitted to the Comptes Rendus de l'Académie des Sciences by Boussinesq, Flamant determined the expressions for the stress and the displacement fields under the application point of the force by using the potentials previously defined by Boussinesq in 1885 [6]. He also recognized the singularity of the compressive stress field at the application point of the force using rectangular coordinates *OXY* (see Fig. 1a).

Later on, the same problem was independently solved by Michell [7], using a mathematical formulation based on the Airy stress function and considering polar coordinates. In this more recent mathematical framework, the logarithmic singularity in the radial displacement field and the hyperbolic singularity in the radial compressive stresses were put into evidence:

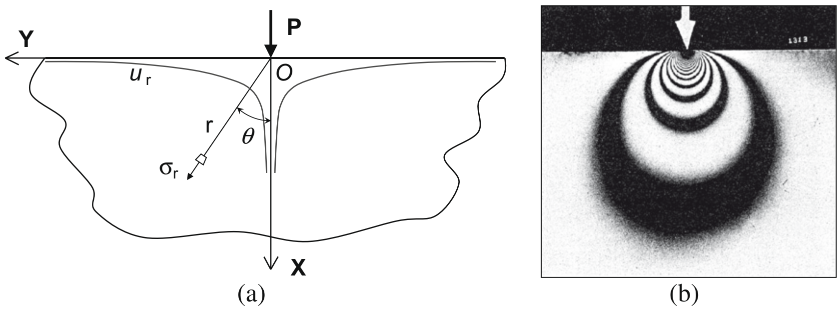

Fig. 1. (a) Scheme of the Flamant's problem and (b) isochromatic fringe pattern obtained according to photoelasticity.

$$u_r(\vartheta = 0) = \frac{2P}{\pi E}\log r + B, \tag{1}$$

$$\sigma_r = -\frac{2P}{\pi}\frac{\cos\vartheta}{r}, \tag{2}$$

where P is the applied line loading, E is the Young's modulus of the half-space and B is a constant of integration that can be determined by assuming that a point on the X axis at a given distance from the origin does not undergo a vertical displacement.

Moreover, the solution of this problem not only shows that the singularity is present in both the stress and displacement fields at the application point of the force, but also leads to a nonintegrable strain energy in the area around the same point:

$$W = \lim_{\varepsilon\to 0}\left(\int_{-\pi/2}^{\pi/2}\int_{\varepsilon}^{R}\frac{\sigma_r^2}{2E} r\,\mathrm{d}r\,\mathrm{d}\vartheta\right) = \lim_{\varepsilon\to 0}\left(\frac{P^2}{\pi E}\log\frac{R}{\varepsilon}\right) \to \infty. \tag{3}$$

Therefore, the Flamant's problem represents a notable example where the solution of a boundary value problem in Linear Elasticity leads to a singular stress field. Mathematically, it represents a paradox for the theory of Linear Elasticity, since it complies with all of the field equations and, at the same time, violates the simplified assumptions made in Linear Elasticity: the stresses do not exceed the limits of elastic material response; the displacement gradients are small; the loads act on the undeformed boundary throughout the entire loading process, that is, the deflections are small. To overcome the observed paradoxes, it is suggested to take away a small portion of material under the point O, which actually undergoes plastic deformation and flows under the action of the applied force.

Another peculiarity in the Flamant's problem relies in the fact that it can be considered as the basis for the subsequent developments of asymptotic analyses related to stress-singularities. In 1907, Wieghardt [8] (see also the English translation operated by Rossmanith [9] in 1995) firstly addressed the problem of finding the stress distribution around an elastic wedge-shaped domain loaded by concentrated forces (see Fig. 2). Solving this *generalized Flamant's problem*, he recognized the factorization of the stresses into radial terms and angular functions. Specifically, for the crack geometry, Wieghardt demonstrated the square-root singularity of the stresses at the crack tip and proposed a long discussion concerning the reasonability of infinite stresses within the Theory of Elasticity.

In 1933, Brahtz [10] published an important research paper on the *stress distribution in a re-entrant corner*, which seems to be not adequately recognized by the Fracture Mechanics Community. At that time, the Wieghardt work was completely forgotten and Brahtz, in his treatment, made reference to the work by von Kármán [11] on the Airy stress function. According to his approach, taking advantage of the relationships existing between the Airy stress function, its first partial derivatives and the resultant force and moment of the applied external loads, it is possible to determine the stress field components within a half-space with a generic distribution of tractions or concentrated forces acting along the boundaries of the elastic domain. Brahtz generalized the von Kármán's method by observing that it can be applied not only to the half-space, but also to any angular portion of the plane with any distribution of normal and tangential forces over the straight boundaries. In the various examples he proposed, special emphasis was given to the problem of a re-entrant corner with notch angle equal to $\pi/2$ and stress-free boundaries, where the power of the stress-singularity and the corresponding angular function were correctly determined. At that time, a deep analysis of the influence of the notch angle on the power of the stress-singularity was not proposed.

Again in 1933, Westergaard [12] published a paper on the stress analysis at the crack tip in reinforced concrete members in bending. In that contribution, which appears to be one of the first attempts to apply fracture mechanics concepts to quasi-brittle materials, Westergaard modeled the cracked beam as two completely disconnected half-beams with an elliptical boundary along the crack front. These portions are in contact along the so-called ligament zone and the equilibrium is restored by the presence of the reinforcement in tension. Taking advantage of the Hertzian solution for the evaluation of the contact pressures between two cylinders [13], he proposed the following expression for the Airy stress function Φ:

$$\Phi = \sum_{m=1}^{\infty}\frac{K_m r^{m+3/2}}{2\left(m+\frac{1}{2}\right)\left(m+\frac{3}{2}\right)}\left[\left(m-\frac{1}{2}\right)\sin\left(m+\frac{3}{2}\right)\vartheta - \left(m+\frac{3}{2}\right)\sin\left(m-\frac{1}{2}\right)\vartheta\right]. \tag{4}$$

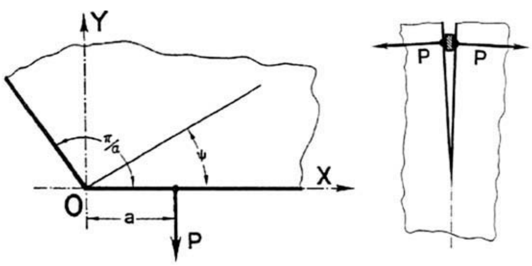

Fig. 2. Original drawings by Wieghardt [8] of the *generalized Flamant's problem*.

Although he mistakenly used only the terms leading to a nonsingular stress field, it is interesting to note that the dropped singular terms corresponding to $m = 0$ would yield a square-root singularity in compression, as for the well-known problem of normal indentation of the elastic half-plane by a flat-ended rigid punch.

In the meantime, a novel mathematical technique based on complex analysis for the study of plane elastic problems and developed by Kolosoff [14] in 1909 inspired a group of co-workers in the USSR. The resulting developments were described by Muskhelishvili in an outstanding work originally published in Russian [15], whose second edition written in English was available only in 1953. In 1939, Westergaard [16] developed a specific *Complex Potential Method* to handle with symmetric plane problems and applied this approach to the asymptotic analysis of the stress field at the tips of a crack embedded into an infinite plate. In his treatment, no reference was made to the work of Kolosoff [14] and Muskhelishvili [15], although we cannot exclude a possible influence. It is also remarkable to observe that the stress analysis at the crack tip for symmetric (Mode I) conditions was reproposed in 1942 by Muskhelishvili [17], who also extended the analysis to the skew-symmetric (Mode II) problem. These results had been preliminary published in Russian and then included by Muskhelishvili in the second edition of his fundamental book. Also in this case, no reference to the work by Westergaard was made. Only in 1966, Sih [18] revisited the Westergaard method in comparison with the Muskhelishvili solution, establishing appropriate and final modifications.

The asymptotic analysis of the singular stress field due to re-entrant corners or cracks in homogeneous, isotropic, linear elastic plates in extension was definitely solved by Williams [19] in the beginning of the 1950s (see Fig. 3). Fully recognizing the separable variable nature of the solution of the biharmonic equation, he proposed the *Eigenfunction Expansion Method*, in which the Airy stress function Φ is decomposed into a radial and an angular part [19,20]:

$$\Phi = \sum_j r^{\lambda_j+1} f_j(\vartheta, \lambda_j), \tag{5}$$

where λ_j and f_j are determined according to an eigenproblem and are, respectively, the eigenvalues and the eigenfunctions. The relative simplicity and ease of application of this mathematical approach as compared to the preceding methods was certainly an important aspect which contributed to its large diffusion.

On the other hand, the separable variable nature of the solution of the biharmonic equation was already an established fact in the context of Fluid Dynamics, where the two-dimensional Stokes flow of a viscous incompressible fluid can be described in terms of a stream function Ψ which obeys a biharmonic equation [21], as the Airy stress function in two-dimensional Linear Elasticity. In this context, Dean and Montagnon [22] had formulated an eigenfunction expansion method for the Stream function 4 years before Williams. Moreover, they applied this method to the analysis of the motion of a viscous fluid around a corner, which is a problem mathematically analogous to that of the re-entrant corner in plane elasticity, as shown in [4] (see Fig. 4).

Such developments in the asymptotic characterization of the stress field at the singular points provided the basis for a discussion on the fundamental issues of crack initiation and propagation. Wieghardt [8] firstly observed that the exact knowledge of the radial and angular variation of the components of the singular stress field can "... provide answers to more questions one might pose regarding the strength of our crack against the action of the force. One may ask: given the strength parameters of our elastic material, what is the magnitude of the force P necessary for material fracture? And furthermore, at which place and in which direction will the fracture initiate?".

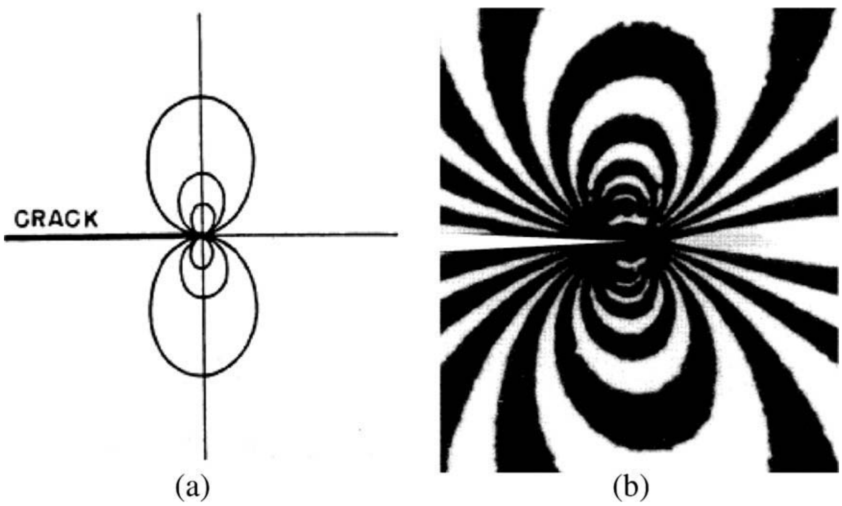

Fig. 3. Isochromatic fringe pattern at the crack tip for the symmetric case: (a) analytical prediction by Williams [20] and (b) experimental results according to photoelasticity.

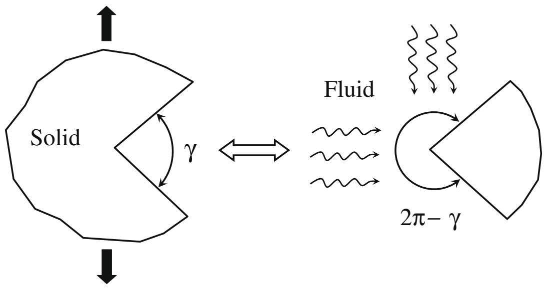

Fig. 4. Analogy between the asymptotic analysis of the stress field at the tip of a re-entrant corner in Linear Elasticity and the asymptotic analysis of the fluid velocity in a viscous fluid around a corner.

The Williams's paper [19] on stress-singularities was considered by Neuber [23] in 1958, who addressed the problem of material strength in the presence of a singular stress field by introducing the concepts of fictitious notch rounding and micro-structural support length. These concepts have influenced the standards for fatigue strength assessment of welded joints [24], recommending the use of a fictitious notch radius of 1 mm when the real radius of the weld toe is equal to zero. Indeed, the concepts of micro-structural support length and elementary structural volume received a great attention from the scientific community. Sih [25] proposed a failure criterion based on the strain energy density factor, given by the product of the strain energy density and a critical distance from the point of stress-singularity. Further developments of this theory for cracked and notched components under fatigue loads were provided by Lazzarin and Berto [26]. The concept of volume energy density function was recently revisited by Sih and Tang [27,28], by developing a multiscale approach where the local damage at the different scales is modeled by different orders of the stress-singularities related to the different constraint associated with the damage.

Another fundamental advancement in this research field is due to Griffith [29], who introduced in 1921 energy considerations into the analysis of the fracture phenomenon. He showed that the elastic strain energy released by a uniformly extended plate of unit thickness when a crack of length $2a$ is formed, and the displacements at infinity are kept constant, is proportional to the energy contained in the circle of radius a before the crack originates. Thus, he established the well-known relationship among the crack length, a, the unit surface energy connected with the creation of traction-free crack surfaces, $\gamma = \mathcal{G}_{IC}/2$, and the critical stress, σ:

$$\sigma = \left(\frac{\gamma E}{\pi a}\right)^{1/2}. \tag{6}$$

A reconciliation between stress and energy approaches was achieved by Irwin [30] in 1957. Using the Westergaard's solution for the asymptotic stress field at the crack tip, he showed that the stress field in that area is completely determined by the stress-intensity factor, ending up with the following fundamental relationship between strain energy release rate, $\mathcal{G}_I$, and stress-intensity factor, K_I (Mode I):

$$\mathcal{G}_I = \frac{K_I^2}{E}, \tag{7}$$

where, in plane stress condition, the constant of proportionality is represented by Young's modulus of the material. The relevance of this equation was well described by a sentence of V.V. Bolotin: "... the elegance of this formula is only comparable to $E = mc^2$ ..." (CISM Course on "Nonlinear Fracture Mechanics" held in Udine, 1989).

Another important progress in the study of Fracture Mechanics regarded the concept of crack propagation vs. plastic flow collapse, and thus the reasonability of infinite stresses in the Theory of Elasticity. Considering the actual distribution of stresses ahead of the crack tip, Irwin [31] proposed an accurate formula for the evaluation of the plastic radius, r_p. Hence, brittle fracture is usually expected when the extension of the plastic zone is small as compared to the crack length and to the reference structural dimension, i.e. when $r_p \ll \{a, h\}$. This result, which led to the fundamental concept of *small scale yielding*, was put at the basis for the subsequent developments in fracture mechanics testing of materials. A step forward with respect to the small scale yielding condition was provided by Carpinteri [32–34], who introduced a brittleness number, $s = K_{IC}/\sigma_p h^{1/2}$, to describe the scale transition from brittle failure to plastic flow collapse. Brittle failure predicted according to Linear Elastic Fracture Mechanics would prevail over plastic flow collapse in bending when $s \leqslant s_0$, where s_0 is the lowest brittleness number corresponding to the unconditioned formation of a plastic hinge. This formulation shows that brittle failure tends to occur with low fracture toughness, high yield strength and/or large structural sizes. Therefore, it is not the individual values of these three parameters that are responsible for the nature of the collapse mechanism, but rather only their combination expressed by the parameter s. It is interesting to note that, as s accounts for the effect of the specimen size and rules the transition from small-scale yielding to large-scale yielding, another dimensionless number can be introduced accounting for the effect of initial crack length. In this instance, we should consider the ratio between the initial crack length

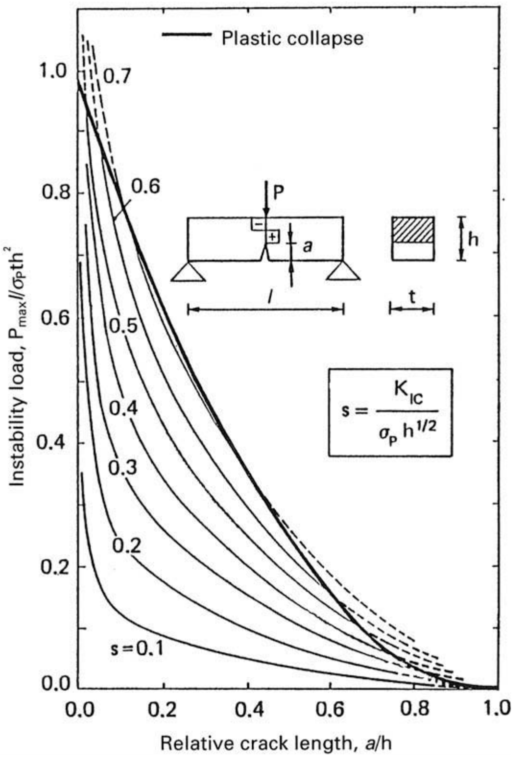

Fig. 5. Ductile collapse vs. brittle failure for three-point bending beams [34].

and the El Haddad material length scale a_0 [35]. This number, called Π_6 in [36], governs the transition from short-cracks to long-cracks in fatigue.

As an example, let us consider the case of three-point bending of a beam (Fig. 5). The condition for brittle crack propagation ($K_I = K_{IC}$) is given by:

$$K_{IC} = \frac{P_{max} l}{t h^{3/2}} f\left(\frac{a}{h}\right), \tag{8}$$

which can be recast in nondimensional form:

$$\frac{P_{max} l}{\sigma_p t h^2} = \frac{s}{f\left(\frac{a}{h}\right)}, \tag{9}$$

where σ_p is the yield strength, h is the beam depth, t is the beam thickness, l is the beam span, f is a shape function depending on the relative crack length and s is the brittleness number. On the other hand, the force P which potentially produces plastic collapse is that corresponding to the plastic hinge formation at the ligament:

$$\frac{P_{max} l}{4} = \sigma_p t \frac{(h-a)^2}{4}, \tag{10}$$

which can also be recast in nondimensional form:

$$\frac{P_{max} l}{\sigma_p t h^2} = \left(1 - \frac{a}{h}\right)^2. \tag{11}$$

Eqs. (9) and (11) are plotted in Fig. 5 as functions of the relative crack depth, a/h. For this structural geometry, the brittleness number which marks the transition from ductile collapse to brittle failure is $s_0 \cong 0.75$. For this value, the fracture curve is tangent to the plastic collapse curve.

Regarding the recent research trends on asymptotic techniques, it has to be remarked that the crack-contact analogies received a great attention from the scientific community. The original analysis of Williams [19], relating to the corner of a single wedge, may be used to analyze the fretting problem when the coefficient of friction is sufficiently high to cause adhesion (see also [37]). An early attempt to employ asymptotic solutions to fretting fatigue was the work by Giannakopoulos et al. [38,39]. They established an analogy between the crack tip stress field and that arising at the corner of a complete contact [38]. Further developments of the asymptotic solutions have been carried out for punches elastically dissimilar to the contacting surface [40], for frictionless contact [41] and frictional contact [42,43] (see also [44] for a detailed review of the different types of asymptotic solutions and how they may be usefully employed for the quantification of nucleation of fretting fatigue cracks). An asymptotic approach to crack initiation in fretting fatigue of complete contacts with a detailed analysis of the spatial variation of the stresses was provided by Mugadu et al. [45]. Asymptotic results for sliding contacts are available in [46]. The characterization of the process zone in almost complete frictional contacts involving sharp square-ended finite rigid punch or with a small edge radius were proposed in [47,48]. Asymptotic solutions of contact problems for bi- or three-dimensional elastic bodies were recently provided by Argatov [49,50], using the method of matched asymptotic expansions.

Another recent line of research worth mentioning is the development of analytical methods for the study of perturbation problems of interfacial cracks. Willis and Movchan [51] extended the pioneering studies on 3D interfacial cracks by Willis [52,53] when the front of the crack deviates from its original canonical shape. Further results on high-order asymptotics and perturbation problems have recently been proposed by Bercial-Velez et al. [54]. The variation of the stress-intensity factor arising from an infinitesimal coplanar perturbation of the crack front of an interfacial wavy crack has also been recently examined in [55].

3. Attenuation of the elastic singularities

In this section we shall briefly summarize the main geometrical configurations and mechanical conditions that can be used to relieve or remove the elastic singularities.

3.1. Re-entrant corners

Considering a two-dimensional elastic structure with a re-entrant corner of amplitude γ, Williams [19] proved that, when both notch surfaces are stress-free, the Mode I stress field components at the notch tip behave as follows:

$$\sigma_{ij} = K_{\mathrm{I}}^{*} r^{-(1-\lambda_{\mathrm{I}})} f_{ij}(\theta), \tag{12}$$

where K_{I}^{*} was referred to as *generalized stress-intensity factor* in [56,57], with the anomalous physical dimensions of $[K_{\mathrm{I}}^{*}] = [F][L]^{-(1+\lambda_{\mathrm{I}})}$. The exponent λ_{I} ranges between 1/2 (when $\gamma = 0$) and 1 (when $\gamma = \pi$), whereas $f_{ij}(\theta)$ is an angular function. Therefore, the power of the stress-singularity is a decreasing function of the re-entrant corner angle γ, as illustrated in Fig. 6a.

Applying Buckingham's theorem for physical similitude and scale modeling and considering the applied stress, σ, and the characteristic structural size, h, as fundamental quantities, Carpinteri [57] found that the generalized stress-intensity factor behaves as $K_{\mathrm{I}}^{*} \sim \sigma h^{(1-\lambda_{\mathrm{I}})}$. When the angle γ vanishes, the stress-singularity corresponds to that of a crack in a homogeneous material and the physical dimensions of K_{I}^{*} coincide with those of an ordinary stress-intensity factor, i.e. $[K_{\mathrm{I}}^{*}] = [F][L]^{-3/2}$. On the other hand, when $\gamma = \pi$, the stress-singularity disappears and the generalized stress-intensity factor K_{I}^{*} assumes the physical dimensions of stress, i.e. $[K_{\mathrm{I}}^{*}] = [F][L]^{-2}$.

This transition in the physical dimensions by varying the notch angle has an important consequence for the size-scale effects. In fact, the criticality condition corresponding to $K_{\mathrm{I}}^{*} = K_{\mathrm{IC}}^{*}$ would yield the following scaling law for the stress of failure, σ_f:

$$\log \sigma_f \sim -(1-\lambda_{\mathrm{I}}) \log h. \tag{13}$$

Therefore, if we keep material and structural shape constant and take into consideration a set of geometrically similar structures, then the strength $\log \sigma_f$ results into a linear decreasing function with negative slope $(1-\lambda_{\mathrm{I}})$ of the scale-parameter $\log h$ (see Fig. 6b). When $\gamma \to \pi$, i.e. when $(1-\lambda_{\mathrm{I}}) \to 0$, any scale effect vanishes and the straight line becomes horizontal.

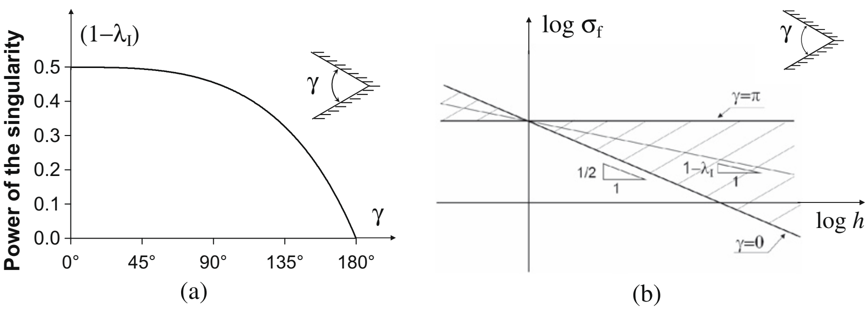

Fig. 6. (a) Power of the singularity for a re-entrant corner vs. notch angle. (b) Failure stress vs. reference structural size for different notch angles.

In the middle 1980s, dealing with high cycle fatigue regime, a stress field criterion was proposed also by Atzori [58] to estimate the strength of mechanical components weakened by V-notches with a root radius equal to zero or close to zero. The fatigue strength was estimated on the basis of straight lines in log–log diagrams relating the stresses to the distance from the point of singularity. A comparison between geometrically similar notched components was performed on the basis of the obtained parallel lines, whose slope made it possible to quantify the size-scale effect.

Finally, we remark that Gómez and Elices [59,60] developed a procedure based on the cohesive crack model to estimate the critical loads for V-notched specimens with a nonzero notch-root radius. Their numerical procedure is able to recover sharp V-notched results, obtained using Linear Elastic Fracture Mechanics, as an asymptotic limit when the notch-root radius tends to zero. Further results on size-scale effects on strength of quasi-brittle structural components with re-entrant corners have also been recently provided by Bažant and Yu [61].

3.2. Power-law hardening materials

Cracks propagating into power-law hardening materials are another notable example where the square-root stress-singularity is modified. These materials are characterized by the Ramberg–Osgood stress–strain power-law:

$$\tilde{\epsilon} = \tilde{\sigma}^n, \tag{14}$$

where $\tilde{\epsilon}$ and $\tilde{\sigma}$ denote the nondimensional stress and strain tensor components with respect to the yield stress and the yield strain, respectively. A variation in the exponent n from unity to infinity permits to cover the whole range from a linear elastic material up to a rigid perfectly-plastic material in a nondimensional stress–strain diagram (see Fig. 7a).

In this case, Rice and Rosengren [62] and Hutchinson [63] demonstrated that the governing equation for the Airy stress function is no longer biharmonic, as in the previous situation, but it depends on the exponent n:

$$\begin{aligned} &\frac{E}{E^*\sigma_y L^2}\nabla^4\Phi - \frac{\beta}{3\sigma_y L^2}\left\{\left[\frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2} - \frac{1}{\rho}\frac{\partial}{\partial\rho}\right] \times \left[\frac{1}{\tilde{\sigma}_e}\left(\frac{2}{\rho^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{2}{\rho}\frac{\partial\Phi}{\partial\rho} - \frac{\partial^2\Phi}{\partial\rho^2}\right)\right]\right. \\ &\left. + \frac{1}{\rho}\frac{\partial^2}{\partial\rho^2}\left[\frac{1}{\tilde{\sigma}_e}\left(2\rho\frac{\partial^2\Phi}{\partial\rho^2} - \frac{\partial\Phi}{\partial\rho} - \frac{1}{\rho}\frac{\partial^2\Phi}{\partial\theta^2}\right)\right] + \frac{6}{\rho^2}\frac{\partial^2}{\partial\rho\partial\theta}\left[\frac{\rho}{\tilde{\sigma}_e}\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial\Phi}{\partial\theta}\right)\right]\right\} = 0, \end{aligned} \tag{15}$$

where E^* is the tangent Young's modulus of the yielded or damaged material, σ_y is the yield strength, $\tilde{\sigma}_e$ is the Von Mises equivalent stress, L is a reference length, $\rho = r/L$ is a nondimensional radial coordinate and β is given by:

$$\begin{aligned} &\beta = \tfrac{3}{2}\left(\tfrac{E}{E^*} - 1\right) && \text{if } \tilde{\sigma}_e > 1, \\ &\beta = 0 && \text{if } \tilde{\sigma}_e \leqslant 1. \end{aligned}$$

Considering a separable variable solution, Rice and Rosengren [62] and Hutchinson [63] found the following expression for the singular stress field components:

$$\sigma_{ij} \sim K_{\mathrm{I}}^* r^{-1/(n+1)}, \tag{16}$$

where K_{I}^* was referred to as *plastic stress-intensity factor* in [64]. The anomalous physical dimensions of this parameter were also recognized in [64], i.e. $[K_{\mathrm{I}}^*] = [F][L]^{-(2n+1)/(n+1)}$. Also in this case, a transition from the physical dimensions of a stress-intensity factor ($[K_{\mathrm{I}}^*] = [F][L]^{-3/2}$) to those of a stress ($[K_{\mathrm{I}}^*] = [F][L]^{-2}$) are observed when n increases from unity to infinity (see also Fig. 7b showing the relationship between the power of the singularity and the exponent n). Correspondingly, an attenuation of the scale effects is expected, in close analogy with the behavior of re-entrant corners.

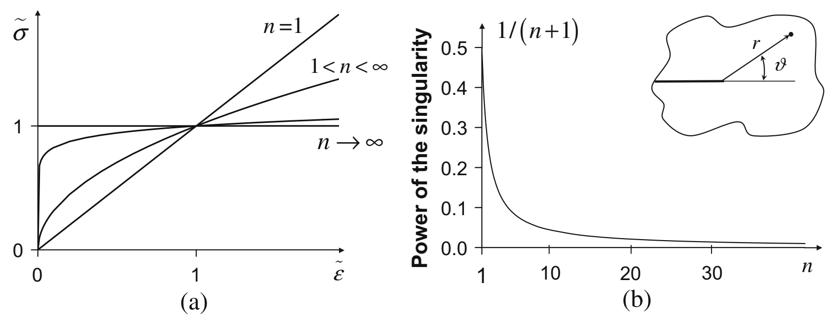

Fig. 7. (a) Stress–strain relationship for a power-law hardening material. (b) Power of the singularity as a function of the Ramberg–Osgood exponent n.

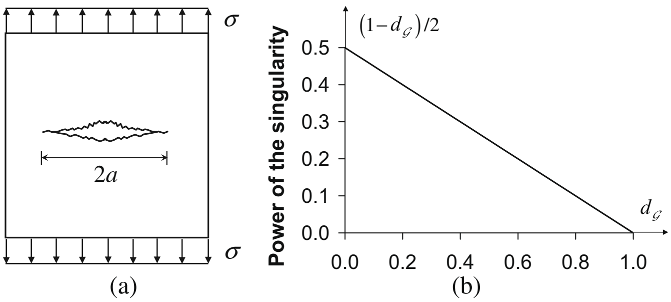

Fig. 8. (a) Fractal crack in a plate in tension. (b) Power of the singularity as a function of the dimensional increment $d_{\mathscr{G}}$.

3.3. Fractal character of the crack surfaces

Another notable example of attenuation of the square-root singular behavior can be observed by considering the fractal character of the crack surfaces [65,66]. Modeling the crack surface as an invasive self-similar fractal, Goldstein and Mosolov [67] and Carpinteri [65,66] independently extended the Griffith's energetic approach to the problem of an infinite plate of unit thickness, containing a fractal crack of projected length $2a$ and subjected to a remote tensile stress σ (see Fig. 8a).

The elastic strain energy release caused by the crack depends on its Euclidean length, that is:

$$W_e = \frac{\pi\sigma^2 a^2}{E}. \tag{17}$$

On the other hand, the energy dissipated on the surface of the fractal crack can be expressed using the renormalized fracture energy $\mathscr{G}_F^*$ [65,66]:

$$W_s = 2\mathscr{G}_F^* a^*, \tag{18}$$

where $2a^*$ is the fractal crack measure equal to $2a^{1+d_{\mathscr{G}}}$, where $d_{\mathscr{G}}$ ranges between 0 and 1 and is directly related to the fractal dimension of the crack profile, $(1 + d_{\mathscr{G}})$. Consequently, the variation $\mathrm{d}W_e$ in the elastic strain energy release for an infinitesimal crack extension is defined with respect to the Euclidean crack length $(2a)$, while the variation $\mathrm{d}W_s$ in the energy dissipation refers to the fractal crack measure $(2a^*)$. Differentiating W_e and W_s with respect to a and a^*, respectively, and considering the energy balance, Carpinteri [65,66] obtained the following expression:

$$\mathrm{d}W_e = \mathrm{d}W_s \Rightarrow \sigma^2\pi a^{1-d_{\mathscr{G}}} = (1 + d_{\mathscr{G}})\mathscr{G}_F^* E. \tag{19}$$

The renormalized critical condition for crack growth is therefore:

$$(K_{\mathrm{I}}^*)^2 = (K_{\mathrm{IC}}^*)^2, \tag{20}$$

where the following renormalized (scale-invariant) quantities can be defined [65]:

$$K_{\mathrm{I}}^* = \sigma\sqrt{\pi a^{1-d_{\mathscr{G}}}} = K_{\mathrm{I}} a^{-d_{\mathscr{G}}/2}, \tag{21}$$

$$K_{\mathrm{IC}}^* = \sqrt{(1 + d_{\mathscr{G}})\mathscr{G}_F^* E}. \tag{22}$$

The renormalized quantity K_{I}^* has the physical dimensions $[F][L]^{-(3+d_{\mathscr{G}})/2}$. Therefore, also in this case we have a transition from the physical dimensions of a stress-intensity factor $([K_{\mathrm{I}}^*] = [F][L]^{-3/2})$ when $d_{\mathscr{G}} \to 0$, to those of a stress $([K_{\mathrm{I}}^*] = [F][L]^{-2})$ when $d_{\mathscr{G}} \to 1$. From dimensional analysis considerations, the asymptotic stress field becomes:

$$\sigma_{ij} \sim K_{\mathrm{I}}^* r^{-\left(\frac{1-d_{\mathscr{G}}}{2}\right)} \tag{23}$$

and the power of the stress-singularity depends on $d_{\mathscr{G}}$, as shown in Fig. 8b.

4. Multi-material wedges and junctions

The geometry of a plane elastostatic problem consisting of n dissimilar isotropic, homogeneous wedges of arbitrary angles perfectly bonded along their interfaces which converge to the same vertex O is depicted in Fig. 9 and is another notable example of occurrence of stress-singularity. Each of the material sectors is denoted by Ω_i with $i = 0, \ldots, n-1$, and it is comprised between the interfaces Γ_i and Γ_{i+1}. The first and the last interfaces, defined by $\theta = 0$ and $\theta = 2\pi$, coincide and are referred to as Γ_0. When the material wedges are joined together with a total wedge angle less than 2π, this situation is usually referred to as *multi-material wedge*. On the contrary, the terminology of *multi-material junction* is preferred when the total

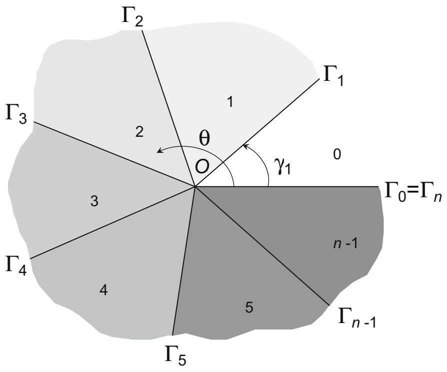

Fig. 9. Scheme of a multi-material wedge or junction.

wedge angle formed by the material regions equals 2π, i.e. the whole plane is occupied by the elastic materials without any empty sector.

Multi-material wedges and junctions are very commonly observed in composite materials and are the subject of extensive research (see e.g. [68–72,4], among others). Different mathematical methods have been employed for the determination of the power of the stress-singularity at the vertex *O* of the material wedges, namely the *Eigenfunction Expansion Method*, the *Complex Function Representation* and the *Mellin Transform Technique*. Although it is not the purpose of this paper to compare all of these mathematical formulations, it is remarkable to notice that there has been a long debate on their equivalence (see e.g. [4,73,72]). In these configurations, the singular components of the stress field at the vertex *O* behave as follows:

$$\sigma_{ij} = K^* r^{-(1-\mathrm{Re}\lambda)} f_{ij}(\theta), \tag{24}$$

where λ is a complex eigenvalue and f_{ij} is the corresponding eigenfunction.

Several experimental studies have investigated the use of an interface corner stress-intensity factor to predict the failure of bi-material joints. Gradin [74] tested three different types of three-layers laminates subjected to various loading conditions and proposed a static failure criterion for edge-bonded bi-material joints. Hattori et al. [75] formalized a method based on the notch stress-intensity factors for evaluating the adhesive strength of bi-material joints used in electronic devices and subjected to thermo-elastic stresses. On the basis of these results, Reedy [76] and Reedy and Guess [77] found the size of the regions dominated by stress-singularities and quantified the ability of the notch stress-intensity factors to characterize the extend of adhesive yielding for butt tensile joints.

The full appreciation of the power of the stress-singularity for these problems is particularly important from the engineering point of view, since a proper design of the joint geometry, as well as the choice of the materials may permit to relieve

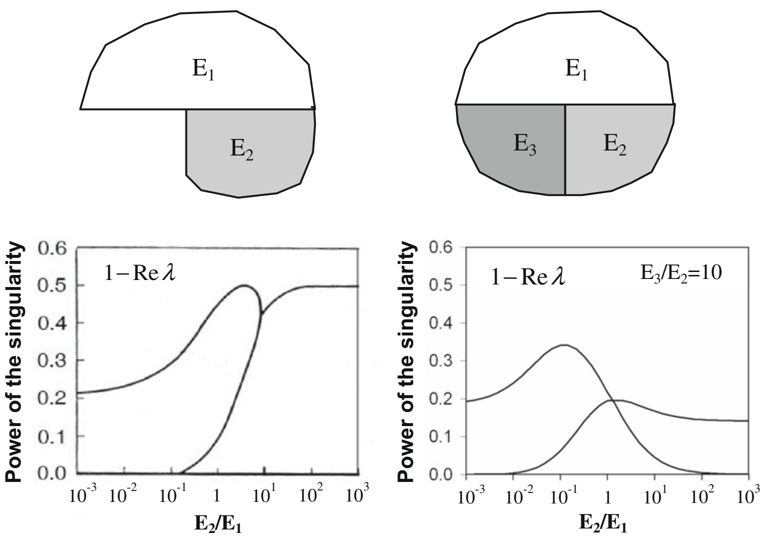

Fig. 10. Powers of the stress-singularity for a bi-material wedge and a tri-material junction as functions of the ratio between Young's moduli of materials 1 and 2.

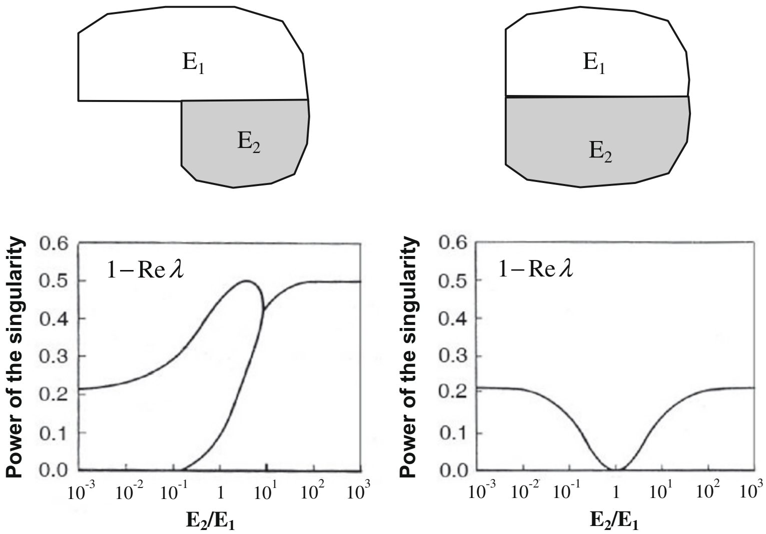

Fig. 11. Powers of the stress-singularity for two bi-material wedges as functions of the ratio between Young's moduli of materials 1 and 2.

the singularities. As illustrative examples, let us consider the material junctions and wedges shown in Figs. 10 and 11. In the former case, joining a third material with $E_3/E_2 = 10$ is favorable, since it permits to reduce the power of the stress-singularity for any Young's modular ratio E_2/E_1. On the other hand, in the latter situation, the reduction of the power of the stress-singularity is achieved by modifying the bi-material wedge geometry, i.e. by changing the wedge angle of the material 1 from π to $\pi/2$.

5. Nonhomogeneous materials

Functionally graded materials (FGMs) are an illustrative example of advanced two-phases synthesized materials recently designed in such a way that the volume fractions of the constituents vary continuously along a certain direction to give a predetermined composition profile [78–82]. According to this cutting-edge technology, any desired grading on the mechanical properties, such as Young's modulus or the thermal expansion coefficient, can potentially be obtained. In 1983, Erdogan [83] stated "...if the crack is embedded into a nonhomogeneous medium with smoothly varying elastic properties, the square-root nature of the stress-singularity seems to remain unchanged". The square-root singularity was mathematically proven in 1987 by Eischen [84] by assuming the following particular expression for the elastic modulus variation:

$$E(r,\theta) = E_0\left(1 + rE_1(\theta) + \frac{r^2}{2}E_2(\theta) + O(r^3)\right), \quad r \to 0, \tag{25}$$

where $E_0 =$ constant and $E_1(\theta), E_2(\theta)$ are smooth, bounded functions of θ.

Carpinteri and Paggi [85] showed that the case of a pure angular grading on Young's modulus, i.e. $E = E(\theta)$, is not considered in the expression proposed by Eischen. In fact, when the elastic modulus in Eq. (25) is independent of r, it turns out to be equal to E_0, that is a constant. For this class of materials, that were referred to as *angularly nonhomogeneous materials* in [85],

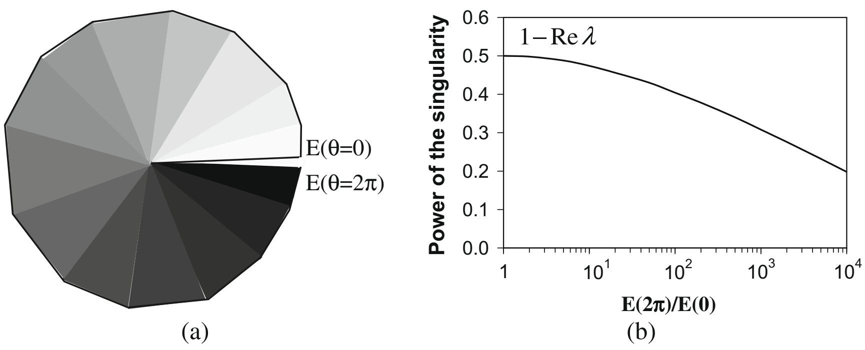

Fig. 12. (a) Scheme of a crack embedded into an angularly nonhomogeneous material. (b) Power of the stress-singularity as a function of the ratio between Young's moduli evaluated at the opposite crack surfaces.

the governing PDE for the Airy stress-function is not biharmonic as for homogeneous materials, but depends on the angular variation of the Young's modulus:

$$\nabla^4\Phi + \left[\frac{2}{E^2}\left(\frac{dE}{d\theta}\right)^2 - \frac{1}{E}\left(\frac{d^2E}{d\theta^2}\right)\right]\left[\frac{1}{r^3}\frac{\partial\Phi}{\partial r} + \frac{1}{r^4}\frac{\partial^2\Phi}{\partial\theta^2} - \frac{\nu}{r^2}\frac{\partial^2\Phi}{\partial r^2}\right] + \frac{1}{E}\frac{dE}{d\theta}\left[\frac{2\nu}{r^2}\frac{\partial^3\Phi}{\partial r^2\partial\theta} - \frac{2}{r^4}\frac{\partial^3\Phi}{\partial\theta^3} - \frac{2}{r^3}\frac{\partial^2\Phi}{\partial r\partial\theta}\right]$$
$$+ \frac{1+\nu}{E}\left[\frac{dE}{d\theta}\left(-\frac{2}{r^2}\frac{\partial^3\Phi}{\partial r^2\partial\theta} + \frac{2}{r^3}\frac{\partial^2\Phi}{\partial r\partial\theta} - \frac{2}{r^4}\frac{\partial\Phi}{\partial\theta}\right)\right] = 0. \tag{26}$$

The application of the variable separation method permits to determine the singular stress components as $r \to 0$ and it was found that the power of the stress-singularity is influenced by the angular grading. Considering an exponential variation of Young's modulus with respect to θ, Carpinteri and Paggi [85] analytically determined the power of the stress-singularity as a function of the ratio between Young's moduli evaluated at the crack surfaces, i.e. at $\theta = 2\pi$ and at $\theta = 0$ (see Fig. 12a). This power ranges from 1/2 in correspondence of $E(2\pi)/E(0) = 1$, down to zero for $E(2\pi)/E(0) \to \infty$ (see Fig. 12b). Therefore, the use of the angular grading seems to be very promising in reducing the power of the stress-singularity. Further results on angularly nonhomogeneous materials were proposed in [86], as far as the plane wedge problem loaded by a concentrated force at its apex is concerned.

6. Conclusions

The state-of-the-art of stress-singularity identification in classical Elasticity includes a large variety of problems concerning in-plane loading of a plate, antiplane shear of a wedge, plate bending, axisymmetric torsion of a cylinder, axisymmetric axial loading at vertex and axisymmetric axial loading at a cylindrical boundary. For all of these classes of configurations, the asymptotic expression of the singular stress field can be found in the fundamental review articles by Sinclair [2,3]. A unification of the available mathematical techniques for the computation of the power of the stress-singularity in two-dimensional interface problems involving multi-material junctions and wedges has recently been proposed by Paggi and Carpinteri [4].

In the present paper, we have provided the reader with a self-consistent review of the main efforts and research developments towards a full appreciation of the mathematical and engineering relevance of stress-singularities. To keep the scope of the present article within reasonable limits, we have also restricted our attention to the main geometrical configurations and mechanical conditions that can be used to effectively relieve the power of the stress-singularities with respect to the well-known square-root singularity typical of Linear Elastic Fracture Mechanics. Among them, we have shown the effect of the notch angle, the influence of the exponent of the stress–strain relationship in power-law hardening materials, as well as the effect of the roughness of crack surfaces. Special attention has also been devoted to the problem of stress-singularities in multi-material junctions and wedges, a situation frequently observed in mechanical and composite engineering. Finally, a section on stress-singularities in nonhomogeneous materials has been presented, which is a new emerging research field, more and more appealing for the Scientific Community due to the enormous advances in materials processing achieved during the last few years. Clearly, much work remains to be done in this research area in order to fully elucidate the potentials of the use of functionally graded materials for the removal of stress-singularities in junction problems.

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