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A unified interpretation of the power laws in fatigue and the analytical correlations between cyclic properties of engineering materials

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ABSTRACT

A phenomenological description of the fatigue life of engineering components can be given either by plotting the applied stress range as a function of the total number of cycles to failure, i.e., according to the Wöhler's curve, or, after the advent of fracture mechanics, by plotting the crack growth rate in terms of the stress-intensity factor range, i.e., using the Paris' curve. In this work, an analytical approach is proposed for the study of the relationships existing between the Wöhler's and the Paris' representations of fatigue. According to dimensional analysis and the concepts of complete and incomplete self-similarity, generalized Wöhler and Paris equations are determined, which provide a rational interpretation to a majority of empirical power-law criteria used in fatigue. Then, by integration of the generalized Paris' law, the relationship between the aforementioned generalized representations of fatigue is established, providing the link between the cumulative fatigue damage and the fatigue crack propagation approaches. Moreover, paying attention to the limit points defining the range of validity of the classical Wöhler and Paris power-law relationships, whose co-ordinates are referred to as *cyclic* or *fatigue properties*, alternative expressions for the classical laws of fatigue are proposed. Finally, the correlations between such fatigue properties are determined according to theoretical arguments, giving an interpretation of the empirical trends observed in the material property charts.

1. Introduction

As admitted by Paris in a recent review [1], "a specific accumulation damage model for the computation of damage growth under a wide variety of service loads is still lacking" and "no computational model is entirely satisfactory today", although a general understanding of many aspects of fatigue crack growth was established since the early 1960s. We know that fatigue damage increases with applied cycles in a cumulative way, which may eventually lead to failure. To model this physical phenomenon, the existing approaches for the prediction of fatigue life can be distinguished in two main categories: those related to the Cumulative Fatigue Damage (CFD) approach, which is the traditional framework for fatigue strength assessment, and those based on the Fatigue Crack Propagation (FCP) approach, developed since the 1960s after the advent of fracture mechanics.

At present time, the CFD analysis based on the Wöhler or *S–N* curves [2] still plays a key role in predicting the life of components and structures subjected to field-load histories (a state-of-the-art review of the available models can be found in the review papers [3–5]). In the empirical *S–N* curve, the fatigue life, N, is related to the applied stress range, $\Delta\sigma$ or S, and a reasonable power-law approximation was discovered since 1910 by Basquin [6]. A schematic representation of a typical Wöhler's curve is shown in Fig. 1, where the cyclic stress range, $\Delta\sigma = \sigma_{\max} - \sigma_{\min}$, is plotted as a function of the number of cycles to failure, N. Note that the cyclic stress amplitude is defined as half the stress range, $\sigma_a = \Delta\sigma/2$, and that the loading ratio is the ratio between the minimum and the maximum applied stresses, $R = \sigma_{\min}/\sigma_{\max}$. In this diagram, we also introduce the range of stress at static failure, $\Delta\sigma_u = \sigma_{\max} - \sigma_{\min} = \sigma_u - \sigma_{\min} = (1 - R)\sigma_u$, where σ_u is the material *tensile strength*, and we define the *endurance* or *fatigue limit*, $\Delta\sigma_{fl}$, as the stress range that a sample will sustain without failure for $N_\infty = 1 \times 10^7$ cycles, which is a conventional value that can be thought of as "infinite" life. Fatigue criteria based on the CFD approach have the advantage that can be used for the fatigue life assessment of unnotched or welded specimens, but suffer from the significant deficiency that there is no consistent definition of failure. It may correspond to the appearance of the first detectable crack, although it may also be defined as when the actual failure of the structural component takes place.

With the advent of fracture mechanics, a more ambitious task was undertaken, i.e., to predict, or at least understand, the propagation of cracks. Plotting the crack growth rate, da/dN, as a function of the stress-intensity factor range, $\Delta K = K_{\max} - K_{\min}$, most of the experimental data can be well-interpreted in terms of a power-law relationship, i.e., according to the so-called Paris' law

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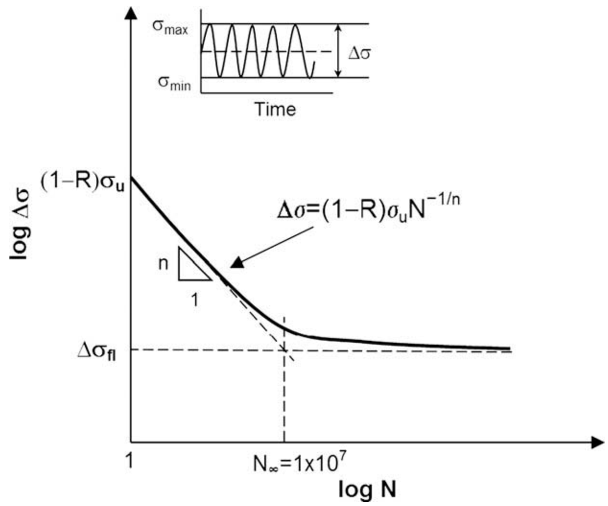

Fig. 1. Scheme of the Wöhler's curves with the corresponding fatigue parameters.

[7,8]. A schematic representation of a typical Paris' curve is shown in Fig. 2. Due to Linear Elastic Fracture Mechanics (LEFM), the loading ratio still corresponds to the ratio between the minimum and the maximum applied stresses, i.e., $R = K_{min}/K_{max} = \sigma_{min}/\sigma_{max}$. Note that the power-law representation presents some deviations for very high values of ΔK approaching $\Delta K_{cr} = (1 - R)K_{IC}$ [9,10], where K_{IC} is the material *fracture toughness*, or for very low values of ΔK approaching the *threshold stress-intensity factor range*, ΔK_{th}. Again, in close analogy with the concept of fatigue limit, the fatigue threshold is defined in a conventional way as the value of ΔK below which the crack grows at a rate of less than 1×10^{-9} m/cycle. The main drawback of this approach relies in the fact that the Paris' law is far from providing a universal representation of fatigue, since several deviations have been noticed in the last decades. Among them, the anomalous behaviour of short cracks is probably the most important aspect, which led to the development of more complicated fatigue crack growth criteria (see e.g. [11–17], among others). The behaviour of small fatigue cracks is a matter of importance not only because a major portion of the total fatigue life is spent in the propagation of small defects, but also because the boundary between propagation and nonpropagation separates the safe from the potentially unsafe fatigue regimes.

For a long time, the CFD and the FCP approaches have been considered as totally independent. The CFD criteria have been mainly confined to the fatigue life assessment of unnotched or welded components, where the elasto-plastic nature of damage, crack nucleation and crack initiation are important aspects, whereas the FCP models have been mainly applied to the long-crack regime, when the concept of small scale yielding holds and LEFM applies reasonably well. In spite of this, the differentiation of the two stages of crack nucleation and crack growth still remains "qualitatively distinguishable but quantitatively ambiguous", as recently stated by Jiang and Feng [18].

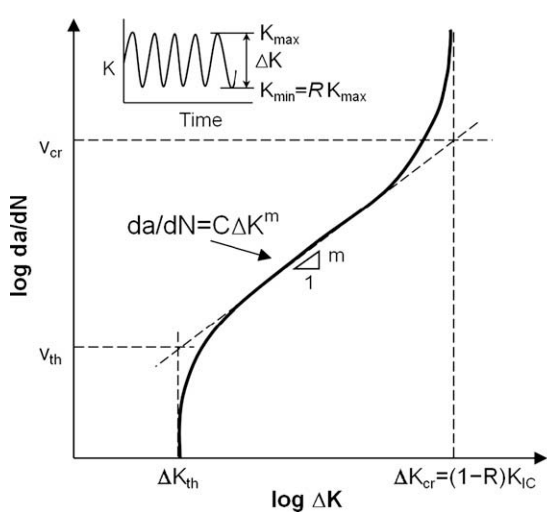

Fig. 2. Scheme of the Paris' curves with the corresponding fatigue parameters.

In the last few decades, the researchers have attempted to extend the field of application of the FCP approach. Among the various efforts, it is worth mentioning the contribution by McEvily and co-workers [17], who proposed a modified Paris' law dealing with the elasto-plastic behaviour of small cracks, and that by Atzori et al. [19], who proposed a method for the fatigue life prediction of welded joints based on the notch stress-intensity factor, overcoming some difficulties inherent in the fatigue life concept based on fracture mechanics. The effect of surface roughness was also modelled by Spagnoli [20,21] according to a fractal model, and a unified interpretation of the anomalous scaling laws in fatigue due to short cracks has recently been provided by Paggi and Carpinteri [22] according to fractal geometry. These advances in understanding the complex phenomenon of fatigue crack growth shed a new light on the possibility to unify the CFD and the FCP approaches, and to solve the challenging task of interpreting the Paris and Wöhler power-law regimes within a unified theoretical framework. A recent effort in this direction was given by Pugno et al. [23,24], who proposed a generalized Paris' law based on Quantized Fracture Mechanics for a unified treatment of long cracks, short cracks and fully yielded regimes. Their proposed generalized Paris' law would correspond to an intermediate asymptotic matching of the well-known empirical fatigue laws of Wöhler and Paris, obtained by using an increased crack size. Always in the same framework, Ciavarella and Monno [25] generalized the Kitagawa-Takahashi diagram and the El Haddad equation for nonpropagating cracks to finite life predictions.

In the present paper, we extend the dimensional analysis approach originally proposed by Barenblatt and Botvina [26–28] for the study of the size-scale effects on the Paris' law to derive generalized mathematical representations of the phenomenon of fatigue. Considering the crack growth rate as the representative parameter, a generalized Paris' law is derived by assuming incomplete self-similarity in the dimensionless variables governing fatigue crack growth. On the other hand, choosing the number of cycles as the representative parameter, a generalized Wöhler's curve is obtained by assuming again the condition of incomplete self-similarity in the dimensionless variables governing the fatigue response. It will be shown that such generalized representations cover almost all the main deviations from the empirical fatigue laws of Wöhler and Paris. For instance, the generalized Paris' law taking into account the effect of roughness of crack surfaces recently proposed in [22] can be regarded as one of the possible deviations from the classical power-law regime.

Moreover, by integration of the generalized Paris' equation and comparison with the generalized Wöhler's representation, the relationships existing between these two approaches is obtained. This will permit to interpret both FCP and CFD approaches within a unified theoretical framework. This is a step forward with respect to [23,24], since it will be proven that the relationship between the two regimes does exist on the basis of purely dimensional analysis arguments.

Finally, alternative expressions to the Paris' and Wohler's curves are provided in the corresponding fields of variation, replacing the parameters entering the power-law equations by the so-called static and fatigue properties, such as the *tensile strength*, the *fracture toughness*, the *fatigue limit* and the *threshold stress-intensity factor* *range*. In doing so, analytical correlations between the fatigue properties of engineering materials are determined and compared with the empirical trends proposed by Fleck et al. [29], giving a rational interpretation to the fundamental fatigue property charts.

2. Generalized mathematical representations of fatigue and their relationships

2.1. Generalized Paris' law

According to dimensional analysis, the phenomenon of fatigue crack growth can be regarded as a *black box* connecting the external variables (called input or governing parameters) with the mechanical response (output parameters). Following the pioneering work by Barenblatt and Botvina [26], we assume that the mechanical response of the system can be fully represented by the crack growth rate, $q_0 = \mathrm{d}a/\mathrm{d}N$, which is the parameter to be determined. This output parameter is a function of a number of variables:

$$q_0 = F(q_1, q_2, \ldots, q_n; s_1, s_2, \ldots, s_m; r_1, r_2, \ldots, r_k), \tag{1}$$

where q_i are quantities with independent physical dimensions, i.e., none of these quantities has a dimension that can be represented in terms of a product of powers of the dimensions of the remaining quantities. Parameters s_i are such that their dimensions can be expressed as products of powers of the dimensions of the parameters q_i. Finally, parameters r_i are nondimensional quantities. More specifically, the following functional dependence can be considered:

$$\frac{\mathrm{d}a}{\mathrm{d}N} = F(\sigma_u, K_{\mathrm{IC}}, \omega; \Delta K, \Delta K_{\mathrm{th}}, h, d, a; 1-R), \tag{2}$$

where the governing variables are summarized in Table 1, along with their physical dimensions expressed in the Length–Force–Time class (LFT).

From this list it is possible to distinguish between three main categories of parameters. The first category regards the material properties, such as the tensile strength, σ_u, and the fracture toughness, K_{IC}. In addition to such static properties, in this category we also include the threshold stress-intensity factor range which was not considered by Barenblatt and Botvina [26] in their treatment, although this parameter can be considered as a fundamental fatigue property [29]. The second category comprises the variables governing the testing conditions, such as the stress-intensity factor range, ΔK, the loading ratio, R, and the frequency of the loading cycle, ω. Concerning environmental conditions and chemical phenomena, they are not considered as primary variables in this formulation and their influence on fatigue crack growth can be taken into account as a degradation of the material properties. Finally, the last category includes geometric parameters related to the material microstructure, d, and to the tested geometry, such as the characteristic structural size, h, and the crack length, a. The microstructural dimension d corresponds to the grain size in metals, whereas it represents the size of the aggregates in concrete.

Table 1
Governing variables of the fatigue crack growth phenomenon.

Variable definition	Symbol	Dimensions
Ultimate tensile strength	σ_u	FL^{-2}
Fracture toughness	K_{IC}	$FL^{-3/2}$
Frequency of the loading cycle	ω	T^{-1}
Stress-intensity factor range	ΔK	$FL^{-3/2}$
Threshold stress-intensity factor range	ΔK_{th}	$FL^{-3/2}$
Stress range	$\Delta\sigma$	FL^{-2}
Fatigue limit	$\Delta\sigma_{fl}$	FL^{-2}
Characteristic structural size	h	L
Microstructural dimension (grain or aggregate size)	d	L
Crack length	a	L
Loading ratio	R	–

Considering a state with no explicit time dependence, it is possible to apply the Buckingham's Π Theorem [30] to reduce the number of parameters involved in the problem (see e.g. [31–35] for some relevant applications of this method in Solid Mechanics). As a result, we have:

$$\begin{aligned}\frac{\mathrm{d}a}{\mathrm{d}N} &= \left(\frac{K_{\mathrm{IC}}}{\sigma_u}\right)^2 \Phi\left(\frac{\Delta K}{K_{\mathrm{IC}}}, \frac{\Delta K_{\mathrm{th}}}{K_{\mathrm{IC}}}, \frac{\sigma_u^2}{K_{\mathrm{IC}}^2}h, \frac{\sigma_u^2}{K_{\mathrm{IC}}^2}d, \frac{\sigma_u^2}{K_{\mathrm{IC}}^2}a; 1-R\right)\\ &= \left(\frac{K_{\mathrm{IC}}}{\sigma_u}\right)^2 \Phi(\Pi_1, \Pi_2, \Pi_3, \Pi_4, \Pi_5, \Pi_6),\end{aligned} \tag{3}$$

where $\Pi_i\,(i = 1, \ldots, 6)$ are dimensionless numbers. Note that Π_3 corresponds to the square of the dimensionless number Z introduced by Barenblatt and Botvina [26] and to the inverse of the square of the brittleness number s introduced by Carpinteri [31–35]. The number Π_5 was firstly considered by Spagnoli [21] for the analysis of the crack-size dependence of the Paris' law parameters.

At this point, we want to see if the number of quantities involved in the relationship (3) can be reduced further from six. This can occur either in the case of *complete* or *incomplete self-similarities* in the corresponding dimensionless variables. In the former situation, the dependence of the mechanical response on a given dimensionless number, say Π_i, disappears and we can say that Π_i is *nonessential* for the representation of the physical phenomenon. In the latter situation, a power-law dependence on Π_i can be proposed, which usually characterizes a physical situation intermediate between two asymptotic behaviours. Considering the nondimensional number $\Pi_1 = \Delta K/K_{\mathrm{IC}}$, it has to be noticed that it rules the transition from the asymptotic behaviours characterized by the condition of nonpropagating cracks, when $\Delta K \to \Delta K_{\mathrm{th}}$, to the pure Griffith–Irwin instability, when $\Delta K \to \Delta K_{\mathrm{cr}}$. Moreover, incomplete self-similarity in Π_1 would correspond to a power-law dependence of the crack growth rate on the stress-intensity factor range, which is experimentally confirmed by the Paris' law [7,8]. Therefore, complete self-similarity in Π_1 cannot be accepted, whereas incomplete self-similarity gives:

$$\frac{\mathrm{d}a}{\mathrm{d}N} = \left(\frac{K_{\mathrm{IC}}}{\sigma_y}\right)^2 \left(\frac{\Delta K}{K_{\mathrm{IC}}}\right)^{\alpha_1} \Phi_1(\Pi_2, \Pi_3, \Pi_4, \Pi_5, \Pi_6), \tag{4}$$

where the exponent α_1 and, consequently, the nondimensional function Φ_1, cannot be determined from considerations of dimensional analysis alone. Moreover, the exponent α_1 may depend on the nondimensional parameters Π_i.

Incomplete self-similarity can also be assumed for the nondimensional variables Π_4, Π_5 and Π_6, obtaining the following generalized representation of fatigue crack growth:

$$\begin{aligned}\frac{\mathrm{d}a}{\mathrm{d}N} &= \left(\frac{K_{\mathrm{IC}}}{\sigma_u}\right)^2 \left(\frac{\Delta K}{K_{\mathrm{IC}}}\right)^{\alpha_1} \left(\frac{\sigma_u^2}{K_{\mathrm{IC}}^2}d\right)^{\alpha_2} \left(\frac{\sigma_u^2}{K_{\mathrm{IC}}^2}a\right)^{\alpha_3} (1-R)^{\alpha_4} \Phi_2(\Pi_2, \Pi_3)\\ &= \Delta K^{\alpha_1} d^{\alpha_2} a^{\alpha_3} (1-R)^{\alpha_4} \frac{\Phi_2(\Pi_2, \Pi_3)}{K_{\mathrm{IC}}^{\alpha_1+2\alpha_2+2\alpha_3-2}\sigma_u^{2(1-\alpha_2-\alpha_3)}}.\end{aligned} \tag{5}$$

As far as Π_1 is concerned, the assumption of incomplete self-similarity holds whenever ΔK is far lower than K_{IC}, i.e., for $\Pi_1 \ll 1$. As regards Π_4, we note that, according to Irwin, the ratio $(K_{\mathrm{IC}}/\sigma_u)^2$ is proportional to the plastic zone size, r_{p}. Hence, this number compares the microstructural length scale d with the plastic zone size. A power-law dependence on Π_4 is usually found in the LCF regime, where plastic deformations take place. This implies that incomplete self-similarity is attained for $d/r_{\mathrm{p}} \cong 1$, i.e., for $\Pi_4 \cong 1$. The same reasoning applies for Π_5, that is when the crack length is comparable with the process zone size in quasi-brittle materials or with the plastic zone size in metals (for an experimental confirmation, see

[36]). As regards Π_6, incomplete self-similarity is usually observed for $0 < \Pi_6 < 1$.

Comparing Eq. (5) with the expression of the classical Paris' law:

$$\frac{\mathrm{d}a}{\mathrm{d}N} = C\Delta K^m, \tag{6}$$

we find that our proposed formulation encompasses Eq. (6) as a limit case when:

$$m = \alpha_1, \\ C = d^{\alpha_2} a^{\alpha_3} (1-R)^{\alpha_4} \frac{\Phi_2(\Pi_2, \Pi_3)}{K_{\mathrm{IC}}^{\alpha_1+2\alpha_2+2\alpha_3-2} \sigma_u^{2(1-\alpha_2-\alpha_3)}} \tag{7}$$

Therefore, Eq. (5) can be considered as a generalized Paris' law, in which all the main functional dependencies of the parameter C have been fully explicitated. This generalized mathematical representation encompasses several improved versions of the Paris' law proposed in the past to cover specific anomalous deviations from the simplest power-law regime suggested by Paris. For instance, as far as the grain-size dependence of C is concerned, Chan [37,38] has recently demonstrated that the cycles to failure in many alloys is a decreasing function of the grain size, suggesting a power-law dependence of C on d as in Eq. (7), with an exponent α_2 related to the parameters of the Hall–Petch relationship.

Regarding the crack-size dependence of C, it is important to notice that this situation can be ascribed to all the cases where the power of the stress-singularity is different from 1/2. Notable examples are the roughness of the crack surfaces modelled in the framework of fractality [21,22,39], the asymptotic stress-field behaviour of welded joints [19], as well as the asymptotic field behaviour of re-entrant corners [40]. In the case of rough fractal cracks, the exponent α_3 can be theoretically related to the fractal dimension D of the crack trajectory and to the Paris' exponent m, i.e., $\alpha_3 = -(D-1)(1+m/2)$, where D usually ranges between 1 and 3/2. In the case of welded joints or re-entrant corners, the stress-intensity factor has to be generalized to deal with the asymptotic stress field, i.e., $\Delta K^* \propto \Delta K a^{-\lambda}$, where λ can be determined from an asymptotic analysis of the stress field and ranges between 0 and 1, depending on the amplitude of the notch angle and on the elastic mismatch between the joined materials [41]. Introducing the generalized stress-intensity factor in Eq. (6), one would obtain a generalized Paris' law with a coefficient C scaling with a as in Eq. (7) with $\alpha_3 = -m\lambda$. This anomalous dependence on the crack length cannot be obtained when the stress-intensity factor maintains its usual physical dimensions.

Modified Paris' laws taking into account the effect of the crack length have been proposed both for metals and quasi-brittle materials. For metals, several researchers have questioned the validity of the similitude hypothesis, which states that "two different sized cracks embedded into two different sized bodies subjected to the same stress-intensity factor range should grow at the same rate". As a support to the theories against the similitude hypothesis, we mention the experimental results by Newman et al. [42], who observed that "in the threshold regime there is something missing in the (closure) model", and those by Forth et al. [43], revealing that similitude does not hold in Region I (the near-threshold region) and also in the lower portion of Region II. To solve this problem, Molent et al. [44] and Jones et al. [45] have recently proposed a generalized Frost and Dugdale [46] crack growth law, assuming that the crack growth rate is proportional to the accumulated plastic strain, averaged over a characteristic length ahead of the crack tip, $\mathrm{d}a/\mathrm{d}N = C' a^{(1-m'/2)} \Delta K^{m'}$, where C' and m' are regarded as material constants. This equation states that $\mathrm{d}a/\mathrm{d}N$ is not only a function of the stress-intensity factor range, but also of the crack length. Such a generalized Frost and Dugdale crack growth equation was successfully used to predict the growth of near micron sized cracks in both coupon and full scale aircraft fatigue tests and interpret a large amount of experimental data that could not be modelled using the Paris' law. For concrete, a detailed experimental examination of crack propagation in flexural fatigue [47] has shown that the crack growth rate is not a monotonic increasing function of the crack length. For cracks shorter than the crack length at peak load in quasi-static monotonic loading, a *deceleration stage* was found, where $\mathrm{d}a/\mathrm{d}N$ is a decreasing function of a. Afterwards, an *acceleration stage* takes place and $\mathrm{d}a/\mathrm{d}N$ can be well approximated according to the classical Paris' law [47]. To model the deceleration stage, Kolloru et al. [47] proposed an empirical relationship between $\mathrm{d}a/\mathrm{d}N$ and the crack length, apparently independent of the Paris' law. Actually, it can be interpreted as a particular case of our proposed generalized Paris' law, simply allowing a crack-size dependence of the coefficient C, i.e., setting $C \propto a^{(n_1-n_2/2)}$, where n_1 and n_2 are the power-law exponents for the two regimes found in [47].

Finally, as far as the loading ratio is concerned, several Authors have proposed to include in the fatigue crack growth criterion both R and ΔK on an empirical basis [48–53]. They obtained the so-called "two-parameters" formulations with an exponent α_4 less than zero (for instance, $\alpha_4 = -1.38$ in [53]), confirming the experimental evidence that the crack propagation rate is an increasing function of the loading ratio.

2.2. Generalized Wöhler curve

In the previous section, the crack growth rate has been chosen as the main output parameter characterizing the phenomenon of fatigue crack growth. However, we can also consider the number of cycles, N, as the parameter representative of fatigue. Following this route, we can repeat the dimensional analysis approach of the previous section, considering the following functional dependence:

$$N = F(\sigma_u, K_{\mathrm{IC}}, \omega; \Delta\sigma, \Delta\sigma_{fl}, h, d, a; 1-R), \tag{8}$$

where the definitions of the governing variables are summarized in Table 1, along with their physical dimensions expressed in the Length–Force–Time class (LFT). In this list we have considered all the parameters already defined in the previous section, with the exception of the fatigue threshold stress-intensity factor range, ΔK_{th}, which has been replaced by its analogous counterpart used in the CFD approach, represented by the fatigue limit, $\Delta\sigma_{fl}$.

Considering a state with no explicit time dependence, it is possible to apply the Buckingham's Π Theorem [30] to reduce the number of parameters involved in the problem. As a result, we have:

$$N = \Psi\left(\frac{\Delta\sigma}{\sigma_u}, \frac{\Delta\sigma_{fl}}{\sigma_u}, \frac{\sigma_u^2}{K_{\mathrm{IC}}^2} h, \frac{\sigma_u^2}{K_{\mathrm{IC}}^2} d, \frac{\sigma_u^2}{K_{\mathrm{IC}}^2} a; 1-R\right) \\ = \Psi(\Pi_1, \Pi_2, \Pi_3, \Pi_4, \Pi_5; \Pi_6), \tag{9}$$

where Ψ is a nondimensional function. At this point, we want to see if the number of the quantities involved in the relationship (9) can be reduced further from six. In close analogy with the procedure discussed in the previous section for the Paris' law, we assume incomplete self-similarity in Π_1, Π_4, Π_5 and Π_6 (the same conditions apply to the dimensionless numbers as in the derivation of Eq. (5)), obtaining:

$$N = \left(\frac{\Delta\sigma}{\sigma_u}\right)^{\beta_1} \left(\frac{\sigma_u^2}{K_{\mathrm{IC}}^2} d\right)^{\beta_2} \left(\frac{\sigma_u^2}{K_{\mathrm{IC}}^2} a\right)^{\beta_3} (1-R)^{\beta_4} \Psi_2(\Pi_2, \Pi_3) \\ = \Delta\sigma^{\beta_1} d^{\beta_2} a^{\beta_3} (1-R)^{\beta_4} \frac{\Psi_2(\Pi_2, \Pi_3)}{K_{\mathrm{IC}}^{2(\beta_2+\beta_3)} \sigma_u^{\beta_1-2\beta_2-2\beta_3}}. \tag{10}$$

The exponents β_i cannot be determined from considerations of dimensional analysis alone and may depend on the nondimensional parameters Π_i. Equation (10) represents a generalized Wöhler relationship of fatigue and encompasses the empirical S–N curves as limit cases. For instance, the S–N curve in Fig. 1 can be approximated by the Basquin power law in the High Cycle Fatigue (HCF) regime, stating that:

$$1 \times \Delta\sigma_u^n = N_\infty \Delta\sigma_{fl}^n = N\Delta\sigma^n = k, \tag{11}$$

where $\Delta\sigma_u$ is the range of stress at static failure, $\Delta\sigma_{fl}$ is the fatigue limit corresponding to a conventional fatigue life of $N_\infty = 1 \times 10^7$ cycles, $\Delta\sigma$ is the stress range corresponding to a fatigue life N and k is a constant. Equating the first and the third terms in Eq. (11), we obtain he following power-law equations:

$$N = \left(\frac{\Delta\sigma_u}{\Delta\sigma}\right)^n = \frac{(1-R)^n \sigma_u^n}{\Delta\sigma^n}. \tag{12}$$

Similarly, the Coffin-Manson relationship $\Delta\varepsilon_p N^m = k_1$ used to describe the Low Cycle Fatigue (LCF) regime, where $\Delta\varepsilon_p$ is the plastic strain range, can be converted into a stress-based equation as Eq. (11). In fact, as shown by Murakami and Miller [54], during plastic deformation, it is possible to experimentally determine a stress–strain constitutive relationship of the type $\Delta\sigma/2 = k_3\Delta\varepsilon_p^l$, where l is a positive exponent equal to 0.26 for carbon steel. Introducing this expression into the Coffin-Manson relationship, the plastic strain range can be eliminated, obtaining again a stress-based power-law equation $N\Delta\sigma^n = k$, where we have set $n = 1/(lm)$. Clearly, the exponent n and the constant k can assume different values in the two regimes, that is, in the most general case, they may depend on $\Pi_1 = \Delta\sigma/\sigma_u$. Comparing the generalized expression of the S–N curve in Eq. (10) with the empirical one in Eq. (12), we find that a perfect correspondence exists when $\beta_1 = -n$, $\beta_2 = \beta_3 = 0$ and $\beta_4 = n$. From a dimensional analysis point of view, this result implies that the empirical Basquin power-law or the Coffin-Manson equation correspond to an incomplete self-similarity in Π_1 and Π_4, and a complete similarity in Π_2 and Π_3. It is important to notice that more recent theories have also attempted to generalize the S–N curve by including the effect of the initial crack size [25] (see also an experimental confirmation in [55]), or by incorporating the grain size in the mathematical formulation [36,37], giving rise to power-law representations that can be interpreted in the framework of Eq. (10) with a suitable choice of the exponents β_i.

2.3. Relationships between FCP and CFD approaches: from the Paris to the Wöhler curve

In the previous sections we have determined the generalized representations of fatigue according to dimensional analysis arguments. We have also shown that the empirical power-law relationships proposed in the Literature can be deduced by assuming incomplete self-similarity in the corresponding dimensionless variables. Therefore, if the classical Paris' and Wöhler curves apply for a specific range of variation in the dimensionless parameters, the observed experimental deviations invoke the use of more general expressions as those reported in Eqs. (5) and (10). However, such representations of fatigue are not uncorrelated, and we will show in this section that they are intimately connected to each other.

The cornerstone for determining the relationships existing between the CFD and the FCP approaches is represented by the integration of the generalized Paris' law in Eq. (5) between an initial defect size, a, and a generic final crack length, a_f, corresponding to a given fatigue life N. Recalling that $\Delta K = \Delta\sigma\sqrt{\pi a}$ for a Griffith crack, then we have:

$$\int_a^{a_f} a^{-\left(\frac{\alpha_1}{2}+\alpha_3\right)} \mathrm{d}a = \Delta\sigma^{\alpha_1} d^{\alpha_2} (1-R)^{\alpha_4} B \int_0^N \mathrm{d}N, \tag{13}$$

where we have set $B = \pi^{\alpha_1/2} \frac{\Phi_2(\Pi_2,\Pi_3)}{K_{IC}^{\alpha_1+2\alpha_2+2\alpha_3-2}\sigma_u^{2(1-\alpha_2-\alpha_3)}}$, which is constant with respect to the crack length and the cycles number. The integration gives the following result:

$$N = \frac{\Delta\sigma^{-\alpha_1} d^{-\alpha_2} (1-R)^{-\alpha_4}}{\left[1-\left(\frac{\alpha_1}{2}+\alpha_3\right)\right]B} \left[a_f^{1-\left(\frac{\alpha_1}{2}+\alpha_3\right)} - a^{1-\left(\frac{\alpha_1}{2}+\alpha_3\right)}\right], \tag{14}$$

which can be simplified by noting that $a_f^{1-\left(\frac{\alpha_1}{2}+\alpha_3\right)} \ll a^{1-\left(\frac{\alpha_1}{2}+\alpha_3\right)}$, since the exponent of the crack length is negative valued and $a_f \ll a$. Under such conditions, the fatigue life can be approximated as follows:

$$N \cong \frac{\Delta\sigma^{-\alpha_1} d^{-\alpha_2} (1-R)^{-\alpha_4} a^{1-\left(\frac{\alpha_1}{2}+\alpha_3\right)}}{\left[\left(\frac{\alpha_1}{2}+\alpha_3\right)-1\right]B}. \tag{15}$$

A comparison between Eq. (10), obtained according to dimensional analysis, and Eq. (15), obtained through the integration of the generalized Paris' law in Eq. (5), leads to the following relationships between the powers entering the two representations:

$$\begin{aligned} \beta_1 &= -\alpha_1 \\ \beta_2 &= -\alpha_2 \\ \beta_3 &= 1-\left(\frac{\alpha_1}{2}+\alpha_3\right) \\ \beta_4 &= -\alpha_4 \end{aligned} \tag{16}$$

and, after some manipulation, we also establish the relationship between the nondimensional functions Ψ_2 and Φ_2:

$$\Psi_2 = \left[\frac{\pi^{-\alpha_1/2}}{\left(\frac{\alpha_1}{2}+\alpha_3\right)-1}\right]\frac{1}{\Phi_2}. \tag{17}$$

An example of derivation of the S–N equation by integrating a crack-size dependent Paris' law can be found in [54], in perfect agreement with the procedure herein discussed.

3. Alternative representations in the classical power-law regimes and analytical correlations between the fatigue properties of engineering materials

3.1. Wöhler's curve and correlation for the fatigue strength exponent

Let us consider the limit points in the Wöhler's curve defining the range of validity of the power-law approximation relating the stress range, $\Delta\sigma$, to the cycles to failure, N, i.e., the points corresponding to the *cyclic stress at static failure*, $\Delta\sigma_u$, and to the *fatigue limit*, $\Delta\sigma_{fl}$. In this range, the S–N curve can be approximated by a simple equation fully characterized by its exponent n:

$$\Delta\sigma = \Delta\sigma_u N^{-1/n}. \tag{18}$$

Evaluating the S–N curve in correspondence of the fatigue limit, $\Delta\sigma_{fl} = \Delta\sigma_u N_\infty^{-1/n}$, a one-to-one relationship between the exponent n and the co-ordinates of this special point of the Wöhler's curve can be determined:

$$\frac{1}{n} = -\frac{\log\Delta\sigma_{fl} - \log\Delta\sigma_u}{\log N_\infty}, \tag{19}$$

where by definition, $N_\infty = 1 \times 10^7$ cycles corresponds to an "infinite" fatigue life. As a result, an alternative expression for the classical Wöhler's curve can be considered, where the exponent n is now written in terms of the fatigue properties $\Delta\sigma_u$ and $\Delta\sigma_{fl}$. In formulae, we have:

$$\Delta\sigma = (1-R)\sigma_u N^{\frac{\log\Delta\sigma_{fl}-\log\Delta\sigma_u}{\log N_\infty}}, \tag{20}$$

where the dependence on the loading ratio R has been explicitated.

3.2. Paris' curve and correlations for the parameters C and m

Let us consider the limit points of the Paris' curve defining the range of validity of the power-law approximation relating the crack growth rate, da/dN, to the stress-intensity factor range, ΔK. They correspond, respectively, to the points with horizontal co-ordinates equal to the *fatigue threshold*, ΔK_{th}, and to ΔK_{cr}, where the Paris' instability coincides with the Griffith–Irwin crack growth instability when K_{max} tends to the *fracture toughness*. In this range, the Paris' curve is usually defined in terms of the parameters C and m (see Eq. (6) and Fig. 2).

Now, let us consider the useful construction added with dashed line to Fig. 2, as proposed by Fleck et al. [29]. If a tangent is drawn at the mid-point of the central linear region of the curve and extrapolated, it is found empirically that it intersects the vertical line $\Delta K = \Delta K_{th}$ in correspondence to a crack growth rate of approximately $v_{th} = 1 \times 10^{-9}$ m/cycle, and it intersects the line $\Delta K = \Delta K_{cr} = (1 - R)K_{IC}$ at about $v_{cr} = 1 \times 10^{-5}$ m/cycle. Evaluating the Paris' law in correspondence of the second point, the following correlation between the parameters C and m of the Paris' curve can be obtained:

$$C = \frac{v_{cr}}{[(1-R)K_{IC}]^m}. \tag{21}$$

This result demonstrates that the phenomenon of fatigue crack growth can be described by only one independent parameter, such as the exponent m. This correlation proposed in [10,24] represents a step forward with respect to the empirical relationships previously proposed in the Literature and established only for certain materials and loading conditions (see e.g. the correlations by Tanaka [56] for steels and for a vanishing loading ratio, $R = 0$, and those determined by Radhakrishnan [53] for Al alloys and steels).

Repeating this reasoning for the point defined by the fatigue threshold, the following relationship can be obtained:

$$C = \frac{v_{th}}{(\Delta K_{th})^m}, \tag{22}$$

which establishes a link between the Paris' law parameter C and the co-ordinates of the point defining the condition of nonpropagating cracks. This relationship has also important consequences for the anomalous crack-size dependence of the fatigue threshold [22]. In fact, if we determine ΔK_{th} by inverting the Paris' law in correspondence of a conventional value of the crack growth rate, v_{th}, then we have $\Delta K_{th} \propto C^{-1/m}$, as comes from Eq. (22). Considering the crack-size dependence of the Paris' law parameter C, i.e., $C(a) \propto a^{\alpha_3}$ in the short crack regime, we have $\Delta K_{th} \propto a^{-\alpha_3/m}$. For $\alpha_3 = -1/2$, as proposed by Frost in 1966 [57], and $m = 3$ as a typical value for metals, we find $\Delta K_{th} \propto a^{1/6}$, which corresponds to the scaling law for ΔK_{th} suggested by Frost [57] and Murakami and Endo [58]. Actually, the exponent α_3 of the power-law can vary as a function of the scale of observation and a multi-fractal scaling law should be considered, as recently proposed in [22], which gives a rational interpretation to the empirical Kitagawa diagram [12,13].

In the long-crack regime, where the complete self-similarity in the initial crack size leads to $\alpha_3 = 0$, the equation set consisting in Eqs. (21) and (22) permits to express the remaining Paris' law parameter m as a function of the fatigue properties:

$$m = \frac{\log v_{th} - \log v_{cr}}{\log \Delta K_{th} - \log[(1-R)K_{IC}]}. \tag{23}$$

Moreover, equating the second members of Eqs. (21) and (22), we find that the ratio between the fatigue threshold and the fracture toughness is a function of the Paris' law parameter m, i.e.:

$$\frac{\Delta K_{th}}{K_{IC}} = (1-R)\sqrt[m]{\frac{v_{th}}{v_{cr}}}, \tag{24}$$

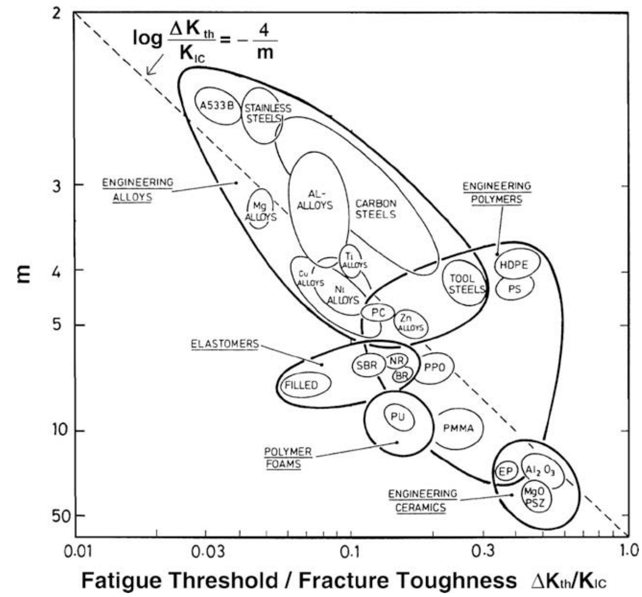

Fig. 3. Paris' law parameter m vs. $\Delta K_{th}/K_{IC}$ (material property chart adapted from [29]).

or, in logarithmic form:

$$\log\left(\frac{\Delta K_{th}}{K_{IC}}\right) = \log(1-R) + \frac{1}{m}\log\left(\frac{v_{th}}{v_{cr}}\right). \tag{25}$$

Eq. (25) establishes a one-to-one correspondence between ΔK_{th}, K_{IC} and m in the long-crack regime and was experimentally confirmed by Fleck et al. [29] for a wide range of materials. Considering the fatigue property chart reported in Fig. 3, we observe a very good agreement between the experimental trend and the proposed correlation, being $R = 0$ and $\log\ (v_{th}/v_{cr}) \cong \log\ (1 \times 10^{-9}/1 \times 10^{-5}) = -4$.

3.3. Correlation between the fatigue threshold and the fatigue limit

A relationship between the fatigue stress-intensity factor threshold and the fatigue limit can be obtained by considering the propagation of a Griffith crack of length $2a_0$ in an infinite elastic plate subjected to cyclic loading with $\Delta\sigma = \Delta\sigma_{fl}$ acting at infinity and $R = 0$. The initial crack length is chosen as representative of the size of the existing microdefects, i.e., $a_0 = (K_{IC}/\sigma_u)^2/\pi$.

According to LEFM, the ratio of the fatigue threshold range to the fracture toughness is equal to the ratio between the fatigue limit and the tensile strength of the material:

$$\left.\begin{aligned} \Delta K_{th} &= \Delta\sigma_{fl}\sqrt{\pi a_0} \\ K_{IC} &= \sigma_u\sqrt{\pi a_0} \end{aligned}\right\} \Rightarrow \frac{\Delta K_{th}}{K_{IC}} = \frac{\Delta\sigma_{fl}}{\sigma_u}. \tag{26}$$

Considering the classical Paris' equation written in terms of applied stress-range instead of the stress-intensity factor range and with $\Delta\sigma = \Delta\sigma_{fl}$:

$$\frac{da}{dN} = C\Delta K^m = C\left(\Delta\sigma\sqrt{\pi a}\right)^m = C\left(\Delta\sigma_{fl}\sqrt{\pi a}\right)^m, \tag{27}$$

we can integrate this expression from a_0 up to the length corresponding to the onset of rapid crack growth, i.e., up to $a_f = (K_{IC}/\Delta\sigma_{fl})^2/\pi$:

$$\int_{a_0}^{a_f} (\pi a)^{-m/2} da = C\int_0^{N_\infty} \Delta\sigma_{fl}^m dN, \tag{28}$$

where N_∞ denotes the number of fatigue cycles to failure, which is also the horizontal co-ordinate of the point corresponding to $\Delta\sigma_{fl}$ in the S–N curve. After integration we have ($m \neq 2$):

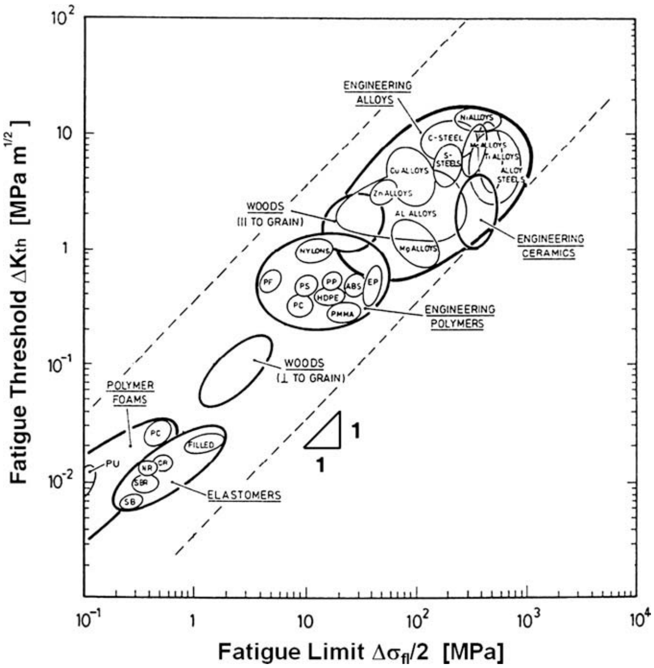

Fig. 4. Fatigue threshold vs. fatigue limit (material property chart adapted from [29]).

$$\frac{2\pi^{-m/2}}{2-m}\left(a_f^{\frac{2-m}{2}} - a_0^{\frac{2-m}{2}}\right) = C(\Delta\sigma_{fl})^m N_\infty. \tag{29}$$

Noting that usually $a_f \gg a_0$, and that $m > 2$, Eq. (29) can be simplified as follows:

$$C(\Delta\sigma_{fl})^m N_\infty \cong -\frac{2\pi^{-m/2}}{2-m} a_0^{\frac{2-m}{2}} = \frac{2}{\pi(m-2)}\left(\frac{K_{IC}}{\sigma_u}\right)^{2-m}, \tag{30}$$

where the definition of a_0 has been suitably introduced. Note that Eq. (30) can also be obtained from the generalized Wöhler curve in Eq. (10) in the short-crack regime, i.e., by setting $\alpha_2 = \alpha_3 = \alpha_4 = 0$. Eq. (30) permits to obtain a closed-form relationship between the fatigue limit and the fatigue threshold. In fact, considering Eqs. (22) and (26), we can relate, respectively, the parameter C to ΔK_{th} and the ratio K_{IC}/σ_u to the ratio $\Delta K_{th}/\Delta\sigma_{fl}$. Introducing such expressions into Eq. (30), we obtain, after some manipulation, the following approximate correlation between the fatigue threshold and the fatigue limit, strictly holding for $R = 0$:

$$\Delta K_{th} \cong \sqrt{\frac{\pi(m-2)\,\upsilon_{th} N_\infty}{2}}\Delta\sigma_{fl} \tag{31}$$

or, in logarithmic form:

$$\log \Delta K_{th} \cong \frac{1}{2}\log\left[\frac{\pi(m-2)\,\upsilon_{th} N_\infty}{2}\right] + \log \Delta\sigma_{fl}. \tag{32}$$

A direct comparison between this proposed correlation and the experimental trend observed for a wide range of materials and collected in the fatigue property chart by Fleck et al. [29] is proposed in Fig. 4. As can be seen, the analytically predicted linear relation between the fatigue threshold and the fatigue limit is correctly reproduced.

4. Conclusions

The Wöhler and Paris curves were originally thought as "universal laws", in the sense that they should be able to provide a universal description of fatigue. Actually, the experimentally observed deviations led to a proliferation of modified fatigue criteria, very often represented by power laws. Therefore, if on the one hand the research efforts were directed towards the extension of the original fields of application of the Wöhler and Paris representations of fatigue, on the other hand the fundamental problem of finding the link between the cumulative fatigue damage and the fatigue crack propagation approaches remained largely unsolved.

In the present contribution, a dimensional analysis approach and the concepts of complete and incomplete self-similarity have been applied to the Paris' curve, extending and generalizing the pioneering work by Barenblatt and Botvina [26], and, for the very first time, to the Wöhler curve. As a main conclusion, it has been shown that the large number of power laws used in fatigue are the result of incomplete self-similarity in the corresponding dimensionless variables. This gives a rational interpretation to such empirically-based fatigue criteria, towards a unified description of fatigue and a possible standardization. Moreover, the integration of the proposed generalized Paris' law and the comparison with the generalized Wöhler curve has permitted to find the relationship between these two representations of fatigue.

Alternative expressions of the classical Wöhler and Paris equations have also been proposed, where the parameters entering the power laws are rewritten in terms of the cyclic properties of engineering materials, that are true material parameters. In doing so, analytical correlations between the cyclic properties have been established, providing an analytical interpretation to the empirical correlations existing in the Literature and to the well-known fatigue property charts.

Unsolved aspects, which may require further investigation, are those related to the size-scale effects and to the influence of the environmental variables on fatigue. As regards the first aspect, Barenblatt and Botvina [26] have shown that the Paris' law parameter m is dependent on the dimensionless number Π_3 and Ciavarella et al. [59] have recently analyzed the size-scale effects on the Paris' law parameter C. The dimensionless number Π_3 appears also in the generalized Wöhler representation of fatigue and therefore it may influence the parameters of the Basquin and Coffin-Manson fatigue criteria. Concerning the second aspect, the Literature on thermal effects on fatigue and fatigue-creep interactions at elevated temperatures is enormous. The dimensional analysis interpretation of the empirical criteria proposed in these fields is still an open point.

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