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Static-kinematic fractional operators for fractal and non-local solids

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Two fractional calculus approaches in the framework of continuum mechanics are revisited and compared. The former is a local approach, which has been proposed to investigate the behaviour of fractal media. The latter is a non-local approach, according to which long-range interactions between material particles are opportunely modelled in the equilibrium equations. Analogies and differences between the two models are outlined.

1 Introduction

Aim of the paper is to revisit and compare two applications of fractional calculus to continuum mechanics. For the sake of simplicity, only one-dimensional problems will be dealt with.

The former approach is based on the local fractional calculus introduced by Kolwankar [14,15] to address the problem of fractal media, i.e. solids where the deformation is localized on a fractal subset. By such assumption, the displacement field is represented by devil staircase-like functions [4–7]. As well known, these functions have zero first derivative, except in an infinite number of points where they are not differentiable. On the other hand, they admit fractional derivatives of order less than the fractal dimension of the set where strain localizes. It is hence possible to express the fractal strain as the local fractional derivative of the displacement field.

The latter approach was introduced to model non-local continua [9], i.e. solids characterized by non-local interactions [1,10]. The novelty is that internal forces are described by fractional integrals and derivatives [9]. One of the most remarkable achievements of this approach is that, by the Marchaud definition of fractional derivative [20], the fractional operators have a clear mechanical interpretation, i.e. springs connecting non-adjacent points of the body. The related stiffness decays along with the distance among the material points. However, it is seen that, in order to have a consistent mechanical model, only the integral part of Marchaud derivative has to be retained.

The two approaches are finally discussed and compared.

For the sake of clarity, it is worth recalling the definitions of the left and right Riemann-Liouville integrals, $\mathrm{I}^{\alpha}_{a+}f$ and $\mathrm{I}^{\alpha}_{b-}f$, and derivatives, $\mathrm{D}^{\alpha}_{a+}f$ and $\mathrm{D}^{\alpha}_{b-}f$, respectively [20]. In particular, given a Lebesgue function $f(x)$ on the closed interval $[a,b]$, the fractional integrals are defined by ($\alpha > 0$)

$$\mathrm{I}^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{f(\xi)}{(x-\xi)^{1-\alpha}}\mathrm{d}\xi\,, \tag{1}$$

$$\mathrm{I}^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b \frac{f(\xi)}{(\xi-x)^{1-\alpha}}\mathrm{d}\xi\,, \tag{2}$$

while the fractional derivatives are given by ($0 < \alpha < 1$)

$$\mathrm{D}^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x \frac{f(\xi)}{(x-\xi)^{\alpha}}\mathrm{d}\xi\,, \tag{3}$$

$$\mathrm{D}^{\alpha}_{b-}f(x) = -\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}\int_x^b \frac{f(\xi)}{(\xi-x)^{\alpha}}\mathrm{d}\xi\,. \tag{4}$$

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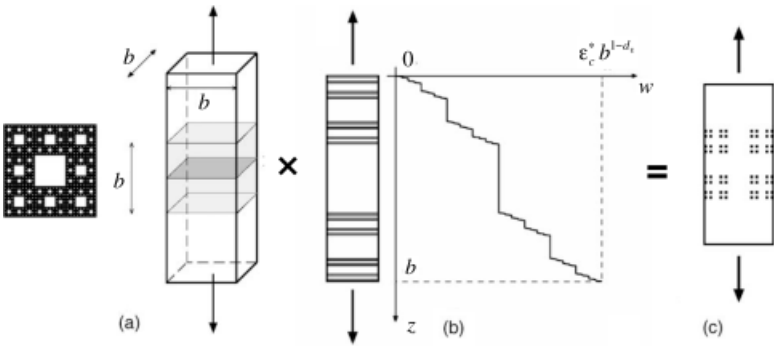

Fig. 1 Fractal localization of: stress (a), strain (b), energy dissipation (c).

2 Local approach

The mechanics of solids deformable over fractal subsets has been widely investigated [2–6]. When dealing with fractal media, mechanical quantities with anomalous physical dimensions, related to the fractal dimensions of the domain upon which they are defined, must be taken into account [2, 3]. Then, by means of the local fractional operators, the static and kinematic equations (as well as the principle of virtual work) may be derived [4]. Before entering into details, let us remind that we are referring to a generic body whose stress flux and deformation pattern have fractal characteristics, whereas the material itself does not need to present a fractal microstructure.

2.1 Fractal mechanics quantities

The singular stress flux through fractal media can be modelled by means of lacunar fractal sets of dimension $\Delta_\sigma = 2 - \mathrm{d}_\sigma \leq 2$ (Fig. 1a) and measure A^*, representing damaged cross sections. An original definition of the fractal stress σ^* was put forward in [3] by applying the renormalization group procedure to the nominal stress tensor $[\sigma]$. The fractal stress σ^*, whose dimensions are $[F][L]^{-(2-d_\sigma)}$, is a scale-invariant quantity.

The kinematical conjugate of the fractal stress σ^* is the fractal strain ε^*. The basic assumption is that displacement discontinuities can be localized on an infinite number of cross-sections, spreading throughout the body. This hypothesis has been suggested by several experimental investigations, for instance in metals [13] and in highly stressed rock masses [19]. Considering the simplest uniaxial model, a slender bar subjected to tension, it can be argued that the projection (over the horizontal axis z) of the cross sections where deformation localizes, is a lacunar fractal set, with dimension $\Delta_\varepsilon = 1 - d_\varepsilon$ comprised between zero and one. If the Cantor set ($\Delta_\varepsilon \cong 0.631$) is assumed as the archetype of damage distributions, we may speak of the fractal Cantor bar (Fig. 1b). The dilatation strain tends to localize into singular stretched regions, while the rest of the body is considered as undeformed. The displacement function can be represented by a devil's staircase graph, that is, by a singular fractal function which is constant everywhere except at the points corresponding to a lacunar fractal set of zero Lebesgue measure (Fig. 1b). By applying the renormalization group procedure [4], the fractal strain ε^*, whose physical dimension $[L]^{d_\varepsilon}$ lies between that of pure strain $[L]^0$ and that of a displacement $[L]^1$, reveals to be the scale-independent parameter describing the kinematics of the fractal bar.

Eventually, observe that the lacunar fractal domain Ω^*, with dimension $\Delta_\omega = 3 - d_\omega$, where the strain energy Φ is stored during a generic loading process, must be equal to the Cartesian product of the lacunar cross-section with dimension $2 - d_\sigma$ times the cantorian projection set, with dimension $1 - d_\varepsilon$ (Fig. 1c). Thus, a fundamental relationship among the exponents is achieved [4]

$$d_\omega = d_\sigma + d_\varepsilon \,. \tag{5}$$

2.2 Local fractional calculus

Local fractional derivatives (LFDs) were introduced with the motivation of studying the local properties of fractal structures and processes [15, 16], since no fractal function can be the solution of a classical differential equation [8]. The LFD definition is obtained from Eq. (3) introducing two "corrections" in order to avoid some physically undesirable features of the classical definition. In fact, if one wishes to analyze the local behaviour of a function, both the dependence on the lower limit a and the fact that adding a constant to a function yields a different fractional derivative should be avoided. This can be obtained subtracting from the function the value of the function at the point where we want to study the local scaling property and

choosing as the lower limit that point itself. Therefore, restricting our discussion to an order q comprised between 0 and 1, the LFD is defined as the following limit (if it exists and is finite)

$$D^q f(x) = \lim_{t \to x} \mathrm{D}^q_{x+}[f(t) - f(x)]\,, \qquad 0 < q \leq 1\,. \tag{6}$$

Note that italic symbols denote local fractional operators to distinguish them from the Riemann-Liouville ones.

In [14] the Weierstrass function was shown to be locally fractionally differentiable up to a *critical order* α between 0 and 1. More precisely, the LFD is zero if the order is lower than α, does not exist if greater, while exists and is finite only if equal to α. Thus, the LFD shows a behaviour analogous to that of the Hausdorff measure of a fractal set. Furthermore, the critical order is strictly linked to the fractal properties of the function itself. In fact, the critical order α coincides with the local Hölder exponent (which depends, as is well-known, on the fractal dimension), as it was demonstrated by proving the following local fractional Taylor expansion of function $f(x)$ of order $q < 1$ for $x \to x_0$ [14]

$$f(x) = f(x_0) + \frac{D^q f(x_0)}{\Gamma(q+1)}(x - x_0)^q + R_q(x - x_0)\,, \tag{7}$$

where $R_q(x - x_0)$ is a remainder, negligible if compared with the other terms. Let us observe that the terms in the right hand side of Eq. (7) are nontrivial and finite only if q is equal to the critical order α. Moreover, for $q = \alpha$, the fractional Taylor expansion (7) gives us the geometrical interpretation of the LFD. When q is set equal to unity, one obtains from (7) the equation of a tangent. All the curves passing through the same point x_0 with the same first derivative have the same tangent. Analogously, all the curves with the same critical order α and the same D^α form an equivalence class modelled by x^α. This is how it is possible to generalize the geometric interpretation of derivatives in terms of "tangents".

The solution of the simple differential equation $\mathrm{d}f/\mathrm{d}x = 1_{[0,x]}$ gives the length of the interval $[0, x]$. The solution is nothing but the integral of the unit function. Wishing to extend this idea to the evaluation of the measure of fractal sets, it can be seen immediately that the classical fractional integral does not work, as it fails to be additive because of its nontrivial kernel. On the other hand, a fractional measure of a fractal set can be obtained through the inverse of the LFD defined as [16]

$$I^\alpha_{[a,b]} f = \lim_{N \to \infty} \sum_{i=0}^{N-1} f(x_i^*) \mathrm{I}^\alpha_{x_i+} 1_{\mathrm{d}x_i}(x_{i+1})\,, \tag{8}$$

where $[x_i, x_{i+1}]$, $i = 0, \ldots, N-1$, $x_0 = a$ and $x_N = b$, provide a partition of the interval $[a, b]$ and x_i^* is some suitable point chosen in the subinterval $[x_i, x_{i+1}]$, while $1_{\mathrm{d}x_i}$ is the unit function defined on the same subinterval. Kolwankar [16] called $I^\alpha_{[a,b]} f$ the *fractal integral* of order α of $f(x)$ over the interval $[a, b]$. Observe that both the fractal integral $I^\alpha_{[a,x]}$ (Eq. 8) and the LFD D^α (Eq. 6) are defined by means of the Riemann-Liouville right fractional integral I^α_{a+} (Eq. 1) and derivative D^α_{a+} (Eq. 3), although different from each other.

The simple local fractional differential equation $D^\alpha f(x) = g(x)$ has not a finite solution when $g(x)$ is constant and $0 < \alpha < 1$. Interestingly, the solution exists if $g(x)$ has a fractal support whose Hausdorff dimension d is equal to the fractional order of derivation α. Consider, for instance, the triadic Cantor set C, built on the interval $[0, 1]$, whose dimension is $d = \ln 2/\ln 3$. Let $1_C(x)$ be the function whose value is one in the points belonging to the Cantor set upon $[0, 1]$, zero elsewhere. Therefore, the solution of $D^\alpha f(x) = 1_C(x)$ when $\alpha = d$ is $f(x) = I^\alpha_{[0,x]} 1_C(t)$. Applying Eq. (8) with $x_0 = 0$ and $x_N = x$ and choosing x_i^* to be such that $1_C(x_i^*)$ is maximum in the interval $[x_i, x_{i+1}]$, one gets [16]

$$f(x) = I^\alpha_{[0,x]} 1_C = \lim_{N \to \infty} \sum_{i=0}^{N-1} F_C^i \frac{(x_{i+1} - x_i)^\alpha}{\Gamma(1+\alpha)} = \frac{S_C(x)}{\Gamma(1+\alpha)}\,, \tag{9}$$

where F_C^i is a flag function that takes value 1 if the interval $[x_i, x_{i+1}]$ contains a point of the set C and 0 otherwise. $S_C(x)$ is the Cantor (devil's) staircase (Fig. 1b), i.e. a function almost everywhere flat except on an infinite number of singular points corresponding to the underlying Cantor set where it grows from 0 to 1 [11]. Moreover, Eq. (9) introduces the *fractional measure* of a fractal set: for the Cantor set C it is defined as $\mathcal{F}^\alpha(C) = I^\alpha_{[0,1]} 1_C(x)$. In fact $\mathcal{F}^\alpha(C)$ is infinite if $\alpha < d$, and 0 if $\alpha > d$. For $\alpha = d$, we find $\mathcal{F}^\alpha(C) = 1/\Gamma(1+\alpha)$, since $S_C(1) = 1$. This measure definition yields the same value as that predicted by the Hausdorff measure, the difference being represented only by a different value of the normalization constant. Eventually, from Eq. (9), it follows that the fractional measure of a generalized Cantor set $C_\alpha^{[a,b]}$ of dimension α built over the interval $[a, b]$ of the x-axis is

$$\mathcal{F}^\alpha(C_\alpha^{[a,b]}) = I^\alpha_{[a,b]} 1_{C_\alpha^{[a,b]}} = \frac{(b-a)^\alpha}{\Gamma(1+\alpha)}\,, \tag{10}$$

where $1_{C_\alpha^{[a,b]}}$ is the function equal to 1 if $x \in C_\alpha^{[a,b]}$, to 0 elsewhere.

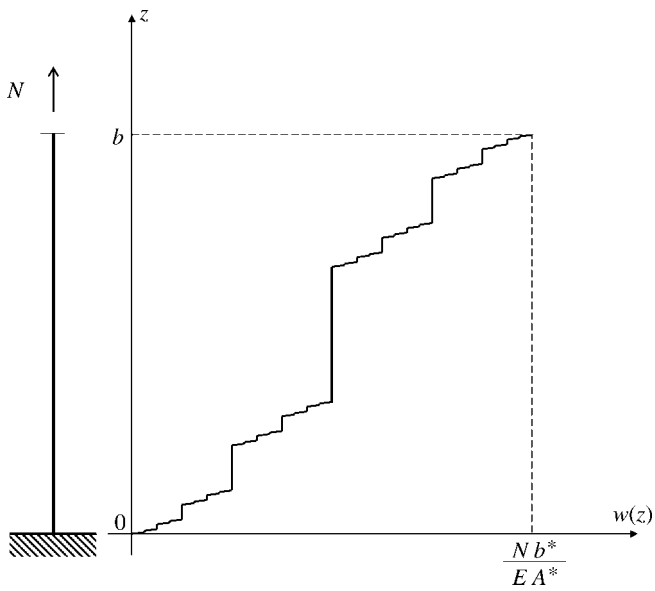

Fig. 2 The fractal bar subjected to an axial load: the displacement field.

2.3 Fractal bar

In this section, we intend to solve a simple structural problem using the mathematical tools presented in the previous section. Our aim is to show that experimental diagrams (see, for instance, [13]) such as the one of Fig. 1b can be obtained also analytically. More details can be found in [7].

Thus, let us consider a uniaxial model, hereafter called *fractal Cantor bar*, according to Feder's terminology [11], i.e. a bar of length b deformable on a fractal subset of dimension $(1-d_\varepsilon)$. The longitudinal axis is z. The bar is clamped in $z=0$, whereas a tensile load N is applied at its end $z=b$ (Fig. 2). A strain field will arise that is zero almost everywhere except in an infinite number of points (corresponding to the deformable subset) where it is singular. The displacement singularities can be characterized by the LFD of order equal to the fractal dimension $\alpha = 1-d_\varepsilon$ of the domain of the singularities, the unique value for which the LFD is finite and different from zero (the critical value). Therefore, we can define analytically the fractal strain ε^* as the LFD of order α of the displacement

$$\varepsilon^*(z) = D^\alpha w(z)\,. \tag{11}$$

Let us observe that, in Eq. (11), the noninteger physical dimensions $[L]^{d_\varepsilon}$ of ε^* are introduced by the LFD, whereas in Sect. 2.1 they are a geometrical consequence of the fractal dimension of the localization domain.

Without losing generality, let us assume the deformable subset to be the triadic Cantor set $C_\alpha^{[0,b]}$ built on $[0,b]$, $\alpha = \ln 2/\ln 3$. In order to compute the displacement function $w(z)$, we need the proper constitutive law. Here, for the sake of simplicity, we use a linear elastic relation and assume $d_\sigma = d_\varepsilon$: in this case the coefficient of proportionality between fractal stress and fractal strain coincides with the one between the nominal quantities, i.e. it is the Young's modulus E. In symbols: $\sigma^* = E\varepsilon^*$.

For equilibrium reasons, the internal axial force is constant and equal to N throughout the bar. Hence, we get a fractal strain ε^* equal to N/EA^* over the deformable subset, 0 elsewhere. Hence the kinematic Eq. (11) becomes

$$D^\alpha w(z) = \frac{N}{EA^*} 1_{C_\alpha^{[0,b]}}(z)\,. \tag{12}$$

Introducing the dimensionless quantities $\tilde{w} = w/b$, $\tilde{z} = z/b$ ($\tilde{z} \in [0,1]$), we can apply the scaling property expressed by ($a=0$, [18])

$$\mathrm{D}_{0+}^q f(bx) = \frac{\mathrm{d}^q f(bx)}{[\mathrm{d}x]^q} = b^q \frac{\mathrm{d}^q f(bx)}{[\mathrm{d}(bx)]^q}\,, \tag{13}$$

which is valid also for the LFD, to get $D^\alpha w(z) = b^{1-\alpha} D^\alpha \tilde{w}(\tilde{z})$. Eq. (12) can therefore be expressed in dimensionless form as follows

$$D^\alpha \tilde{w}(\tilde{z}) = \frac{N}{EA^* b^{1-\alpha}} 1_C(\tilde{z})\,, \tag{14}$$

where C is the triadic Cantor set built on $[0,1]$ as indicated in Sect. 2.1. In this form, the solution of the differential Eq. (14) can be obtained directly from Eq. (9)

$$\tilde{w}(\tilde{z}) = \frac{N}{EA^* b^{1-\alpha}} \frac{S_C(\tilde{z})}{\Gamma(1+\alpha)} , \tag{15}$$

where $S_C(x)$ is the Cantor staircase built on the interval $[0,1]$ and rising from 0 to 1. Recovering the physical quantities yields

$$w(z) = \frac{Nb^*}{EA^*} S_C\left(\frac{z}{b}\right) , \tag{16}$$

where $b^* = \frac{b^\alpha}{\Gamma(1+\alpha)}$ is the fractal measure of the deformable subset. Eq. (16) is plotted in Fig. 2. Let us emphasize that the Cantor staircase, introduced geometrically in Sect. 2.2, is now obtained analytically. Furthermore, notice that Eq. (16) provides important information about the size effect affecting the global deformation. In fact we find that the free end displacement $w(b)$ is equal to $\frac{Nb^*}{EA^*}$, i.e. $w(b) \sim b^\alpha$. This means that the displacement increases less than linearly with the bar length, as occurs with classical elastic bodies. From the point of view of the overall deformation $\varepsilon = w(b)/b$, we get $\varepsilon \sim b^{-(1-\alpha)}$: it decreases with size as a consequence of the strain localization on a lacunar fractal subset.

2.4 Energy aspects

The principle of virtual work is a fundamental identity of solid mechanics. Independently of the material constitutive law, it states the equality between the internal virtual work and the external virtual work. Its expression will now be given for the fractal bar problem, starting from the kinematic analysis of the previous section.

Let us now denote the axial load per unit of fractal length acting on the bar with $p^*(z)$ and take into consideration a strain-displacement field (ε^*, u) satisfying Eq. (11) and a force-stress field (p^*, N) satisfying the axial equilibrium equation

$$D^\alpha N(z) + p^*(z) = 0 . \tag{17}$$

Note that the two fields are not necessarily related to each other. The internal virtual work is given by the fractal α-integral of the product of the axial force N times the fractal strain ε^* performed over the interval $[0,b]$. By means of Eqs. (11) and (17), we may prove the following equality chain

$$I^\alpha_{[0,b]}\left[N(z)\varepsilon^*(z)\right] = I^\alpha_{[0,b]}\left[N(z)D^\alpha w(z)\right] = \left[N(z)w(z)\right]_{z=0}^{z=b} + I^\alpha_{[0,b]}\left[w(z)p^*(z)\right] , \tag{18}$$

where the fractal integration by parts

$$I^\alpha_{[a,b]}[f(x)D^\alpha g(x)] = [f(x)g(x)]_{x=a}^{x=b} - I^\alpha_{[a,b]}[g(x)D^\alpha f(x)] \tag{19}$$

has been exploited (for details, see [7, 15]). The right hand side of Eq. (18) is easily recognized as the external virtual work; hence, Eq. (18) itself represents the principle of work for the fractal bar.

It is worth observing that the load p^* has the anomalous dimensions $[F][L]^{-\alpha}$, since it considers forces acting on a fractal medium. On the other hand, both sides of Eq. (18) possess the classical dimensions of work ($[F][L]$).

Finally, according to Clapeyron's Theorem, the strain energy Φ can be evaluated as one half of the work of external forces. Hence, by Eq. (18) and exploiting the constitutive equation

$$\Phi = \frac{1}{2}\left[N(z)w(z)\right]_{z=0}^{z=b} + \frac{1}{2} I^\alpha_{[0,b]}\left[w(z)p^*(z)\right] = \frac{EA^*}{2} I^\alpha_{[0,b]}\left[(\varepsilon^*(z))^2\right] . \tag{20}$$

What has been done in the one-dimensional case can be formally extended to the three-dimensional case for a generic fractal medium [4]. Moreover, the finite element method can be applied also to such a medium by means of the so-called fractal splines. The interested reader is referred to [5, 6] for further details.

3 Non-local approach

Several approaches have been proposed to include the effects of non-local interactions into the classical continuum mechanics framework since the pioneering work by Eringen [10]. New models based on fractional calculus have recently been suggested [9, 17] and they will be discussed in the next sections.

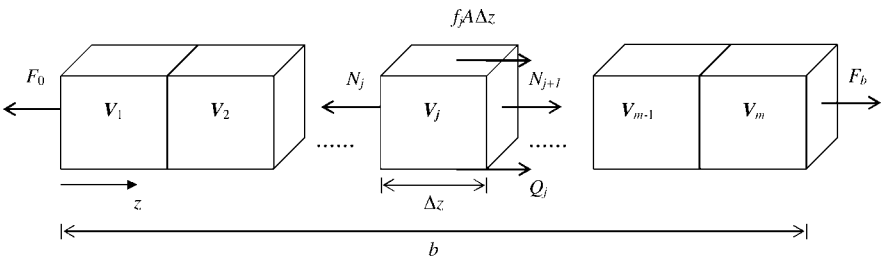

Fig. 3 Discretized elastic bar: equilibrium of the generic volume element V_j.

3.1 Elastic bar with long-range interactions

Let us now consider an elastic bar of length b and subjected to a distributed longitudinal force $f(z)$. Let us discretize the bar in m equal elements, each with volume $V_j = A\Delta z$ $(j = 1, \ldots, m)$, where A is the cross-section area and $\Delta z = b/m$. The generic volume element V_j is located at the abscissa $z_j = j\Delta z$. Its equilibrium equation can be written taking into account external loads, contact forces provided by adjacent volumes, V_{j-1} and V_{j+1}, which are denoted by N_j and N_{j+1}, respectively, and long range interactions Q_j applied on V_j by the surrounding non-adjacent elements of the bar (Fig. 3). In formulae

$$\Delta N_j + Q_j = \Delta N_j + \sum_{h=1}^{j-1} Q^{(h,j)} - \sum_{h=j+1}^{m} Q^{(h,j)} = -f_j A\Delta z\,, \tag{21}$$

where $f_j = f(z_j)$, $\Delta N_j = N_{j+1} - N_j$ and $Q^{(h,j)}$ $(h \neq j)$ are the long-range forces that surrounding volume elements apply on element V_j. These forces can be modelled as

$$Q^{(h,j)} = \mathrm{sgn}(z_h - z_j)[w(z_h) - w(z_j)]g(z_h, z_j)V_jV_h\,, \tag{22}$$

being $g(z_h, z_j)$ a real-valued monotonically decreasing function expressed as

$$g(z_h, z_j) = \frac{Ec_\alpha\alpha}{A\Gamma(1-\alpha)|z_h - z_j|^{1+\alpha}}\,, \quad (0 \le \alpha \le 1) \tag{23}$$

and $\mathrm{sgn}(z)$ the classical signum function

$$\mathrm{sgn}(z) = \begin{cases} -1 & \text{if } z < 0 \\ 0 & \text{if } z = 0 \\ +1 & \text{if } z > 0 \end{cases}\,. \tag{24}$$

Note that the constant c_α in Eq. (23) has the anomalous dimensions $[L]^{\alpha-2}$. The classical continuum mechanics is recovered not only for $\alpha \to 0$ but also for $\alpha \to 1$, since the Gamma function is infinite for a vanishing argument. Substituting Eq. (22) into (Eq. (21)), yields

$$\Delta N_j - \frac{Ec_\alpha\alpha A}{\Gamma(1-\alpha)}\Delta z\left[\sum_{h=1}^{j-1}\frac{w(z_h) - w(z_j)}{(z_j - z_h)^{1+\alpha}}\Delta z - \sum_{h=j+1}^{m}\frac{w(z_j) - w(z_h)}{(z_h - z_j)^{1+\alpha}}\Delta z\right] = -f_jA\Delta z\,. \tag{25}$$

Dividing Eq. (25) by $EA\Delta z$ and assuming a linear elastic material i.e., $\sigma(z) = E\varepsilon(z) = E\mathrm{d}w/\mathrm{d}z$, the following differential equation is obtained for $\Delta z \to 0$ [9]

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} - c_\alpha\left(\widehat{\mathbf{D}}^\alpha_{0+}w + \widehat{\mathbf{D}}^\alpha_{b-}w\right) = -\frac{f(z)}{E}\,. \tag{26}$$

$\widehat{\mathbf{D}}^\alpha_{0+}w$ and $\widehat{\mathbf{D}}^\alpha_{b-}w$ in Eq. (26) represent the integrals terms in the Marchaud fractional derivatives on a finite interval (note that $a = 0$), which have been proved to coincide [20], for a certain class of functions, with the Riemann-Liouville fractional derivatives (Eqs. (3) and (4))

$$\mathbf{D}^\alpha_{a+}f(x) = \frac{f(x)}{\Gamma(1-\alpha)(x-a)^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)}\int_a^x \frac{f(x) - f(\xi)}{(x-\xi)^{1+\alpha}}\mathrm{d}\xi = \frac{f(x)}{\Gamma(1-\alpha)(x-a)^\alpha} + \widehat{\mathbf{D}}^\alpha_{a+}f(x)\,, \tag{27}$$

$$\mathbf{D}^\alpha_{b-}f(x) = \frac{f(x)}{\Gamma(1-\alpha)(b-x)^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)}\int_x^b \frac{f(x) - f(\xi)}{(\xi-x)^{1+\alpha}}\mathrm{d}\xi = \frac{f(x)}{\Gamma(1-\alpha)(b-x)^\alpha} + \widehat{\mathbf{D}}^\alpha_{b-}f(x)\,. \tag{28}$$

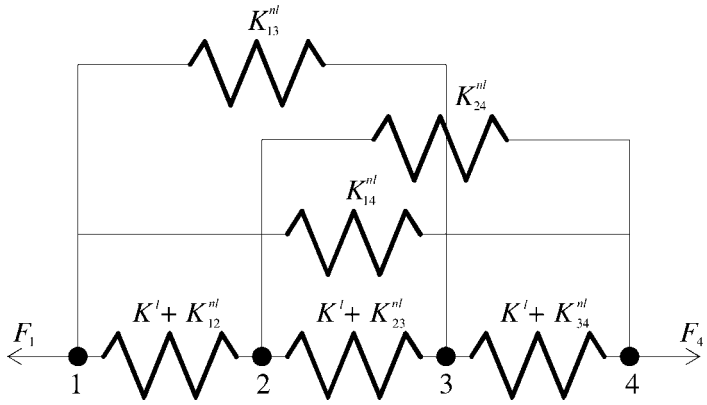

Fig. 4 Discretized elastic bar: spring-point model.

Eq. (26) can be rewritten, equivalently, in terms of the Marchaud fractional derivatives (Eqs. (27) and (28)) as

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} - c_\alpha[\left(\mathbf{D}^\alpha_{0+} w + \mathbf{D}^\alpha_{b-} w\right) - w\left(\mathbf{D}^\alpha_{0+} 1 + \mathbf{D}^\alpha_{b-} 1\right)] = -\frac{f(z)}{E}\,, \tag{29}$$

which represents a nonhomogeneous ordinary fractional differential equation with non-constant coefficients, since the derivatives of the unit function are neither zero nor constant.

It is worth observing that the fractional formula of integration by parts [20]

$$\int_a^b [(\mathbf{D}^\alpha_{a+} f)\cdot g]\mathrm{d}x = \int_a^b [(\mathbf{D}^\alpha_{b-} g)\cdot f]\mathrm{d}x \tag{30}$$

can be exploited to verify that the resultant of the non-local forces, expressed by the second term in the left-hand side of Eq. (29), is zero. In fact, by choosing either f or g equal to 1, it is easy to check that

$$\int_0^b [(\mathbf{D}^\alpha_{0+} w + \mathbf{D}^\alpha_{b-} w) - w(\mathbf{D}^\alpha_{0+} 1 + \mathbf{D}^\alpha_{b-} 1)]\mathrm{d}z = 0\,. \tag{31}$$

Eq. (31) shows that the contribution to the overall stress due to the non-local terms vanishes at the boundaries of the bar ($z = 0, b$). In other words, the forces F_0 and F_b (Fig. 3) acting at the bar extremes have to be carried by the local stress $\sigma = F/A$. Kinematic and static boundary conditions associated to Eq. (29) are hence the classical ones. At $z = 0$ we may prescribe either

$$w(0) = w_0 \quad \text{or} \quad \left.\frac{\mathrm{d}w}{\mathrm{d}z}\right|_{z=0} = -\frac{F_0}{EA}\,, \tag{32}$$

while at $z = b$:

$$w(b) = w_b \quad \text{or} \quad \left.\frac{\mathrm{d}w}{\mathrm{d}z}\right|_{z=b} = \frac{F_b}{EA}\,. \tag{33}$$

Obviously, a kinematical boundary condition has to be assigned at least to one of the extremes, otherwise the solution is defined up to an irrelevant rigid motion (provided that the overall bar equilibrium is satisfied). Note that the simplicity of Eqs. (32) and (33) can be considered a strength point of the present fractional model, since the difficulty to determine the boundary conditions and the related physical meaning represents a drawback for several non-local models, such as the gradient theory [1].

Unfortunately, analytical solutions of Eq. (29) (i.e., Eq. (26)) are not available in the literature at the authors best knowledge. Eventually, observe that the fractional terms in Eq. (29) (i.e. Eq. (26)) vanish both for $\alpha = 0$ and $\alpha = 1$, as already stated about Eq. (23). In fact, for $\alpha = 0$ the left and right Marchaud derivatives provide the function itself (and hence the term in the square brackets becomes $2w - 2w$); on the other hand, for $\alpha = 1$ the left derivative coincides with the classical first derivative, whereas the right one with its opposite.

3.2 Equivalent mechanical model

The physical validity of Eq. (21) can be explained by considering a simple discrete spring-point model of the bar [9], as reported in Fig. 4 only for four points. Local forces between adjacent particles are taken into account by springs with elastic

stiffness $K^l = EA/\Delta z$. On the other hand, long-distance interactions between particles are modelled by linear springs with distance-decay stiffness as $K^{nl}_{jh} = g(z_j, z_h)$, where function g is provided by Eq. (23).

Thus, the equilibrium equation for the generic node located at the abscissa z_j may be written as

$$K^l(w_2 - w_1) + (A\Delta z)^2 \sum_{h=2}^{m} g(z_h, z_1)(w_h - w_1) = F_1 \,, \qquad j = 1 \tag{34}$$

$$K^l(w_{j-1} - 2w_j + w_{j+1}) + (A\Delta z)^2 \sum_{h=1, h\neq j}^{m} g(z_h, z_j)(w_h - w_j) = -F_j \,, \qquad j = 2, \ldots, m-1 \,, \tag{35}$$

$$K^l(w_{m-1} - w_m) + (A\Delta z)^2 \sum_{h=1}^{m-1} g(z_m, z_h)(w_h - w_m) = -F_m \qquad j = m \,, \tag{36}$$

where the right-hand side of Eq. (35) represents the body forces applied to material particles ($F_j = f_j A\Delta z$), while F_1 and F_m represent the forces acting at the bar extremes.

Equilibrium Eqs. (34)–(36) can be sintethized in a compact form by introducing the non-local coefficient matrix $\mathbf{K} = \mathbf{K}^l + \mathbf{K}^{nl}$ as

$$\mathbf{K}\mathbf{w} = \mathbf{F} \,, \tag{37}$$

where $\mathbf{w}$ and $\mathbf{F}$ are the (m)-dimensional displacement and force vectors, respectively; $\mathbf{K}^l$ is the tri-diagonal matrix related to contact contributions due to adjacent elements

$$\mathbf{K}^l = \begin{bmatrix} K^l & -K^l & \ldots & \ldots & 0 \\ -K^l & 2K^l & -K^l & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & -K^l & 2K^l & -K^l \\ 0 & \ldots & \ldots & -K^l & K^l \end{bmatrix} \,, \tag{38}$$

while $\mathbf{K}^{nl}$ is the fully populated, non-local stiffness matrix

$$\mathbf{K}^{nl} = \begin{bmatrix} K^{nl}_{11} & -K^{nl}_{12} & \ldots & -K^{nl}_{1m} \\ -K^{nl}_{12} & K^{nl}_{22} & \ldots & -K^{nl}_{2m} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ -K^{nl}_{1m} & \ldots & \ldots & K^{nl}_{mm} \end{bmatrix} \,, \tag{39}$$

with

$$K^{nl}_{jh} = (A\Delta z)^2 g(z_j, z_h) \,, \quad j \neq h \tag{40}$$

and

$$K^{nl}_{jj} = \sum_{h=1, h\neq j}^{m} K^{nl}_{jh} \,. \tag{41}$$

3.3 Numerical applications

The non-local stiffness matrix obtained by the point-spring model (Eq. (39)) is found to coincide, for $\Delta z \to 0$, with that obtained by discretizing Eq. (26) with fractional finite differences [12].

Let us consider a clamped-free bar, loaded at the free end by a tensile force $F = 10^4$ N. The bar dimensions are selected as follows: $b = 200$ mm and $A = 100\,\text{mm}^2$. The following material characteristics are considered: $E = 72$ GPa, $\alpha = 0.5$ and $c_\alpha = 0.03\,\text{mm}^{-1.5}$. The axial strain related to such a structure is displayed in Fig. 5, showing a perfect agreement between the continuum fractional approach (continuous line) and the discrete spring-point model presented in the previous section (dots). Note that the axial strains are uniform along the bar core and increase at the boundary, as from experimental evidences. Similar results are predicted also by different non-local models (see, for instance, [1]).

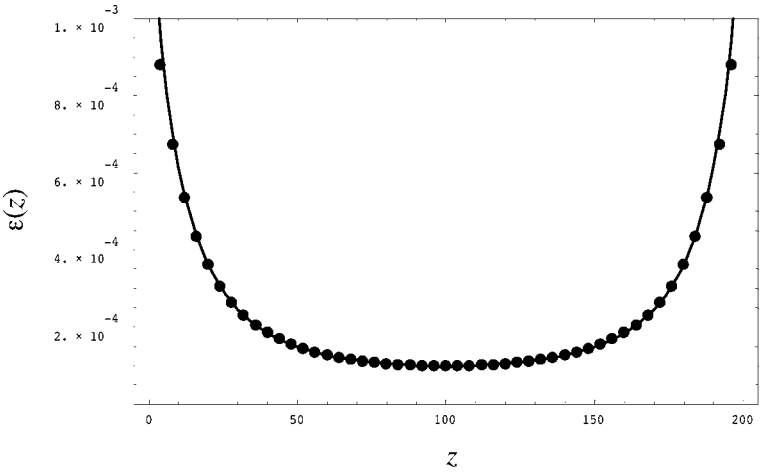

Fig. 5 Axial strain in a clamped-free bar: continuum fractional approach (continuous line) vs. discrete spring-point model (dots).

3.4 Energy aspects

The strain energy Φ stored in an elastic bar with long-range interactions can be written as the sum of two terms

$$\Phi = \Phi^l + \Phi^{nl}\,, \tag{42}$$

the former related to local interactions

$$\Phi^l = \frac{A}{2}\int_0^b (\sigma\varepsilon)\mathrm{d}z \tag{43}$$

and the latter obtained by summing up the strain energy of each non-local spring

$$\begin{aligned}\Phi^{nl} &= \frac{1}{2}\left(\frac{1}{2}\frac{c_\alpha \alpha E A}{\Gamma(1-\alpha)}\int_0^b\int_0^b \frac{[w(z)-w(\xi)]^2}{|z-\xi|^{1+\alpha}}\mathrm{d}\xi\mathrm{d}z\right)\\ &= \frac{1}{4}\frac{c_\alpha \alpha E A}{\Gamma(1-\alpha)}\int_0^b\left(\int_0^b \frac{w^2(z)-w^2(\xi)}{|z-\xi|^{1+\alpha}}\mathrm{d}\xi + 2\int_0^b \frac{w^2(\xi)-w(z)w(\xi)}{|z-\xi|^{1+\alpha}}\mathrm{d}\xi\right)\mathrm{d}z\,.\end{aligned} \tag{44}$$

By exploiting the fractional integration by parts (Eq. (30)) with $f = w^2$ and $g = 1$, it is easy to show that the first double integral at the right hand side is null. Thus

$$\Phi^{nl} = \frac{1}{2}\frac{c_\alpha \alpha E A}{\Gamma(1-\alpha)}\int_0^b\left(\int_0^b \frac{w^2(\xi)-w(z)w(\xi)}{|z-\xi|^{1+\alpha}}\mathrm{d}\xi\right)\mathrm{d}z\,. \tag{45}$$

Since the equivalence between the non-local fractional model and the point-spring model has been proved, it can be argued that Clapeyron's theorem still holds true for the fractional model. We may check this statement by evaluating one half of the work exerted by external forces as they were present throughout the loading process with their final value

$$\frac{A}{2}\left[\sigma w\right]_{z=0}^{z=b} + \frac{A}{2}\int_0^b (fw)\mathrm{d}z = \frac{A}{2}\left[\sigma w\right]_{z=0}^{z=b} - \frac{EA}{2}\int_0^b \left[w'' - c_\alpha(\mathbf{D}^\alpha w - w\mathbf{D}^\alpha 1)\right] w\,\mathrm{d}z\,, \tag{46}$$

where we have used the integro-differential equilibrium Eq. (29) and, for the sake of simplicity, the symbol $\mathbf{D}^\alpha = \mathbf{D}^\alpha_{0+} + \mathbf{D}^\alpha_{b-}$ has been introduced. Applying the classical formula of integration by parts as well as its fractional counterpart to the right-hand side of Eq. (46), provides the same local and non-local contributions to the overall strain energy Φ as expressed by Eqs. (43) and (45)

$$\begin{aligned}&\frac{A}{2}\int_0^b (\sigma\varepsilon)\mathrm{d}z + \frac{c_\alpha E A}{2}\int_0^b (w\mathbf{D}^\alpha w - \mathbf{D}^\alpha w^2)\mathrm{d}z\\ &= \frac{A}{2}\int_0^b (\sigma\varepsilon)\mathrm{d}z + \frac{1}{2}\frac{c_\alpha \alpha E A}{\Gamma(1-\alpha)}\int_0^b\left(\int_0^b \frac{w^2(\xi)-w(z)w(\xi)}{|z-\xi|^{1+\alpha}}\mathrm{d}\xi\right)\mathrm{d}z = \Phi^l + \Phi^{nl} = \Phi\,.\end{aligned} \tag{47}$$

Hence, Clapeyron's theorem is valid and can be used to evaluate the strain energy stored in the bar with non-local interactions modelled by Eq. (22).

At present, it does not seem possible to express the non-local strain energy Φ^{nl} as the product of a stress times a strain, even non-local. The difficulty lies in the absence of a chain and composition rule for the fractional operator $\mathbf{D}^{\alpha}$, i.e. $\mathbf{D}^{\alpha}\mathbf{D}^{\alpha} \neq \mathbf{D}^{2\alpha}$. This drawback is not present in the local model, where fractional operators and quantities simply replace the classical ones, leading to the elegant generalization of the principle of virtual works for fractal media (Eq. (19)).

4 Conclusions

Classical continuum theory works properly at the macro-scale, where the effect of material microstructure can be neglected. Recent technological progress (e.g. nano-devices) as well as the possibility to have a deeper insight into material behaviour forced the scientific community to analyze phenomena taking place at the meso/micro level.

One way to address such problems is the use of enriched continuum mechanics models. Among these models, in the present paper we revisited two recently proposed approaches whose common feature is represented by the use of fractional calculus, which turns out to be a powerful tool to handle non-standard continua.

The former approach deals with fractal media, i.e. with solids whose microstructure is such that strain localizes onto fractal subsets. The latter approach deals with solids characterized by non-local long-range interactions. Although they are rather different, it is interesting to highlight the following analogies/differences between them.

1. Both the approaches yield non-standard fractional derivatives, the former one using the local fractional derivative and the latter one the integral part of Marchaud fractional derivative.
2. The local approach is based on kinematic arguments, whilst the non-local one stems from a static analysis.
3. Both the approaches provide a non-uniform strain field under a uniform stress field.
4. Both the models introduce a displacement derivative of order lower than the classical one, differently from what proposed by gradient theory [1], which provides derivatives of the displacement of higher orders. This is a drawback of gradient theory, since higher order differential equations need extra boundary conditions to be solved, whose physical meaning is still unclear.
5. In the local model, the LFD replaces the classical derivative (Eq. (11)), whereas in the non-local model the fractional derivative represents a correction to be added to the classical equilibrium equation (Eq. (25)) in order to take internal forces into account.
6. In the local approach, the kinematic equation appears as the straightforward generalization to fractal media of the definition of strain in continuum mechanics. This provides the key to generalize the principle of virtual work as well as to highlight the static-kinematic duality for fractal media [4]. On the other hand this aspect remains hidden in the non-local approach.

Eventually note that, while an analytical solution is achieved for what concerns the local model, the non-local model can be solved, at the authors best knowledge, only numerically.

References

[1] E. C. Aifantis, Gradient effects at macro micro and nano scales, J. Mech. Behav. Mat. **5**, 355–375 (1994).
[2] A. Carpinteri, Fractal nature of material microstructure and size effects on apparent mechanical properties, Mech. Mater. **18**, 89–101 (1994).
[3] A. Carpinteri, Scaling laws and renormalization groups for strength and toughness of disordered materials, Int. J. Solids Struct. **31**, 291–302 (1994).
[4] A. Carpinteri, B. Chiaia, and P. Cornetti, Static-kinematic duality and the principle of virtual work in the mechanics of fractal media, Comput. Methods Appl. Mech. Eng. **190**, 6053–6073 (2001).
[5] A. Carpinteri, B. Chiaia, and P. Cornetti, The elastic problem for fractal media: basic theory and finite element formulation, Comp. Struct. **82**, 499–508 (2004).
[6] A. Carpinteri, B. Chiaia, and P. Cornetti, A disordered microstructure material model based on fractal geometry and fractional calculus, Z. Angew. Math. Mech. **84** (2004) 128–135.
[7] A. Carpinteri and P. Cornetti, A fractional calculus approach to the description of stress and strain localization in fractal media, Chaos Solitons Fractals. **13**, 85–94 (2002).
[8] A. Carpinteri and F. Mainardi, Fractals and Fractional Calculus in Continuum Mechanics (Springer-Verlag, Wien, 1997).
[9] M. Di Paola and M. Zingales, Long-range cohesive interactions of non-local continuum mechanics faced by fractional calculus, Int. J. Sol. Struct. **45**, 5642–5659 (2008).
[10] A. C. Eringen and D. G. B. Edelen, On non local elasticity, Int. J. Eng. Sci. **10**, 233–248 (1972).
[11] J. Feder, Fractals (Plenum Press, New York, 1988).

[12] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, Theory and Applications of Fractional Differential Equations (Elsevier, Amsterdam, 2006).
[13] T. Kleiser and M. Bocek, The fractal nature of slip in crystals, Z. Metallk. **77**, 582–587 (1986).
[14] K. M. Kolwankar and A. D. Gangal, Fractional differentiability of nowhere differentiable functions and dimensions, Chaos. **6**, 505–523 (1996).
[15] K. M. Kolwankar and A. D. Gangal, Local fractional Fokker-Plank equations, Phys. Rev. Lett. **80**, 214–217 (1998).
[16] K. M. Kolwankar and A. D. Gangal, Local fractional calculus: a calculus for fractal space-time, in: Fractals: Theory and Applications in Engineering (Springer, Delft, 1999), pp. 171–181.
[17] K. A. Lazopoulos, Non-local continuum mechanics and fractional calculus, Mech. Res. Commun. **33**, 751–757 (2006).
[18] K. B. Oldham and J. Spanier, The Fractional Calculus (Academic Press, New York, 1974).
[19] A. B. Poliakov, H. J. Hermann, Y. Y. Podladchikov, and S. Roux, Fractal plastic shearbands, Fractals. **2**, 567–571 (1995).
[20] S. G. Samko, A. A. Kilbas, and O. I. Marichev, Fractional Integrals and Derivatives (Gordon and Breach Science Publishers, Amsterdam, 1993).

