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Fractional calculus in solid mechanics: local versus non-local approach

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Abstract
Several enriched continuum mechanics theories have been proposed by the scientific community in order to develop models capable of describing microstructural effects. The aim of the present paper is to revisit and compare two of these models, whose common denominator is the use of fractional calculus operators. The former was proposed to investigate damage in materials exhibiting a fractal-like microstructure. It makes use of the local fractional derivative, which turns out to be a powerful tool to describe irregular patterns such as strain localization in heterogeneous materials. On the other hand, the latter is a non-local approach that models long-range interactions between particles by means of the Marchaud fractional derivative. Analogies and differences between the two models are outlined and discussed.

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1. Introduction

The aim of this paper is to revisit and compare two applications of fractional calculus to continuum mechanics. For the sake of simplicity, only one-dimensional problems will be dealt with.

The former approach is based on the local fractional calculus introduced by Kolwankar [1, 2] to address the problem of fractal media, i.e. solids where the deformation is localized on a fractal subset. By such an assumption on a fractal subset, the displacement field is represented by devil staircase-like functions [3, 4]. As is well known, these functions have zero first derivative, except in an infinite number of points where they are not differentiable. On the other hand, they admit fractional derivatives of order less than the fractal dimension of the set where the strain localizes. It is hence possible to express the fractal strain as the local fractional derivative (LFD) of the displacement field.

The latter approach [5] was introduced to model non-local continua, i.e. solids characterized by non-local interactions [6, 7]. The novelty is that internal forces are described by fractional integrals and derivatives [5]. One of the most remarkable achievements of this approach is that, by the Marchaud definition of fractional derivative [8], the fractional operators have a clear mechanical interpretation, i.e. springs connecting non-adjacent points of the body. The related stiffness decays along with the distance between the material points. However, it is seen that, in order to have a consistent mechanical model, only the integral part of the Marchaud derivative has to be retained.

The two approaches are finally discussed and compared. For the sake of clarity, it is worth recalling in this section the definitions of the left and right Riemann–Liouville integrals, $\mathrm{I}^{\alpha}_{a+}f$ and $\mathrm{I}^{\alpha}_{b-}f$, and derivatives, $\mathrm{D}^{\alpha}_{a+}f$ and $\mathrm{D}^{\alpha}_{b-}f$, respectively [8]. In particular, given a Lebesgue function $f(x)$ on the closed interval $[a, b]$, the fractional integrals are defined by ($\alpha > 0$):

$$\mathrm{I}^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{f(\xi)}{(x-\xi)^{1-\alpha}}\mathrm{d}\xi, \qquad (1)$$

$$\mathrm{I}^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b \frac{f(\xi)}{(\xi-x)^{1-\alpha}}\mathrm{d}\xi, \qquad (2)$$

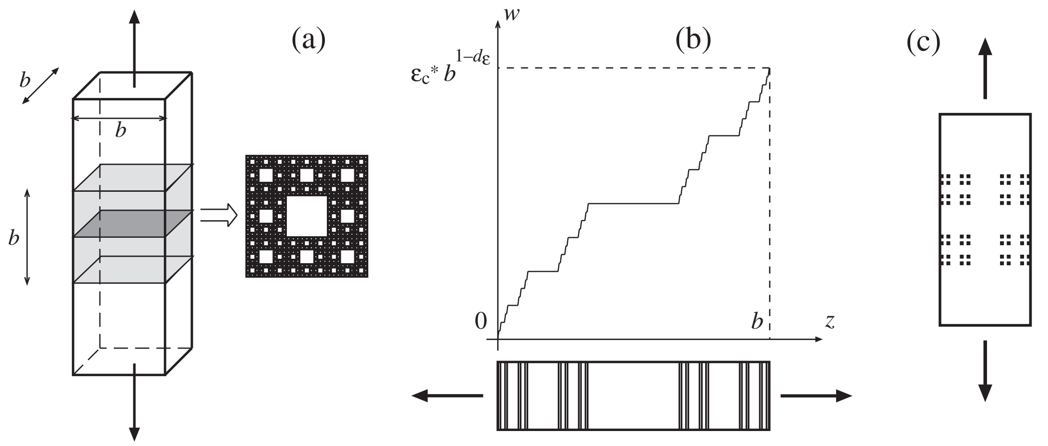

Figure 1. Fractal localization of: (a) stress, (b) strain and (c) energy dissipation.

while the fractional derivatives are given by ($0 < \alpha < 1$):

$$\mathrm{D}_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_a^x \frac{f(\xi)}{(x-\xi)^{\alpha}} \mathrm{d}\xi, \tag{3}$$

$$\mathrm{D}_{b-}^{\alpha} f(x) = -\frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_x^b \frac{f(\xi)}{(\xi-x)^{\alpha}} \mathrm{d}\xi. \tag{4}$$

2. Local approach

The mechanics of solids deformable over fractal subsets has been widely investigated [3, 4, 9]. When dealing with fractal media, mechanical quantities with anomalous physical dimensions, related to the fractal dimensions of the domain upon which they are defined, must be taken into account [10, 11]. Then, by means of the local fractional operators, the static and kinematic equations (as well as the principle of virtual work) may be derived [3, 4]. Before entering into the details, let us recall that we are referring to a generic body whose stress flux and deformation pattern have fractal characteristics, whereas the material itself does not need to present a fractal microstructure.

2.1. Fractal mechanics quantities

The singular stress flux through fractal media can be modelled by means of lacunar fractal sets of dimension $\Delta_\sigma = 2 - d_\sigma \leqslant 2$ (figure 1(a)) and measure A^*, representing damaged cross-sections. An original definition of the fractal stress σ^* was put forward in [10, 11] by applying the renormalization group procedure to the nominal stress tensor $[\sigma]$. The fractal stress σ^*, whose dimensions are $[F][L]^{-(2-d_\sigma)}$, is a scale-invariant quantity.

The kinematical conjugate of the fractal stress σ^* is the fractal strain ε^*. The basic assumption is that displacement discontinuities can be localized on an infinite number of cross-sections, spreading throughout the body. This hypothesis has been suggested by several experimental investigations, for instance in metals [12] and in highly stressed rock masses [13].

Considering the simplest uniaxial model, a slender bar subjected to tension, it can be argued that the projection (over the horizontal axis z) of the cross sections where deformation localizes is a lacunar fractal set, with dimension $\Delta_\varepsilon = 1 - d_\varepsilon$ consisting of between zero and one. If the Cantor set ($\Delta_\varepsilon \cong 0.631$) is assumed as the archetype of damage distributions, we may speak of the fractal Cantor bar (figure 1(b)). The dilatation strain tends to localize into singular stretched regions, while the rest of the body is considered as undeformed. The displacement function can be represented by a devil's staircase graph, that is, by a singular fractal function that is constant everywhere except at the points corresponding to a lacunar fractal set of zero Lebesgue measure (figure 1(b)). By applying the renormalization group procedure [3], the fractal strain ε^*, whose physical dimension $[L]^{d_\varepsilon}$ lies between that of pure strain $[L]^0$ and that of a displacement $[L]^1$, reveals to be the scale-independent parameter describing the kinematics of the fractal bar.

Eventually, let us recall that the lacunar fractal domain Ω^*, with dimension $\Delta_\omega = 3 - d_\omega$, where the strain energy Φ is stored during a generic loading process, must be equal to the Cartesian product of the lacunar cross-section with dimension $2 - d_\sigma$ times the Cantorian projection set, with dimension $1 - d_\varepsilon$ (figure 1(c)). Thus, a fundamental relationship among the exponents is achieved [3]:

$$d_\omega = d_\sigma + d_\varepsilon. \tag{5}$$

2.2. Local fractional calculus

LFDs were introduced with the aim of studying the local properties of fractal structures and processes [2, 14], since no fractal function can be the solution of a classical differential equation [15]. The LFD definition is obtained from equation (3) introducing two 'corrections' in order to avoid some physically undesirable features of the classical definition. In fact, if one wishes to analyse the local behaviour of a function, both the dependence on the lower limit a and the fact that adding a constant to a function yields a different fractional derivative should be avoided. This can be obtained by subtracting from the function the value of the function at the point where we want to study the local scaling property and choosing as the lower limit that point itself. Therefore, restricting our discussion to an order q consisting of between 0 and 1, the LFD is defined as the following limit (if it exists and is finite):

$$D^q f(x) = \lim_{t\to x} D^q_{x+}[f(t) - f(x)], \quad 0 < q \leqslant 1. \tag{6}$$

Note that italic symbols denote local fractional operators to distinguish them from the classical Riemann–Liouville ones.

In [1] the Weierstrass function was shown to be locally fractionally differentiable up to a *critical order* α between 0 and 1. More precisely, the LFD is zero if the order is lower than α, does not exist if the order is greater than α, and exists and is finite only when the order is equal to α. Thus, the LFD shows a behaviour analogous to that of the Hausdorff measure of a fractal set. Furthermore, the critical order is strictly linked to the fractal properties of the function itself. In fact, the critical order α coincides with the local Hölder exponent (which depends, as is well known, on the fractal dimension), as it was demonstrated by proving the following local fractional Taylor expansion of function $f(x)$ of order $q < 1$ for $x \to x_0$ [1]:

$$f(x) = f(x_0) + \frac{D^q f(x_0)}{\Gamma(q+1)}(x - x_0)^q + R_q(x - x_0), \tag{7}$$

where $R_q(x - x_0)$ is the remainder, negligible if compared with the other terms. Let us observe that the terms in the right-hand side of equation (7) are non-trivial and finite only if q is equal to the critical order α. Moreover, for $q = \alpha$, the fractional Taylor expansion (7) gives us the geometrical interpretation of the LFD. When q is set to unity, one obtains from (7) the equation of a tangent. All the curves passing through the same point x_0 with the same first derivative have the same tangent. Analogously, all the curves with the same critical order α and the same D^α form an equivalence class modelled by x^α. This is how it is possible to generalize the geometric interpretation of derivatives in terms of 'tangents'.

The solution of the simple differential equation $\mathrm{d}f/\mathrm{d}x = 1_{[0,x]}$ gives the length of the interval $[0, x]$. The solution is nothing but the integral of the unit function. Wishing to extend this idea to the evaluation of the measure of fractal sets, it can be seen immediately that the classical fractional integral does not work, as it fails to be additive because of its nontrivial kernel. On the other hand, a fractional measure of a fractal set can be obtained through the inverse of the LFD defined as [14]:

$$I^\alpha_{[a,b]} f = \lim_{N\to\infty} \sum_{i=0}^{N-1} f(x_i^*) \mathrm{I}^\alpha_{x_i+} 1_{\mathrm{d}x_i}(x_{i+1}), \tag{8}$$

where $[x_i, x_{i+1}]$, $i = 0, \ldots, N - 1$, $x_0 = a$ and $x_N = b$ provide a partition of the interval $[a, b]$ and x_i^* is some suitable point chosen in the subinterval $[x_i, x_{i+1}]$, while $1_{\mathrm{d}x_i}$ is the unit function defined on the same subinterval. Kolwankar called $I^\alpha_{[a,b]} f$ the *fractal integral* of order α of $f(x)$ over the interval $[a, b]$ [14]. The simple local fractional differential equation $D^\alpha f(x) = g(x)$ has not a finite solution when $g(x)$ is constant and $0 < \alpha < 1$. Interestingly, the solution exists if $g(x)$ has a fractal support whose Hausdorff dimension d is equal to the fractional order of derivation α. Consider, for instance, the triadic Cantor set C, built on the interval [0, 1], whose dimension is $d = \ln 2/\ln 3$. Let $1_C(x)$ be the function whose value is 1 in the points belonging to the Cantor set upon [0, 1] and is 0 elsewhere. Therefore, the solution of $D^\alpha f(x) = 1_C(x)$ when $\alpha = d$ is $f(x) = I^\alpha_{[0,x]} 1_C(t)$. Applying equation (8) with $x_0 = 0$ and $x_N = x$ and choosing x_i^* to be such that $1_C(x_i^*)$ is maximum in the interval $[x_i, x_{i+1}]$, one gets [14]

$$f(x) = I^\alpha_{[0,x]} 1_C = \lim_{N\to\infty} \sum_{i=0}^{N-1} F_C^i \frac{(x_{i+1} - x_i)^\alpha}{\Gamma(1+\alpha)} = \frac{S_C(x)}{\Gamma(1+\alpha)}, \tag{9}$$

where F_C^i is a flag function that takes value 1 if the interval $[x_i, x_{i+1}]$ contains a point of the set C and 0 otherwise. $S_C(x)$ is the Cantor (devil's) staircase (figure 1(b)), i.e. a function that is flat almost everywhere except on an infinite number of singular points corresponding to the underlying Cantor set where it grows from 0 to 1. Moreover, equation (9) introduces the *fractional measure* of a fractal set: for the Cantor set C it is defined as $\mathcal{F}^\alpha(C) = I^\alpha_{[0,1]} 1_C(x)$. In fact $\mathcal{F}^\alpha(C)$ is infinite if $\alpha < d$, and 0 if $\alpha > d$. For $\alpha = d$, we find $\mathcal{F}^\alpha(C) = \frac{1}{\Gamma(1+\alpha)}$, since $S_C(1) = 1$. This measure definition yields the same value as that predicted by the Hausdorff measure, the difference being represented only by a different value of the normalization constant. Eventually, from equation (9), it follows that the fractional measure of a generalized Cantor set $C_\alpha^{[a,b]}$ of dimension α built over the interval $[a, b]$ of the x-axis is

$$\mathcal{F}^\alpha(C_\alpha^{[a,b]}) = I^\alpha_{[a,b]} 1_{C_\alpha^{[a,b]}} = \frac{(b-a)^\alpha}{\Gamma(1+\alpha)}, \tag{10}$$

where $1_{C_\alpha^{[a,b]}}$ is a function that is equal to 1 if $x \in C_\alpha^{[a,b]}$ and is equal to 0 elsewhere.

2.3. The fractal bar

In the present section, we intend to solve a simple case using the mathematical tools presented in the previous section. Our aim is to show that experimental diagrams (see, for instance, [12]) such as the one of figure 1(b) can also be obtained analytically. More details can be found in [4].

Thus, let us consider a uniaxial model [4], hereafter called a *fractal Cantor bar*, according to Feder's terminology [16], i.e. a bar of length b deformable on a fractal subset of dimension $(1 - d_\varepsilon)$. The longitudinal axis is z. The bar is clamped in $z = 0$, whereas a tensile load N is applied at its end $z = b$ (figure 2). A strain field will arise that is zero almost everywhere except in an infinite number of points (corresponding to the deformable subset) where it is singular. The displacement singularities can be characterized by the LFD of order equal to the fractal dimension $\alpha = 1 - d_\varepsilon$ of the domain of the singularities, the unique value for which the LFD is finite and different from zero (the critical value). Therefore, we can analytically define the fractal strain ε^* as the LFD of order α of the displacement:

$$\varepsilon^*(z) = D^\alpha w(z). \tag{11}$$

Let us observe that, in equation (11), the non-integer physical dimensions $[L]^{d_\varepsilon}$ of ε^* are introduced by the LFD, whereas in section 2.1 they are a geometrical consequence of the fractal dimension of the localization domain.

Without losing generality, let us assume the deformable subset to be the triadic Cantor set $C_\alpha^{[0,b]}$ built on $[0, b]$,

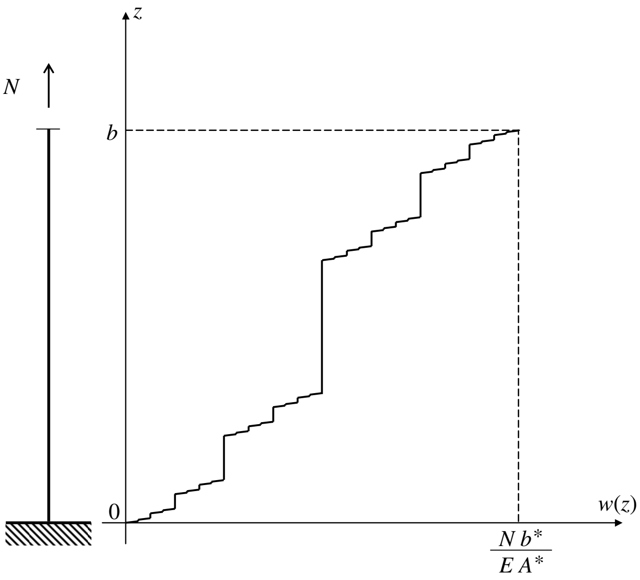

Figure 2. The fractal bar subjected to an axial load: the displacement field.

$\alpha = \ln 2/\ln 3$. In order to compute the displacement function $w(z)$, we need the proper constitutive law. Here, for the sake of simplicity, we use a linear elastic relation and assume $d_\sigma = d_\varepsilon$: in this case the coefficient of proportionality between fractal stress and fractal strain coincides with the one between the nominal quantities, i.e. it is Young's modulus E. In symbols: $\sigma^* = E\varepsilon^*$.

For equilibrium reasons, the internal axial force is constant and equal to N throughout the bar. Hence, we get a fractal strain ε^* that is equal to N/EA^* over the deformable subset and is 0 elsewhere. The kinematic equation (11) becomes

$$D^\alpha w(z) = \frac{N}{EA^*} 1_{C_\alpha^{[0,b]}}(z). \tag{12}$$

Introducing the dimensionless quantities $\tilde{w} = w/b$, $\tilde{z} = z/b$ ($\tilde{z} \in [0,1]$), we can apply the scaling property expressed by (for $a = 0$; [8])

$$\mathrm{D}^q_{0+} f(bx) = \frac{\mathrm{d}^q f(bx)}{[\mathrm{d}x]^q} = b^q \frac{\mathrm{d}^q f(bx)}{[\mathrm{d}(bx)]^q}, \tag{13}$$

which is valid also for the LFD, to get $D^\alpha w(z) = b^{1-\alpha} D^\alpha \tilde{w}(\tilde{z})$. Equation (12) can therefore be expressed in dimensionless form as follows:

$$D^\alpha \tilde{w}(\tilde{z}) = \frac{N}{EA^* b^{1-\alpha}} 1_C(\tilde{z}), \tag{14}$$

where C is the triadic Cantor set built on [0, 1] as indicated in section 2.1. In this form, the solution of the differential equation (14) can be obtained directly from equation (9):

$$\tilde{w}(\tilde{z}) = \frac{N}{EA^* b^{1-\alpha}} \frac{S_C(\tilde{z})}{\Gamma(1+\alpha)}, \tag{15}$$

where $S_C(x)$ is the Cantor staircase built on the interval [0, 1] and rising from 0 to 1. Recovering the physical quantities yields

$$w(z) = \frac{Nb^*}{EA^*} S_C\left(\frac{z}{b}\right), \tag{16}$$

where $b^* = \frac{b^\alpha}{\Gamma(1+\alpha)}$ is the fractal measure of the deformable subset. Equation (16) is plotted in figure 2. Let us emphasize that the Cantor staircase, introduced geometrically in section 2.2, is now obtained analytically. Furthermore, notice that equation (16) provides important information about the size effect affecting the global deformation. In fact we find that the free end displacement $w(b)$ is equal to $\frac{Nb^*}{EA^*}$, i.e. $w(b) \sim b^\alpha$. This means that the displacement increases less than linearly with the bar length, as occurs with classical elastic bodies. From the point of view of the overall deformation $\varepsilon = w(b)/b$, we get $\varepsilon \sim b^{-(1-\alpha)}$: it decreases with size as a consequence of the strain localization on a lacunar fractal subset.

3. Non-local approach

Different approaches have been proposed to take into account the effects of material microstructure on the classical continuum equations [6, 7]. A new model based on fractional calculus has recently been suggested [5], and it will be described in this section.

3.1. Elastic bar with long-range interactions

Let us now consider an elastic bar, of length b, that is subjected to longitudinal forces per unit volume $f(z)$. Let us discretize the bar into m equal elements, each with volume $V_j = A\Delta z$ ($j = 1, \ldots, m$), where A is the cross-section area and $\Delta z = b/m$. The generic volume element V_j is located at the abscissa $z_j = j\Delta z$. Its equilibrium equation can be written taking into account external loads, contact forces provided by adjacent volumes, V_{j-1} and V_{j+1}, which are denoted by N_j and N_{j+1}, respectively, and long-range interactions Q_j applied on V_j by the surrounding non-adjacent elements of the bar (figure 3). In formulae

$$\Delta N_j + Q_j = \Delta N_j + \sum_{h=1}^{j-1} Q^{(h,j)} - \sum_{h=j+1}^{m} Q^{(h,j)} = -f_j A\Delta z, \tag{17}$$

where $f_j = f(z_j)$, $\Delta N_j = N_{j+1} - N_j$ and $Q^{(h,j)}$ ($h \neq j$) are the long-range forces that surrounding volume elements apply on element V_j. These forces can be modelled as

$$Q^{(h,j)} = \mathrm{sgn}(z_h - z_j)[w(z_h) - w(z_j)]g(z_h, z_j)V_j V_h, \tag{18}$$

with $g(z_h, z_j)$ being a real-valued monotonically decreasing function expressed as

$$g(z_h, z_j) = \frac{Ec_\alpha \alpha}{A\Gamma(1-\alpha)|z_h - z_j|^{1+\alpha}}, \quad (0 \leqslant \alpha \leqslant 1) \tag{19}$$

and $\mathrm{sgn}(z)$ being the classical signum function:

$$\mathrm{sgn}(z) = \begin{cases} -1 & \text{if} \quad z < 0, \\ 0 & \text{if} \quad z = 0, \\ +1 & \text{if} \quad z > 0. \end{cases} \tag{20}$$

Note that the constant c_α in equation (19) has the anomalous dimensions $[L]^{\alpha-2}$. The classical continuum mechanics is recovered, not only for $\alpha \to 0$ but also for $\alpha \to 1$, since

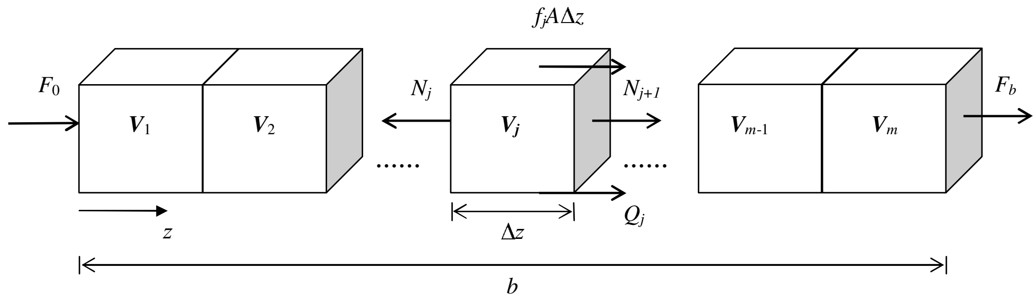

Figure 3. Discretized elastic bar: equilibrium of the generic volume element V_j.

the Gamma function is infinite for a vanishing argument. Substituting equation (18) into equation (17) yields

$$\Delta N_j - \frac{E c_\alpha \alpha A}{\Gamma(1-\alpha)} \Delta z \left[\sum_{h=1}^{j-1} \frac{w(z_h) - w(z_j)}{(z_j - z_h)^{1+\alpha}} \Delta z - \sum_{h=j+1}^{m} \frac{w(z_j) - w(z_h)}{(z_h - z_j)^{1+\alpha}} \Delta z \right] = -f_j A \Delta z. \qquad (21)$$

Dividing equation (21) by $EA\Delta z$ and assuming a linear elastic material, i.e. $\sigma(z) = E\varepsilon(z) = E\,\mathrm{d}w/\mathrm{d}z$, the following differential equation is obtained for $\Delta z \to 0$ [5]:

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} - c_\alpha \left(\widehat{\mathbf{D}}^\alpha_{0+} w + \widehat{\mathbf{D}}^\alpha_{b-} w \right) = -\frac{f(z)}{E}. \qquad (22)$$

$\widehat{\mathbf{D}}^\alpha_{0+} w$ and $\widehat{\mathbf{D}}^\alpha_{b-} w$ in equation (22) represent the integral terms in the Marchaud fractional derivatives on a finite interval (note that $a = 0$), which have been proved to coincide [8], for a certain class of functions, with the Riemann–Liouville fractional derivatives (equations (3) and (4)):

$$\mathbf{D}^\alpha_{a+} f(x) = \frac{f(x)}{\Gamma(1-\alpha)(x-a)^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)} \times \int_a^x \frac{f(x) - f(\xi)}{(x-\xi)^{1+\alpha}} \mathrm{d}\xi, \qquad (23)$$

$$\mathbf{D}^\alpha_{b-} f(x) = \frac{f(x)}{\Gamma(1-\alpha)(b-x)^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)} \times \int_x^b \frac{f(x) - f(\xi)}{(\xi - x)^{1+\alpha}} \mathrm{d}\xi. \qquad (24)$$

Equation (22) can be rewritten, equivalently, in terms of the Marchaud fractional derivatives (equations (23) and (24)) as

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} - c_\alpha \left[\left(\mathbf{D}^\alpha_{0+} w + \mathbf{D}^\alpha_{b-} w \right) - w \left(\mathbf{D}^\alpha_{0+} 1 + \mathbf{D}^\alpha_{b-} 1 \right) \right] = -\frac{f(z)}{E}, \qquad (25)$$

which represents a non-homogeneous ordinary fractional differential equation, with non-constant coefficients, since the derivatives of the unit function are neither zero nor constant.

It is worth observing that the fractional formula of integration by parts [8]:

$$\int_a^b \left[(\mathbf{D}^\alpha_{a+} f) \cdot g \right] \mathrm{d}x = \int_a^b \left[(\mathbf{D}^\alpha_{b-} g) \cdot f \right] \mathrm{d}x \qquad (26)$$

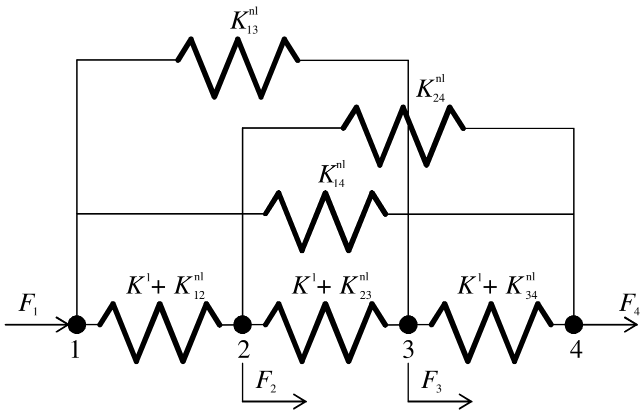

Figure 4. Discretized elastic bar: spring-point model.

can be exploited to verify that the resultant of the non-local forces, expressed by the second term in the left-hand side of equation (25), is zero. In fact, by choosing either f or g equal to 1, it is easy to check that

$$\int_0^b \left[\left(\mathbf{D}^\alpha_{0+} w + \mathbf{D}^\alpha_{b-} w \right) - w \left(\mathbf{D}^\alpha_{0+} 1 + \mathbf{D}^\alpha_{b-} 1 \right) \right] \mathrm{d}z = 0. \qquad (27)$$

Unfortunately, analytical solutions of equation (25) (i.e. equation (22)) are not available in the literature to the best of our knowledge.

Eventually, observe that the fractional terms in equation (25) (i.e. equation (22)) vanish both for $\alpha = 0$ and $\alpha = 1$, as already stated about equation (19). In fact, for $\alpha = 0$ the left and right Marchaud derivatives provide the function itself (and hence the term in the square brackets becomes $2w - 2w$); on the other hand, for $\alpha = 1$ the left derivative coincides with the classical first derivative, whereas the right one coincides with its opposite.

3.2. Equivalent mechanical model

The physical validity of equation (17) can be explained by considering a simple discrete spring-point model of the bar [5], as reported in figure 4 only for four points. Local forces between adjacent particles are taken into account by springs with elastic stiffness $K^{\mathrm{l}} = EA/\Delta z$. On the other hand, long-distance interactions between particles are modelled by linear springs with distance-decay stiffness as $K^{\mathrm{nl}}_{jh} = g(z_j, z_h)$, where function g is provided by equation (19).

Thus, the equilibrium equation for the generic node located at the abscissa z_j may be written as

$j = 1$:

$$K^1(w_2 - w_1) + (A\Delta z)^2 \sum_{h=2}^{m} g(z_h, z_1)(w_h - w_1) = -F_1, \tag{28}$$

$j = 2, \ldots, m-1$:

$$K^1(w_{j-1} - 2w_j + w_{j+1}) + (A\Delta z)^2 \sum_{h=1, h\neq j}^{m} g(z_h, z_j) \times (w_h - w_j) = -F_j, \tag{29}$$

$j = m$:

$$K^1(w_{m-1} - w_m) + (A\Delta z)^2 \sum_{h=1}^{m-1} g(z_m, z_h)(w_h - w_m) = -F_m, \tag{30}$$

where the right-hand sides of equations (28)–(30) represent the body forces applied to material particles ($F_j = f_j A\Delta z$), while F_1 and F_m represent the forces acting at the bar extremes. Equilibrium equations (28)–(30) can be synthesized in a compact form by introducing the non-local coefficient matrix $\mathbf{K} = \mathbf{K}^{\mathrm{l}} + \mathbf{K}^{\mathrm{nl}}$ as

$$\mathbf{K}\mathbf{w} = \mathbf{F}, \tag{31}$$

where $\mathbf{w}$ and $\mathbf{F}$ are the (m)-dimensional displacement and force vectors, respectively; K^{l} is the tri-diagonal matrix related to contact contributions due to adjacent elements:

$$\mathbf{K}^{\mathrm{l}} = \begin{bmatrix} K^{\mathrm{l}} & -K^{\mathrm{l}} & \ldots & \ldots & 0 \\ -K^{\mathrm{l}} & 2K^{\mathrm{l}} & -K^{\mathrm{l}} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & -K^{\mathrm{l}} & 2K^{\mathrm{l}} & -K^{\mathrm{l}} \\ 0 & \ldots & \ldots & -K^{\mathrm{l}} & K^{\mathrm{l}} \end{bmatrix}, \tag{32}$$

while $\mathbf{K}^{\mathrm{nl}}$ is the fully populated, non-local stiffness matrix:

$$\mathbf{K}^{\mathrm{nl}} = \begin{bmatrix} K^{\mathrm{nl}}_{11} & -K^{\mathrm{nl}}_{12} & \ldots & -K^{\mathrm{nl}}_{1m} \\ -K^{\mathrm{nl}}_{12} & K^{\mathrm{nl}}_{22} & \ldots & -K^{\mathrm{nl}}_{2m} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ -K^{\mathrm{nl}}_{1m} & \ldots & \ldots & K^{\mathrm{nl}}_{mm} \end{bmatrix} \tag{33}$$

with

$$\mathbf{K}^{\mathrm{nl}}_{jj} = \sum_{h=1, h\neq j}^{m} K^{\mathrm{nl}}_{jh} \tag{34}$$

and $K^{\mathrm{nl}}_{jh} = (A\Delta z)^2 g(z_j, z_h)$.

3.3. Numerical applications

The non-local stiffness matrix obtained by the point-spring model (equation (33)) is found to coincide, for $\Delta z \to 0$, with that obtained by a discretizing equation (22) with fractional finite differences [5, 17]. Let us consider a clamped-free bar, loaded at the free end by a tensile force $F = 10^4$ N. The bar dimensions are selected as follows: $b = 200$ mm and $A = 100\,\mathrm{mm}^2$. The following material characteristics are considered: $E = 72$ GPa, $\alpha = 0.5$ and $c_\alpha = 0.03\,\mathrm{mm}^{-1.5}$.

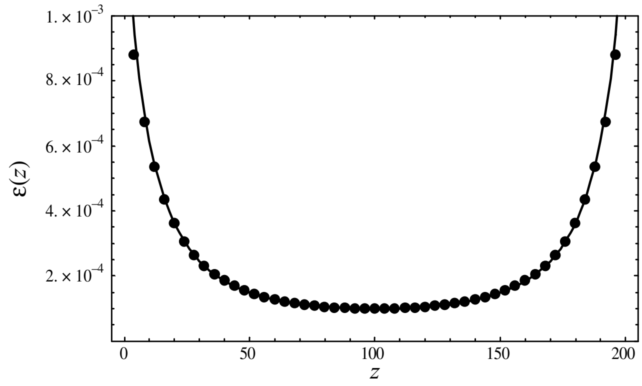

Figure 5. Axial strain in a free–free bar: the continuum fractional approach (continuous line) versus the discrete spring-point model (dots).

The axial strain related to such a structure is displayed in figure 5, showing perfect agreement between the continuum fractional approach (continuous line) and the discrete spring-point model presented in the previous section (dots). Note that the axial strains are uniform along the bar core and increase at the boundary, as from experimental evidence. Similar results are also predicted by different non-local models (see, for instance, [6]).

4. Conclusions

Classical continuum theory works properly at the macro-scale, where the effect of material microstructure can be neglected. Recent technological progress (e.g. nano-devices) as well as the possibility of having a deeper insight into material behaviour forced the scientific community to analyse phenomena taking place at the meso/micro level. One way to address such problems is the use of enriched continuum mechanics models. Among these models, in the present paper we revisited two recently proposed approaches whose common feature is represented by the use of fractional calculus, which turns out to be a powerful tool to handle non-standard continua.

The former approach deals with fractal media, i.e. with solids whose microstructure is such that strain localizes onto fractal subsets. The latter approach deals with solids characterized by non-local long-range interactions. Although they are rather different, it is interesting to highlight the following analogies/differences between them.

1. Both the approaches yield non-standard fractional derivatives, the former one using the LFD and the latter one the integral part of the Marchaud fractional derivative.
2. The local approach is based on kinematic arguments, while the non-local one stems from a static analysis.
3. Both the approaches provide a non-uniform strain field under a uniform stress field.
4. Both the models introduce a displacement derivative of order lower than the classical one, different from what was proposed by gradient theory [6], which provides

derivatives of the displacement of higher orders. This is a drawback of gradient theory, since higher order differential equations need extra boundary conditions to be solved, whose physical meaning is still unclear.

5. In the local model, the LFD replaces the classical derivative (equation (11)), whereas in the non-local model the fractional derivative represents a correction to be added to the classical equilibrium equation (equation (21)) in order to take internal forces into account.
6. In the local approach, the kinematic equation appears as the straightforward generalization to fractal media of the definition of strain in continuum mechanics. This provides the key to generalize the principle of virtual work as well as to highlight the static–kinematic duality for fractal media [3]. On the other hand, this aspect remains hidden in the non-local approach.

References

[1] Kolwankar K M and Gangal A D 1996 *Chaos* **6** 502
[2] Kolwankar K M and Gangal A D 1998 *Phys. Rev. Lett.* **80** 214
[3] Carpinteri A, Chiaia B and Cornetti P 2001 *Comput. Methods Appl. Mech. Eng.* **191** 3
[4] Carpinteri A and Cornetti P 2002 *Chaos Solitons Fractals* **13** 85
[5] Di Paola M and Zingales M 2008 *Int. J. Solids Struct.* **21** 5642
[6] Aifantis E C 1994 *J. Mech. Behav. Mater.* **5** 355
[7] Eringen A C and Edelen D G B 1972 *Int. J. Eng.* **10** 233
[8] Samko S G, Kilbas A A and Marichev O I 1993 *Fractional Integrals and Derivatives* (Amsterdam: Gordon and Breach Science Publishers)
[9] Carpinteri A, Chiaia B and Cornetti P 2004 *Comput. Struct.* **82** 499
[10] Carpinteri A 1994 *Mech. Mater.* **18** 89
[11] Carpinteri A 1994 *Int. J. Solids Struct.* **31** 291
[12] Kleiser T and Bocek M 1986 *Z. Met.kd.* **77** 582
[13] Poliakov A B, Hermann H J, Podladchikov Y Y and Roux S 1995 *Fractals* **2** 576
[14] Kolwankar K M and Gangal A D 1999 Local fractional calculus: a calculus for fractal space-time *Fractals: Theory and Applications in Engineering* (Delft: Springer)
[15] Carpinteri A and Mainardi F 1997 *Fractals and Fractional Calculus in Continuum Mechanics* (Wien: Springer)
[16] Feder J 1988 *Fractals* (New York: Plenum)
[17] Kilbas A A, Srivastava H M and Trujillo J J 2006 *Theory and Applications of Fractional Differential Equations* (Amsterdam: Elsevier)