Strength, Fracture and Complexity 5 (2007/9) 53–62
DOI 10.3233/SFC-2009-0090
IOS Press

Extension of the de Saint Venant and Kirchhoff theories to functionally graded materials

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Abstract. Functionally graded materials (FGMs) are special composites in which the volume fraction of constituent materials varies gradually, giving continuously graded mechanical properties. The aim of the present paper is the extension of the de Saint Venant's theory to axial load, bending and torsional moments, as well as to uniaxial or biaxial shears, from homogeneous members to FGM composite beams. The study is carried out in a simple and systematic way: its main assumption is the conservation of the plane sections. We will focus our attention on stresses and stored energy for FGM beams, the strains being simply derived by the linear elastic isotropic constitutive laws. Such approach is finally extended to plates in flexure, revisiting the Kirchhoff's theory.

Keywords: De Saint Venant, Kirchhoff, beam, plate, functionally graded materials

1. Introduction

Non-homogeneous material systems with gradual variation in properties are collectively referred to as functionally graded materials (FGMs). The spatial variation of the elastic and physical properties makes FGMs an attractive alternative to composite solids opening new possibilities for optimizing both material and component structures to achieve high performance and efficiency. Laminated plates and shells have had until now many structural applications as described in great detail in numerous publications, e.g. [1–3], among many others. Recently there has been a growing interest in FGMs. In fact, gradual variation of properties in FGMs, unlike abrupt changes encountered in discretely layered systems, is known to improve failure performance while preserving the intended thermal, tribological, and/or structural benefits of combining dissimilar materials. A general account, including methods of fabrication, has already been reported [4], whereas strength and fracture of FGMs have recently been treated in a dedicated special issue of Engineering Fracture Mechanics [5]. In general, a huge amount of papers on FGMs has recently been published (e.g. we found $\sim 10^3$ records using FGM as the key word in the paper title); among them two our papers [6,7] present analyses on thermal stresses, delamination, fatigue, cracks and re-entrant corners, being basically devoted to the strength prediction of FGMs, numerically confirmed by finite element calculations [8].

Regarding exact solutions in elasticity of FGM, several pioneer papers must be mentioned [8–21], among others. In particular, the plane stress theory for a moderately thick homogeneous elastic plate

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or for a laminated plate, to include stretching and bending solutions for an inhomogeneous thermoelastic plate, has been extended in [18]. The inhomogeneities, both in the elastic properties and thermal expansion coefficients, were considered to vary symmetrically through the thickness of the plate. The thermoelastic solutions were expressed in terms of the solution of the approximate two-dimensional thin plate equations governing an equivalent homogeneous plate, and were exact solutions of the full equations of three-dimensional thermoelasticity. By considering laminated plates to be a special case of inhomogeneous plates, they derived an exact laminate theory for plates consisting of different homogeneous and isotropic layers, perfectly bonded to each other. For the considered symmetric plates [18–20], the bending and stretching deformation modes uncouple and were treated separately. Such authors have thus presented wide classes of exact isotropic elasticity and thermoelasticity solutions for thick elastic plates such that the Lamé elastic constants vary, either continuously or discontinuously. Laminated plates are frequently constructed so as to be symmetric, but FGMs often do not have this property. For a non-symmetric plate the stretching and bending modes are in general coupled and must be treated concurrently, as further described in [21].

Even if we do not want to compete with such previous and more sophisticated models, we present in this paper a simple extension of the de Saint Venant and Kirchhoff theories (the reader should refer to [22], regarding such classical theories and their related equations) to treat beams and plates composed by FGMs in a simple and engineering way. The main assumption is the conservation of the plane sections during structure deformation. The paper is planned as follows: in Section 2 the de Saint Venant theory is extended to FGM beams, by treating axial load, bending and torsional moments and shear; in Section 3 the Kirchhoff theory is extended to FGM plates under bending; finally, in Section 4 the elastic line or plane equations are formulated and discussed for a FGM beam or plate, just before our concluding remarks. We show that, for the majority of the cases, a FGM structure can be treated as the homogeneous counterpart by simply substituting its rigidity (elastic modulus times area or moment of inertia) with the related elastic weighted rigidity, similarly to the complementary treatment made in [21]. Our results could reveal themselves interesting for engineering applications in designing innovative materials and structures.

2. De Saint Venant beams

2.1. Axial load and bending moments

Assuming the conservation of the plane sections implies that the longitudinal dilation ε_z must be linear with respect to the Cartesian coordinates x, y:

$$\varepsilon_z(x,y) = \frac{1}{E_r}(ax + by + c), \tag{1}$$

where E_r is an arbitrary reference value of Young' modulus, and a, b and c are unknown constants. Imposing the conditions of equivalence for the stress $\sigma_z(x,y) = E(x,y)\varepsilon_z(x,y)$ on the cross-section area A to the applied axial load F and bending moments M_x and M_y implies:

$$\int_A \sigma_z(x,y)\,\mathrm{d}A = F, \tag{2a}$$

$$\int_A \sigma_z(x,y) y \, \mathrm{d}A = M_x, \tag{2b}$$

$$\int_A \sigma_z(x,y) x \, \mathrm{d}A = -M_y. \tag{2c}$$

Introducing Eq. (1) into Eqs (2) gives:

$$\sigma_z(x,y) = \frac{E(x,y)}{E_r}\left(\frac{F}{A^*} + \frac{M_x}{I_x^*} y - \frac{M_y}{I_y^*} x\right), \tag{3}$$

where:

$$A^* = \int_A \frac{E(x,y)}{E_r} \, \mathrm{d}A \tag{4}$$

is the *elastic* weighted area, as well as:

$$I_x^* = \int_A \frac{E(x,y)}{E_r} y^2 \, \mathrm{d}A, \tag{5a}$$

$$I_y^* = \int_A \frac{E(x,y)}{E_r} x^2 \, \mathrm{d}A \tag{5b}$$

represent the *elastic* weighted moments of inertia.

The origin of the reference system, the *elastic* centroid, is defined from the following relationships:

$$x_G = \frac{S_y^*}{A^*}, \qquad y_G = \frac{S_x^*}{A^*} \tag{6a}$$

with:

$$S_x^* = \int_A \frac{E(x,y)}{E_r} y \, \mathrm{d}A = 0, \tag{6b}$$

$$S_y^* = \int_A \frac{E(x,y)}{E_r} x \, \mathrm{d}A = 0, \tag{6c}$$

and the orientation of x, y axes must be principal from an *elastic* viewpoint, i.e.:

$$I_{xy}^* = \int_A \frac{E(x,y)}{E_r} xy \, \mathrm{d}A = 0. \tag{7}$$

As a consequence of Eqs (6) and (7), the elastic energy stored in the FGM beam per unit length is given by:

$$\frac{\mathrm{d}L}{\mathrm{d}z} = \int_A \frac{1}{2} \frac{\sigma_z^2(x,y)}{E(x,y)} \, \mathrm{d}A = \frac{1}{2E_r}\left(\frac{F^2}{A^*} + \frac{M_x^2}{I_x^*} + \frac{M_y^2}{I_y^*}\right), \tag{8}$$

i.e., axial load and bending moments remain energetically orthogonal in FGM beams.

2.2. Torsional moment

2.2.1. Circular cross-section with shear elastic modulus $G = G(r)$

To obtain a solution in closed form, we consider the simplest case of a circular bar with the shear elastic modulus G as a function only of the radial coordinate r, i.e.: $G = G(r)$.

We assume – as for the homogeneous material – the following displacement field of components u, v and w, respectively, along x, y and z:

$$\begin{cases} u = -\varphi_z y, \\ v = \varphi_z x, \\ w = 0. \end{cases} \tag{9}$$

It means that a given cross-section rotates by an infinitesimal angle φ_z around the z axis remaining, at the same time, plane. Introducing the rotation per unit length $\Theta = \varphi_z/z$, from the kinematic and constitutive equations we obtain:

$$\begin{aligned} \tau_{zx}(x,y) &= -\Theta G(r) y, \\ \tau_{zy}(x,y) &= \Theta G(r) x, \end{aligned} \tag{10a}$$

and, therefore,

$$\tau_z = \Theta G(r) r. \tag{10b}$$

As suggested by Eq. (10a), the shearing stress, having modulus given by Eq. (10b), is orthogonal to the corresponding radial vector.

The equilibrium condition:

$$\frac{\partial \tau_{zx}(x,y)}{\partial x} + \frac{\partial \tau_{zy}(x,y)}{\partial y} = 0 \tag{11}$$

implies:

$$\frac{\partial G(r)}{\partial x} y = \frac{\partial G(r)}{\partial y} x, \tag{12}$$

or, equivalently:

$$\frac{\partial G(r)}{\partial r} \frac{\partial r(x,y)}{\partial x} y = \frac{\partial G(r)}{\partial r} \frac{xy}{r} = \frac{\partial G(r)}{\partial r} \frac{\partial r(x,y)}{\partial y} x, \tag{13}$$

that is identically satisfied.

From the conditions of equivalence to vanishing shears, we have:

$$\int_A \tau_{zx}(x,y)\, \mathrm{d}A = 0, \tag{14a}$$

$$\int_A \tau_{zy}(x,y)\, \mathrm{d}A = 0. \tag{14b}$$

Equations (14) define the position of the *elastic* centroid, similarly to Eqs (6).

The equivalence condition to the applied torsional moment is:

$$\int_A \{x\tau_{zx}(x,y) - y\tau_{zy}(x,y)\}\,\mathrm{d}A = M_z \tag{15}$$

and gives:

$$\Theta = \frac{M_z}{G_r I_p^*}, \tag{16}$$

where G_r is an arbitrary reference value of shear modulus and

$$I_p^* = \int_A \frac{G(r)}{G_r} r^2\,\mathrm{d}A \tag{17}$$

is the *elastic* weighted polar moment of inertia.

As a consequence, the shearing stress of Eq. (10b) becomes:

$$\tau_z = \frac{G(r)}{G_r}\frac{M_z}{I_p^*} r, \tag{18}$$

and the elastic energy stored in the FGM beam per unit length is:

$$\frac{\mathrm{d}L}{\mathrm{d}z} = \int_A \frac{1}{2}\frac{\tau_z^2(x,y)}{G(x,y)}\,\mathrm{d}A = \frac{1}{2}\frac{M_z^2}{G_r I_p^*}. \tag{19}$$

2.2.2. Generic cross-section

The kinematic assumptions must be modified as:

$$\begin{cases} u = -\Theta z(y - y_c), \\ v = \Theta z(x - x_c), \\ w = \Theta\omega(x,y), \end{cases} \tag{20}$$

$\omega(x,y)$ being the warping function and $C(x_c, y_c)$ the *elastic* weighted torsional or shear center: a given cross-section rotates by an infinitesimal angle around it and does not remain, at the same time, plane. From the kinematic and constitutive equations the stresses are:

$$\begin{aligned} \tau_{zx}(x,y) &= \Theta G(x,y)\left\{\frac{\partial\omega(x,y)}{\partial x} - (y - y_c)\right\}, \\ \tau_{zx}(x,y) &= \Theta G(x,y)\left\{\frac{\partial\omega(x,y)}{\partial y} + (x - x_c)\right\}. \end{aligned} \tag{21}$$

Assuming the origin of the reference system coincident with the *elastic* centroid, defined by Eqs (6) with

the substitution $E \to G$, the equivalences to the vanishing shears of Eqs (14) give:

$$x_c = \frac{1}{A^*}\int_A \frac{G(x,y)}{G_r}\frac{\partial \omega(x,y)}{\partial y}\,\mathrm{d}A, \tag{22a}$$

$$y_c = -\frac{1}{A^*}\int_A \frac{G(x,y)}{G_r}\frac{\partial \omega(x,y)}{\partial x}\,\mathrm{d}A, \tag{22b}$$

where A^* is defined by Eq. (4) ($E \to G$).

In addition, Eq. (10) becomes:

$$\frac{\partial G(x,y)}{\partial x}\left\{\frac{\partial \omega(x,y)}{\partial x} - (y-y_c)\right\} + \frac{\partial G(x,y)}{\partial y}\left\{\frac{\partial \omega(x,y)}{\partial y} + (x-x_c)\right\} + G(x,y)\left\{\frac{\partial^2 \omega(x,y)}{\partial x^2} + \frac{\partial^2 \omega(x,y)}{\partial y^2}\right\} = 0. \tag{23}$$

The warping function $\omega(x,y)$ can be obtained solving the previous integro-differential equation, in which we have to introduce Eqs (22), coupled with the same boundary conditions for the homogeneous case, i.e., $\tau_{zx}(x,y)n_x + \tau_{zy}(x,y)n_y = 0$, where $\bar{n}(x,y)$ is a vector orthogonal to the cross-section boundary. From Eq. (15) we obtain:

$$\Theta = \frac{M_z}{G_r I_t^*}, \tag{24}$$

where the *elastic* weighted factor of torsional rigidity is:

$$I_t^* = \int_A \frac{G(x,y)}{G_r}\left\{x^2 + y^2 + x\frac{\partial \omega(x,y)}{\partial y} + y\frac{\partial \omega(x,y)}{\partial x}\right\}\mathrm{d}A. \tag{25}$$

As a consequence, the stresses of Eq. (21) become:

$$\begin{aligned}\tau_{zx}(x,y) &= \frac{G(x,y)}{G_r}\frac{M_z}{I_t^*}\left\{\frac{\partial \omega(x,y)}{\partial x} - (y-y_c)\right\},\\ \tau_{zx}(x,y) &= \frac{G(x,y)}{G_r}\frac{M_z}{I_t^*}\left\{\frac{\partial \omega(x,y)}{\partial y} + (x-x_c)\right\},\end{aligned} \tag{26}$$

and the elastic energy stored in the FGM beam per unit length is given by Eq. (19) with $I_p^* \to I_t^*$. In FGM beams, torsion is always energetically orthogonal to eccentric axial load but not necessary to shear.

2.2.3. Thin-walled cross-sections

For FGM thin-walled sections of thickness $b(s)$, defined by a linear coordinate s over a mean line describing a closed area Ω, the well-known Bredt's formula remains valid:

$$\tau_{zs}(s) = \frac{M_z}{2\Omega b(s)} \tag{27}$$

as well as Eq. (19), in which the *elastic* weighted factor of torsional rigidity must be defined as:

$$I_t^* = 4\Omega^2 \Big/ \left(\oint \frac{G_r \, \mathrm{d}s}{G(s)b(s)} \right). \tag{28}$$

2.3. Shear

2.3.1. Shear along a central axis

The well-known Jourawsky's formula, giving the mean value of the shear stresses due to a shear T_y and orthogonal to a line of length b, dividing the section area in a sub-section area A', for FGM becomes:

$$\langle \tau_{zs} \rangle = \frac{T_y S_x^*(A')}{2 I_x^*(A) b}, \quad x \leftrightarrow y, \tag{29}$$

where we must consider Eqs (4) and (6) in which we have to substitute A with A'. In addition:

$$\frac{\mathrm{d}L}{\mathrm{d}z} = \frac{1}{2G_r} t_y \frac{T_y^2}{A^*}, \quad x \leftrightarrow y, \tag{30}$$

where the coefficient t_y can be estimated evaluating the energy per unit length as:

$$\frac{\mathrm{d}L}{\mathrm{d}z} = \int_A \frac{\tau_{zs}^2}{2G(x,y)} \, \mathrm{d}A \approx \int_A \frac{\langle \tau_{zs} \rangle^2}{2G(x,y)} \, \mathrm{d}A. \tag{31}$$

2.3.2. Biaxial shear

For a solution in closed form concerning biaxial shear we have to consider thin cross-sections. The shearing stresses will be:

$$\tau_{zs}(s) = \frac{T_y S_x^*(A')}{2 I_x^* b} + \frac{T_x S_y^*(A')}{2 I_y^* b}, \tag{32}$$

and, from Eq. (31):

$$\frac{\mathrm{d}L}{\mathrm{d}z} = \frac{1}{2G_r A^*} (t_x T_x^2 + t_y T_y^2 + t_{xy} T_x T_y), \tag{33}$$

where:

$$t_x = \frac{G_r A^*}{I_y^{*2}} \oint \frac{S_y^{*2}(A')}{bG(x,y)} \, \mathrm{d}s, \tag{34a}$$

$$t_y = \frac{G_r A^*}{I_x^{*2}} \oint \frac{S_x^{*2}(A')}{bG(x,y)} \, \mathrm{d}s, \tag{34b}$$

$$t_{xy} = \frac{G_r A^*}{I_y^* I_x^*} \oint \frac{S_x^*(A') S_y^*(A')}{bG(x,y)} \, \mathrm{d}s, \tag{34c}$$

the mutual factor t_{xy}, if different from zero, shows that T_x and T_y are in general not energetically orthogonal. If $t_{xy} = 0$, the cross-section is said to be symmetrical from an elastic viewpoint. For *elastic* symmetrical cross-sections, T_x and T_y are energetically orthogonal.

3. Kirchhoff plates

3.1. Bending

Considering a plate of small thickness h along the z axis, loaded by an orthogonal force per unit area $q(x, y)$, with a variable Young's modulus $E(z)$ and a constant Poisson's ratio v, we can assume the well-known Kirchhoff kinematic hypothesis:

$$\begin{cases} u = \varphi_x z = -\dfrac{\partial w}{\partial x} z, \\ v = \varphi_y z = -\dfrac{\partial w}{\partial y} z, \\ w = w(x, y), \end{cases} \tag{35}$$

where the rotation must be considered around the *elastic* centroid, assumed as the origin of the reference systems:

$$S^* = \int_h \frac{E(z)}{E_r} z \, \mathrm{d}z = 0. \tag{36}$$

From the kinematic and constitutive equations it descends:

$$\begin{cases} \sigma_x(z) = \dfrac{E(z)}{1 - v^2}(\chi_x + \nu\chi_y)z, \\ \sigma_y(z) = \dfrac{E(z)}{1 - v^2}(\chi_y + \nu\chi_x)z, \\ \tau_{xy}(z) = \dfrac{E}{2(1 + \nu)}\chi_{xy} z, \end{cases} \tag{37}$$

where:

$$\begin{cases} \chi_x = -\dfrac{\partial^2 w}{\partial x^2}, \\ \chi_y = -\dfrac{\partial^2 w}{\partial y^2}, \\ \chi_{xy} = -2\dfrac{\partial^2 w}{\partial x \, \partial y}. \end{cases} \tag{38}$$

The elastic energy stored per unit area A can be obtained from the stresses of Eq. (37) as:

$$\frac{\mathrm{d}L}{\mathrm{d}A} = \int_h \frac{\sigma_x^2(z) + \sigma_y^2(z) - v\sigma_x(z)\sigma_y(z) + 2(1 + \nu)\tau_{xy}^2(z)}{2E(z)} \, \mathrm{d}z. \tag{39}$$

4. Elastic line or plane equations and applications

Based on the results of the preceding section, the Sophie-Germain's equation becomes:

$$\nabla^4 w(x,y) = \frac{q(x,y)}{D^*}, \qquad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \tag{40}$$

where D^* is the *elastic* weighted rigidity of the plate, defined as:

$$D^* = \int_h \frac{E(z)}{1-\nu^2} z^2 \, \mathrm{d}z. \tag{41}$$

Equation (40) represents the plate extension of the elastic line equation for a FGM beam that, as suggested by Eq. (3), becomes:

$$\frac{\mathrm{d}^4 v(z)}{\mathrm{d}z^4} = \frac{q(z)}{E_r I_x^*}, \tag{42}$$

$q(z)$ being the applied force per unit beam length.

Note that the dynamics of a FGM beam can be treated by Eq. (42) with the formal substitution $q(z) \rightarrow -\mu\partial^2/\partial t^2$ (inertial force per unit beam length) and $d \rightarrow \partial$, where t is the time and μ is the mass per unit beam length. Accordingly, the classical results hold (e.g., the prediction of the fundamental frequency) if the substitution $EI \rightarrow E_r I_x^*$ is made, or $D \rightarrow D^*$ for a plate. Such a substitution could be useful to easily derive predictions also in different contexts, as the critical load due to elastic instability of a beam or plate composed by a FGM.

5. Conclusions

In this paper we have extended the de Saint Venant's and Kirchhoff's theories to FGMs. The study is carried out in a simple and systematic way: its main assumption is the conservation of the plane sections. We have shown that, for the majority of the cases, a FGM structure can be practically treated as the homogeneous counterpart, by simply substituting its rigidity (elastic modulus times area or moment of inertia) with the related elastic weighted rigidity. Without competing with previous and more sophisticated models, we think that our results can be useful for a simple description of FGMs, towards an engineering design of innovative materials and structures.

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