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Fractal and multifractal approaches for the analysis of crack-size dependent scaling laws in fatigue

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Abstract

The enhanced ability to detect and measure very short cracks, along with a great interest in applying fracture mechanics formulae to smaller and smaller crack sizes, has pointed out the so-called anomalous behavior of short cracks with respect to their longer counterparts. The crack-size dependencies of both the fatigue threshold and the Paris' constant C are only two notable examples of these anomalous scaling laws. In this framework, a unified theoretical model seems to be missing and the behavior of short cracks can still be considered as an open problem. In this paper, we propose a critical reexamination of the fractal models for the analysis of crack-size effects in fatigue. The limitations of each model are put into evidence and removed. At the end, a new generalized theory based on fractal geometry is proposed, which permits to consistently interpret the short crack-related anomalous scaling laws within a unified theoretical formulation. Finally, this approach is herein used to interpret relevant experimental data related to the crack-size dependence of the fatigue threshold in metals.

1. Introduction

During the last few decades, the enhanced ability to detect and measure very short cracks, along with a great interest in applying fracture mechanics formulae to smaller and smaller crack sizes, has pointed out the so-called anomalous behavior of short cracks with respect to their longer counterparts (see e.g. [1–3] for a review). Pearson [4] firstly reported the observation that such cracks are characterized by a crack growth rate, da/dN, higher than what would be predicted by the Paris' law for a given stress-intensity factor range, ΔK. Moreover, experimental results by Lankford [5] for a peak-aged 7075 aluminium alloy firstly showed a decrease in da/dN with increasing ΔK, as well as the occurrence of crack growth at ΔK values lower than the long-crack threshold.

On the other hand, most of the fatigue crack growth parameters are experimentally found to be crack-size dependent. Among them, it is worth mentioning the fatigue threshold stress-intensity range, ΔK_{th}, which was introduced by McClintock in 1963 [6] in the framework of the Paris' law [7,8]. By definition, this parameter represents a threshold

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condition of non-propagation of a pre-existing crack or notch subjected to cyclic loading. Therefore, in close analogy with fracture toughness defining the limit of unstable crack propagation, the threshold ΔK_{th} defines the limit of stable fatigue crack growth.

As the fatigue threshold represents a very important parameter in design and failure analysis, much research effort has been devoted during the last decades to reveal underlying principles. Experimentally, the threshold stress-intensity range is calculated in correspondence to a conventional very low propagation rate, da/dN, (usually 10^{-11} m/cycle, see e.g. [9]) and is typically determined for long cracks. For such crack sizes, the threshold stress-intensity range, ΔK_{th}^{∞}, is found to be a material property independent of the crack length.

Frost [10] questioned the validity of LEFM-based threshold stress intensity range in the region of short cracks. He found a crack size-dependence of ΔK_{th} in mild steel, aluminium and copper, where ΔK_{th} becomes a decreasing function of the crack length. A similar trend was also reported for several steels by Usami and Shida [11], and for a high-strength steel by Kitagawa and Takahashi [12]. The same Authors [13] later found that there exists a transition crack length below which ΔK_{th} is smaller than that for long cracks, and that such a length is dependent on the material microstructure. Thus, the short crack problem is essentially an outcome of the inapplicability of the fracture mechanics parameters to uniquely characterize the growth of fatigue cracks independently of their size.

In the last two decades, the properties of *fractality* and *multifractality* of fracture surfaces have been widely recognized in the case of both quasi-brittle [14–19] and ductile materials [20–22]. The former concept is related to *self-similar* domains characterized by a constant fractal dimension, whereas the latter permits to analyze *self-affine* domains where their fractal dimension depends on the scale of observation (see also [23] for a comprehensive collection of technical papers on mathematical aspects and applications of fractals in materials science and engineering). According to the concept of self-affinity, Carpinteri firstly proposed two geometrical multifractal scaling laws for strength and toughness of disordered materials [15]. These results permit to recognize a strong relationship existing between fractal geometry and physics [24,25]. In this framework, defining new mechanical properties with physical dimensions depending on the fractal dimension of the domain where the phenomenon takes place, such new mechanical properties turn out to be scale-dependent. As a notable examples, let us mention the scaling of fracture properties due to fractal cracks [25,26], and the scaling of the mechanical parameters in the contact problems between rough interfaces [27,28].

Starting from these theories, successful applications of fractal geometry to size effect-related fatigue problems have recently been proposed by Carpinteri et al. [29] and by Carpinteri and Spagnoli [30]. A self-similar invasive fractal set has been exploited in [30] to model fracture surfaces, and a size-dependent crack propagation law has been proposed. Recent applications of this model have concerned the analysis of the crack-size dependence of the constant C entering the Paris' law [31]. On the other hand, self-affinity of fracture surfaces was also postulated by Spagnoli [32] to reinterpret the anomalous short-crack behavior of the fatigue threshold according to a multifractal scaling law. In this case, however, an *ad hoc* assumption on the asymptotic value of the fractal dimension for small cracks is put forward, which lies outside the typical experimental range of variation of this parameter. Moreover, since both the Paris' constant C and the fatigue threshold ΔK_{th} are experimentally found to be crack-size dependent, it is reasonable expecting to interpret these anomalous behaviors within the same theoretical framework.

From this brief overview, we notice that fractal geometry emerges as a powerful tool for the explanation of the anomalous scaling laws in fatigue that are usually interpreted according to empirical laws without a theoretical ground. On the other hand, as a criticism, we observe that none of the aforementioned models is able to consistently interpret all the anomalous scaling laws due to short cracks without introducing *ad hoc* assumptions. For all of these reasons, in this paper we propose a reexamination of the fractal models for the analysis of crack-size effects in fatigue. The limitations of the existing models are put into evidence and removed. At the end, a unified theory based on fractal geometry is proposed, which permits to consistently interpret the short crack-related anomalous scaling laws within the same theoretical formulation.

2. Fractal and multifractal scaling laws

Experimental observations have shown that, within a certain range of scales, the fracture surfaces exhibit self-similar characteristics, i.e., they look the same at different magnification levels. Self-similarity implies that a (statistically) similar morphology appears in a wide range of magnifications of the fracture surface. This means that the fractal microstructures are characterized by a low degree, and not by the absence, of order, in the sense that two subsequent points on the surface are not completely uncorrelated. The fractal dimension of these surfaces is then a way to quantify this ''order behind chaos'', i.e., a measure of the correlation in the surface topology. Experimental evidence of self-similarity [33] over a broad range of scales has been reported frequently in the literature for a wide range of materials: fractured surfaces of steel [20], molybdenum [34], natural rocks [35] and also concrete [36].

Formidable advances have been made in the last few years in the study of the fractal aspects of crack morphology and energy dissipation over fractal domains. Carpinteri [14] firstly proposed to model concrete damage by assuming that the rarefied resisting cross-sections in correspondence to the peak load, A^*_{res}, can be represented by stochastic lacunar fractal sets with dimension $2 - d_\sigma$ ($d_\sigma \geqslant 0$). According to this approach, a straightforward application of the Renormalization Group procedure [37,38] permits to derive the following scaling law for the nominal tensile strength (see also [39] for a detailed review):

$$\sigma_u = \sigma_u^* b^{-d_\sigma}, \tag{1}$$

where σ_u^* presents the anomalous physical dimensions $[F][L]^{(2-d_\sigma)}$ and is a scale-invariant mechanical property.

To highlight the scaling on fracture energy, Carpinteri [15] analyzed the work W necessary to break a specimen of nominal cross-section b^2. Considering nominal or apparent quantities, this work is equal to the product of the fracture energy, $\mathscr{G}_{\mathrm{F}}$, times the nominal fracture area $A_0 = b^2$. On the other hand, the surface where energy is dissipated is not a flat cross-section: it is a crack surface, whose area A_{dis} diverges as the measure resolution tends to infinity because of its roughness at any scale. As a consequence, the crack surface can be modeled as an invasive fractal whose topological dimension is equal to $2 + d_{\mathscr{G}}$ ($d_{\mathscr{G}} \geqslant 0$). In this case, a straightforward application of the Renormalization Group procedure permits to derive the following scaling law for the nominal fracture energy:

$$\mathscr{G}_{\mathrm{F}} = \mathscr{G}^*_{\mathrm{F}} b^{d_{\mathscr{G}}}, \tag{2}$$

where $\mathscr{G}^*_{\mathrm{F}}$ is the renormalized fracture energy whose anomalous physical dimensions $[F][L]^{-(1+d_{\mathscr{G}})}$ imply that the energy dissipation is intermediate between a purely surface dissipation and a bulk dissipation.

Considering the well-known Griffith's energetic approach to the problem of an infinite plate of unit thickness, containing a crack of Euclidean length $2a$ and subjected to a remote tensile stress σ, Carpinteri [15] also determined a renormalized expression for the stress-intensity factor within the context of fractal cracks:

$$K_{\mathrm{I}} = K^*_{\mathrm{I}} a^{d_{\mathscr{G}}/2}, \tag{3}$$

where the renormalized quantity K^*_{I} has the following physical dimensions:

$$[F][L]^{-(3+d_{\mathscr{G}})/2}.$$

It has to be remarked that the exponents d_σ and $d_{\mathscr{G}}$ of the aforementioned scaling laws are not uncorrelated, as demonstrated by Carpinteri et al. [40] (see also [41] for more details on the problem of strain localization). At the smaller scales, the following relationship between the exponents holds:

$$d_\sigma + d_{\mathscr{G}} = 1. \tag{4}$$

The scaling variations described by Eqs. (1) and (2) are represented by a constant scaling exponent and therefore can be called *monofractal scaling laws*. If specimens of different sizes, made of the same material, are tested in uniaxial tension, experiments show that the monofractal scaling of σ_u and $\mathscr{G}_{\mathrm{F}}$ is strictly valid only in a limited scale range, where the fractal dimensions of the supporting domains can be considered as constants. As the size increases, in fact, the concept of *geometrical multifractality*, strictly connected with the characteristics of self-affine fractals [15], implies the progressive vanishing of fractality ($d_\sigma \to 0$, $d_{\mathscr{G}} \to 0$) with a corresponding homogenization of the domains. Intuitively, since the microstructure of a disordered material is the same, independently of the macroscopic specimen size, the influence of disorder on the mechanical properties essentially depends on the ratio between a characteristic material length, l_{ch}, and the external size, b, of the specimen. Therefore, the effect of microstructural disorder on the mechanical behavior of the material becomes progressively less important at the larger scales. On the other hand, Carpinteri [15] observed that a Brownian disorder is the highest possible at the smaller scales, yielding to fractal scaling exponents equal to $d_\sigma = d_{\mathscr{G}} = 1/2$ for both invasive and lacunar morphologies. Experimental confirmations of these limit values can be found in [19,36,42,43] for quasi-brittle materials, and in [22,44] for the fracture surfaces of metals.

This transition from a disordered (fractal) regime to an ordered (homogeneous) one can therefore be emphasized in the scaling behavior of any mechanical quantity. The analytical expressions of the *Multifractal Scaling Laws* (MFSL) for tensile strength and fracture energy [15,19,24], which are shown in Fig. 1, are the following:

$$\sigma_u(b) = f_t \left(1 + \frac{l_{\mathrm{ch}}}{b}\right)^{1/2}, \tag{5a}$$

$$\mathscr{G}_{\mathrm{F}} = \mathscr{G}^\infty_{\mathrm{F}} \left(1 + \frac{l_{\mathrm{ch}}}{b}\right)^{-1/2}. \tag{5b}$$

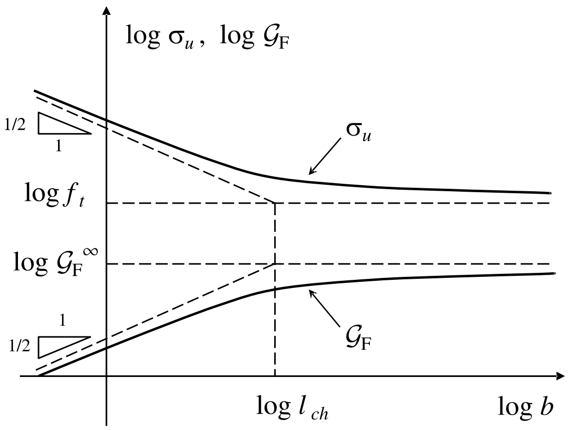


Fig. 1. Multifractal scaling laws for tensile strength and fracture energy, as proposed in [15].

These scaling laws are both two-parameters models, where the asymptotic value of the nominal quantity ($\mathcal{G}_F^\infty$ or f_t), corresponding, respectively, to the highest nominal fracture energy and to the lowest nominal tensile strength, is reached only in the limit case of infinite sizes. The non-dimensional term into round brackets, which is controlled by the characteristic length l_{ch}, represents the variable influence of disorder on the mechanical behavior. In the bilogarithmic diagrams, shown in Fig. 1, the transition from the fractal scaling regime to the Euclidean one is evident, the transition scale being represented by the point of abscissa $\log l_{ch}$.

3. Fractal approaches to fatigue

The well-known Paris' law [8] describes the kinetics of crack propagation in the intermediate range of ΔK_I:

$$\frac{da}{dN} = C(\Delta K_I)^m, \tag{6}$$

where C and m are the Paris' law parameters, N is the number of fatigue cycles, da/dN is the crack propagation rate and ΔK_I is the stress-intensity factor range.

An early application to fatigue of the innovative concepts of fractals and multifractal measures introduced by Mandelbrot [33] can be traced back to the work by Williford [21,45]. He modeled the fracture surfaces near the crack tip as an invasive fractal and proposed a power–law relationship between the J-integral and the crack length increment, Δa, with a non-integer exponent which coincides with that in Eq. (2):

$$J = J^*(\Delta a)^{d_{\mathcal{G}}}. \tag{7}$$

Parameter J^* was considered as a material constant, although its anomalous physical dimensions were not pointed out. On the basis of this scaling law, Williford [45] proposed a modified Paris' law where both the Paris' parameters are functions of the surface fractal dimension. Hence, for the usual range of variation of the exponent m of metals, i.e. $2 \leqslant m \leqslant 6$, he suggested a theoretical variability of $d_{\mathcal{G}}$ in the range $0.2 \leqslant d_{\mathcal{G}} \leqslant 0.6$. For AISI 4340 steel tested under monotonic loading [46], measured values of the fractal dimension using fractographic images led to $d_{\mathcal{G}} = 0.51$, which was also considered by Williford [45] as the representative value for fatigue conditions.

In the 1990s, experimental evidences by Bažant [47,48] pointed out a dependence of the crack growth rate on the specimen size, i.e., a size effect on fatigue crack growth. Thus, exploiting the renormalized quantities related to fractal cracks (whose surfaces can be modelled as invasive fractals according to the results achieved by Carpinteri and summarized in the previous section), Carpinteri and Spagnoli [30] proposed the following size-independent fatigue crack growth law:

$$\frac{da^*}{dN} = C\left(\Delta K_I^*\right)^m, \tag{8}$$

where $a^* = a^{1+d_{\mathcal{G}}}$ and ΔK_I^* is given by Eq. (3).

A scaling law was obtained by Carpinteri and Spagnoli [30] from Eq. (8), by rewriting such a relationship in terms of the nominal crack propagation rate, da/dN, and the nominal stress-intensity factor range, ΔK_I. Using a derivation chain rule to calculate the crack propagation rate da^*/dN of the fractal crack, they obtained:

$$\frac{da}{dN} = \frac{C}{1 + d_{\mathscr{G}}} a^{\beta} \Delta K_{\rm I}^{m} = C_{\rm F}(a) \Delta K_{\rm I}^{m}, \qquad \beta = -d_{\mathscr{G}}\left(1 + \frac{m}{2}\right). \tag{9}$$

For geometrically similar cracked bodies, a is proportional to b. Consequently, Eq. (9) can be regarded either as a structural-size dependent Paris' law [30], or as a crack-size dependent fatigue crack growth law [31]. Note that such an equation is formally identical to the classical Paris' law in Eq. (6), but the new coefficient of proportionality, $C_{\rm F}$, is no longer a material constant. Since $\beta < 0$, Eq. (9) would predict that the crack growth rate da/dN is a decreasing function of the structural dimension or the crack length. In the case of large structural sizes where the transition from disorder to order takes place, $d_{\mathscr{G}} \to 0$, and no size effects occur.

This model was referred to as *monofractal approach* to size effect on fatigue crack growth in [30,31]. It was also noticed that it can be applied within a limited scale range. The use of a *multifractal approach* was also suggested in [30,31] to model the propagation of both short and long fatigue cracks, although this possibility remained unexplored.

On the other hand, Spagnoli [32] proposed an original interpretation of crack-size effects on the fatigue threshold according to geometric multifractality. More specifically, the well-known Kitagawa diagram [12] describes the variation of the threshold stress-intensity range as a function of the crack length, showing the existence of a transition length beyond which the threshold of fatigue crack growth becomes a material constant, $\Delta K_{\rm th}^{\infty}$. For shorter crack lengths, $\Delta K_{\rm th}$ is progressively reduced, as schematically shown in Fig. 2.

To interpret this anomalous trend in the context of fractal geometry, Spagnoli [32] supposed that the monofractal scaling law in Eq. (3) would apply not only to the generalized stress-intensity factor, but also to the threshold stress-intensity range. According to this equation, $\Delta K_{\rm th}$ is expected to be a function of a raised to $d_{\mathscr{G}}/2$ within a limited scale range. Then, he treated fracture surfaces as invasive self-affine fractal sets and postulated the existence of the following multifractal scaling law for $\Delta K_{\rm th}$ to bridge the two experimentally observed asymptotical tendencies of $\Delta K_{\rm th} \propto a^{1/2}$ for $a \to 0$ and $\Delta K_{\rm th} = \text{constant}$ for $a \to \infty$:

$$\Delta K_{\rm th} = \Delta K_{\rm th}^{\infty}\left(1 + \frac{a_0}{a}\right)^{-1/2}, \tag{10}$$

where a_0 denotes a characteristic crack length. This equation, which is identical to the empirical relationship by El Haddad et al. [49], implies that $d_{\mathscr{G}} \to 0$ for long cracks, leading to $\Delta K_{\rm th} = \Delta K_{\rm th}^{\infty}$ and the disappearance of size effects. On the other hand, for short cracks ($a \to 0$), the slope 1/2 of the asymptote can only be predicted in this model by setting $d_{\mathscr{G}} \to 1$ (see Eq. (3) and the diagram in Fig. 2).

As previously pointed out, $d_{\mathscr{G}}$ has to be lower than or equal to 1/2, and therefore the assumption in [32] seems to be in contrast with the common range of variation of this exponent. On the other hand, from the geometrical point of view, fractal dimensions larger than 2.5 would imply overhangs along the surface, which clearly are not admissible in the kinematics of a fracture process.

More importantly, the application of the scaling law (3) to the fatigue threshold is questionable. In fact, $\Delta K_{\rm th}$ is experimentally computed in correspondence of a conventional very low crack propagation rate, and thus it should depend on the parameters defining the kinetics of the fatigue crack propagation phenomenon, such as m and R, as experimentally found in [9].

In this sense, since both the Paris' constant C and the fatigue threshold $\Delta K_{\rm th}$ are experimentally found to be crack-size dependent, it should be reasonable expecting to interpret these anomalous behaviors within the same theoretical framework.

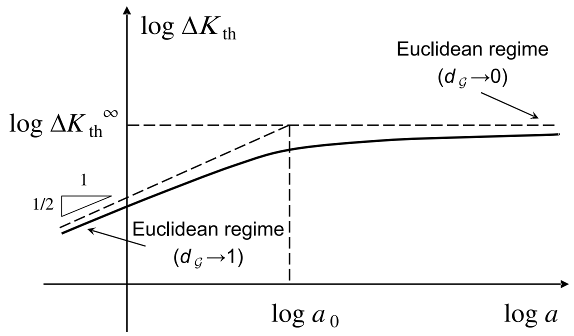


Fig. 2. Multifractal scaling law for the threshold stress-intensity range, as proposed in [32].

4. A unified fractal model for the interpretation of the crack-size dependencies in fatigue

In this section, we propose a unified fractal approach encompassing all the aforementioned size effects related to the anomalous behavior of short cracks observed in fatigue. To this aim, let us consider the crack-size dependent monofractal Paris' law in Eq. (9). Since the threshold stress-intensity factor range is experimentally calculated for a conventional very low propagation rate, $v_{th} \cong 10^{-11}$ m/cycle [9], we can compute the fatigue threshold range by inverting Eq. (9) in correspondence of $da/dN = v_{th}$:

$$\Delta K_{th} = \left(\frac{v_{th}}{C_F}\right)^{1/m} = v_{th}^{1/m}\left[\frac{1+d_{\mathcal{G}}}{C}a^{d_{\mathcal{G}}\left(1+\frac{m}{2}\right)}\right]^{1/m}. \tag{11}$$

For long cracks, a transition from disorder to the Euclidean order is expected, so that $d_{\mathcal{G}} \to 0$. As a consequence, the crack-size dependence disappears and we obtain the asymptotic value of the fatigue threshold for long cracks, ΔK_{th}^{∞}:

$$\Delta K_{th}^{\infty} = \left(\frac{v_{th}}{C}\right)^{1/m}. \tag{12}$$

In this case, recalling the correlation between the Paris' law parameters established in [50], we have

$$\Delta K_{th}^{\infty} = \left(\frac{v_{th}}{v_{cr}}\right)^{1/m} K_{IC}(1-R), \tag{13}$$

where v_{cr} and K_{IC} denote the coordinates of the point of the Paris' curve corresponding to the onset of crack growth instability. Eq. (13) shows that the physical dimensions of the fatigue threshold for long cracks coincide with those of fracture toughness, i.e., $[F][L]^{-3/2}$. For long cracks, Eq. (13) can also be used to establish a useful relationship between the parameter m and the coordinates of two special points of the Paris' curve: the point corresponding to the onset of crack growth instability, and the point defining the threshold condition. Therefore, it should be possible, in principle, to characterize the process of fatigue crack growth in the intermediate regime (Region II) using the coordinates of the aforementioned special points instead of the parameters C and m.

According to Eq. (12), Eq. (11) can now be rewritten in this synthetic form:

$$\Delta K_{th} = \Delta K_{th}^{\infty}\left[(1+d_{\mathcal{G}})a^{d_{\mathcal{G}}\left(1+\frac{m}{2}\right)}\right]^{1/m}. \tag{14}$$

For metals, where $2 \leqslant m \leqslant 6$, this equation would predict a scaling law for the fatigue threshold of the type $\Delta K_{th} \propto a^{\gamma}$, with an exponent γ ranging from $2/3d_{\mathcal{G}}$ for $m = 6$, up to $d_{\mathcal{G}}$ for $m = 2$. Note, incidentally, that this approach leads to $\Delta K_{th} \propto \sqrt{a}$ for short cracks ($d_{\mathcal{G}} \to 1/2$) in the case of $m = 2$, which is far commonly reported for steels. In this case, ΔK_{th} assumes the physical dimensions of stress, i.e. $[F][L]^{-2}$. This is in agreement with the common interpretation of the crack-size effects on the fatigue threshold at the small scales based on the well-known Hall–Petch law [51,52]. In fact, the crack length can be considered as proportional to the average grain size of the material microstructure and we have $\Delta K_{th} \propto \sqrt{d_{grain}}$. Therefore, the dimensional transition from $[F][L]^{-2}$ in the case of short cracks to $[F][L]^{-3/2}$ for long cracks can be physically interpreted as a transition from a dislocation-based diffuse damage to a macroscopic fracturing.

Exploiting the concept of self-affinity, we can also consider a multifractal Paris' law, as suggested by Carpinteri and Spagnoli [30]:

$$\frac{da}{dN} = C_{MF}(a)\Delta K^m. \tag{15}$$

The new Paris' law parameter is now given by the following equation:

$$C_{MF}(a) = C\left(1+\frac{a_0}{a}\right)^{\frac{1}{2}\left(1+\frac{m}{2}\right)}, \tag{16}$$

where a_0 is a characteristic length as that defined in the multifractal scaling laws for tensile strength and fracture energy. This equation, which is a decreasing function of the crack length, is schematically shown in Fig. 3 and bridges the two asymptotic behaviors for the short and the long-crack length regimes. For short cracks, we have the asymptote $\frac{1}{2}(1+m/2)$, which is obtained by setting $d_{\mathcal{G}} \to 1/2$ in Eq. (9). On the other hand, the asymptote for long cracks is characterized by $d_{\mathcal{G}} \to 0$, and therefore the crack-size effect disappears.

Again, inverting Eq. (15) in correspondence of $da/dN = v_{th}$, we obtain the expression for the fatigue threshold stress-intensity range:

$$\Delta K_{th} = \left(\frac{v_{th}}{C_{MF}}\right)^{1/m} = \Delta K_{th}^{\infty}\left(1+\frac{a_0}{a}\right)^{-\frac{1}{2}\left(\frac{1}{2}+\frac{1}{m}\right)}. \tag{17}$$

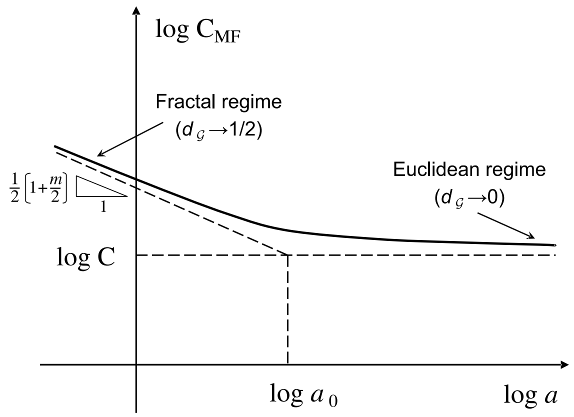


Fig. 3. Multifractal scaling law for the Paris' constant C.

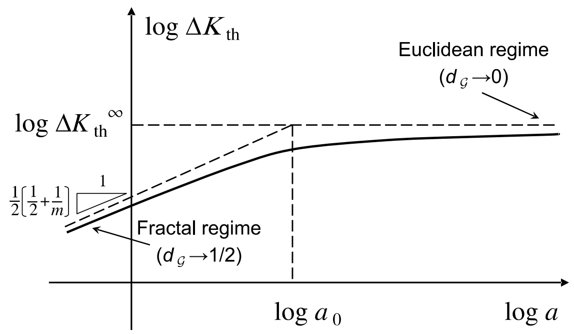


Fig. 4. New multifractal scaling law for the threshold stress-intensity range.

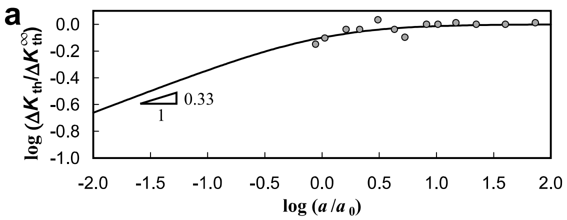
It is worth noting that this equation has been derived as a straightforward consequence of the multifractal scaling law in Eq. (16), without introducing additional *ad hoc* assumptions or limitations on the value of $d_{\mathcal{G}}$ like in [32]. The exponent of the proposed scaling law ranges from 1/3 for $m = 6$, up to 1/2 for $m = 2$ (see Fig. 4). It is important to notice that the variability in the parameter m can also be ascribed to the effect of the loading ratio, $R = K_{min}/K_{max}$, as firstly observed by Radhakrishnan [53].

Considering the data collected in [54], an experimental assessment of Eq. (17) is proposed in Fig. 5 for a variety of metals. By performing a non-linear regression analysis on the experimental data, the value of a_0 and the exponent of the multifractal scaling law are determined. Whereas the parameter a_0 ranges from 1–10 μm for very high strength steels to 100–1000 μm for very low strength steels, the exponent of the scaling law ranges from $0.33 \sim 1/3$ to $0.51 \sim 1/2$, in fair good agreement with the theoretically predicted range of variation for this parameter.

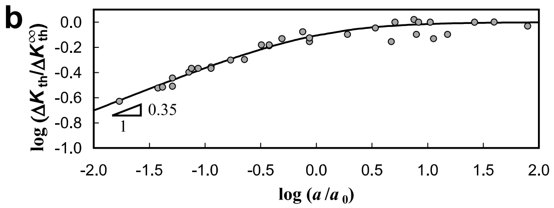
5. Conclusion

In the present paper, we have proposed a unified fractal approach for the interpretation of the anomalous scaling laws in fatigue. Early applications of the concepts of fractality and multifractality to the interpretation of size effects on tensile strength and fracture energy of disordered materials were proposed by Carpinteri since the 1990s. In this framework, the renormalized fracture energy is represented by a dissipation over a surface having a dimension higher than 2. The dimensional increment with respect to the Euclidean dimension represents self-similar tortuosity of the fracture surface due to grains and inclusions present in the material microstructure. In physical reality, fracture surfaces after rupture can be considered as multifractals having dimension 2.5 at small scales and dimension 2 at large scales. This implies that a transition from extreme disorder to extreme order takes place by considering different scales of observation.

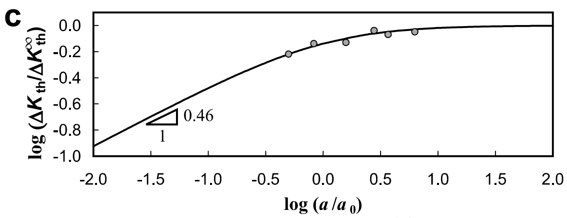
Recent developments concerning the analysis of crack-size effects in fatigue have shown that the coefficient C entering the Paris' law is no longer a material constant, but it depends on the initial crack length. Following the idea



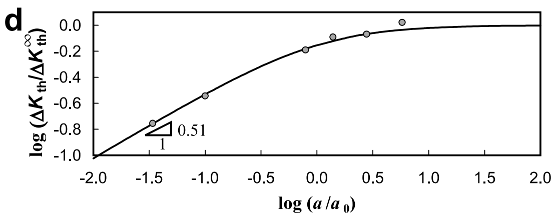
HT80 $\sigma_y = 726\,\text{MPa}$ (Kitagawa, 1979).



Carbon steel (Frost, 1966; Murakami, Endo, 1986).



Copper (Frost, 1966).



S20C $\sigma_y = 366\,\text{MPa}$ (Tanaka, 1981).

Fig. 5. Experimental assessment of the proposed multifractal scaling law for the threshold stress-intensity range vs. crack length (experimental data are taken from the collections reported in [54,32]).

suggested by Carpinteri and Spagnoli [30], a multifractal scaling law for the constant C has been proposed. As a consequence of this scaling law, the threshold stress-intensity factor range, which is computed by definition in correspondence of a conventional low crack propagation rate, is found to be crack-size dependent in its turn. This newly proposed scaling law provides an original interpretation of the so-called Kitagawa diagram, where the stress-intensity range threshold is plotted vs. the crack length in a bilogarithmic plot. In this diagram, ΔK_{th} is an increasing function of the crack length, approaching a constant value in the long-crack regime. For the short crack regime, the slope of the asymptote is found to be dependent on the exponent m of the Paris' law. This dependence sheds a new light on the scatter of the experimental results noticed at the small scales, which cannot be interpreted according to the existing empirical formulae.

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