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ORIGINAL PAPER

Self-similarity in concrete fracture: size-scale effects and transition between different collapse mechanisms

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Abstract Since the pioneering paper by Mandelbrot (Nature, 308:721–722, 1984) on the fractal character of the fracture surfaces in metals, the fractal aspects in the deformation and failure of materials have been investigated by several Researchers (see the reviews by Bouchaud (J Phys Condens Matter, 9:4319–4344) and Carpinteri et al. (Appl Mech Rev, 59:283–305, 2006)) and the attempts to apply fractals to fracture have grown exponentially. Aim of this paper is 2-fold: on one hand, it summarizes in a detailed yet concise fashion the major results of the fractal approach to the scaling of mechanical properties in solid mechanics; on the other hand, it reports some recent results concerning the size effect in the failure of reinforced concrete (RC) beams. These recent findings clearly show that the picture of the size-scale effects is much more complex when interaction among different collapse mechanisms occurs. The consequences on the size-scale effects are discussed in detail.

Keywords Fractals · Size-scale effects

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1 Introduction

Since the discovery of fractal geometry (Mandelbrot 1982), fractals have been widely applied to describe a huge variety of irregular, rough and fragmented natural structures and physical phenomena. After Mandelbrot (1984) paper on the fractal character of fracture surfaces in metals, this happened in fracture mechanics also. As observed by Lu (2007), since the early 1990's, about half a hundred relevant papers has been published each year on the topic of fracture and fractals.

There are different aspects in the study of fractal fracture: among them we could cite the link between fractal dimension of fracture surface and nominal fracture toughness (Weiss 2001), the study of roughness exponents and their connection with universality and critical phenomena (Bouchaud 1997; Bouchaud et al. 1990), the modeling of fracture in non-homogeneous materials (Baia et al. 1994), mesoscale and multiscale approaches to fracture (Panin et al. 2002), fractal modeling of nano- and bio-materials (Lu and Mai 2005), and size-scale effects or the scaling laws on the mechanical properties of solids (Carpinteri 1994a,b). The latter, i.e., the interpretation of the size-scale effects as the result of damage localization of the failure mechanisms on fractal patterns, is one of the most notable advancements provided by the fractal approach in the mechanics of heterogeneous quasi-brittle materials (Carpinteri et al. 2006).

In this paper, we will revisit the fundamental concepts of fractal modeling of the size-scale effects. The fractal explanation of the size-scale effects in the framework of Renormalization Group Theory (Carpinteri 1994a,b) and the formulation of the so-called fractal cohesive crack model (Carpinteri et al. 2002) are briefly presented. The hypotheses, strengths and weaknesses of the model will be also clearly discussed.

As a further advancement, since the analyzes of the fracture surfaces have shown that a single and unique fractal dimension of fracture surface is not sufficient to characterize a fractal crack (Carpinteri 1994b; Carpinteri and Chiaia 1995), the fractal scaling laws have been extended to self-affine sets, leading to the definition of the multifractal scaling laws (Carpinteri 1994b) for the tensile strength (Carpinteri and Chiaia 1996; Carpinteri et al. 1995, 1998), for the fracture energy (Carpinteri and Chiaia 1995; Carpinteri and Chiaia 1996b, 1997), and, more recently, for the critical strain (Carpinteri and Chiaia 2002; Carpinteri and Cornetti 2002a,b). The different demonstrations proposed to explain the slopes of the multifractal scaling laws, the fractal and the statistical one, are both briefly revisited.

The last part of the paper is a summary of recent results concerning the scaling in the failure of RC beams. The fractal scaling laws are limited to type 1 size effect, i.e., only to structures of positive geometry (unnotched), failing at fracture initiation, and not to type 2, which occurs in unnotched or notched structures that reach the peak load only after extended and stable crack growths (Bažant 2002). In the case of type 2 size effect, several other models have been proposed to predict the scaling law. Also these models are often based on a similarity approach, since similar crack patterns are assumed in beams of different sizes. This point is carefully analyzed in the light of recent results by Carpinteri et al. (2007a,b, 2008), who studied the interactions between the three collapse mechanisms occurring in RC beams failing in shear, i.e., the flexural, the diagonal tension and the crushing collapse. As shown by the experiments and confirmed by these recent analyzes, the picture is much more complex: the diagonal shear failure is the intermediate (transitional) failure mode between the other two failure types, i.e., flexural fracturing with yielding of the reinforcement and concrete crushing.

2 The fractal explanation of the size-scale effects

Several experiments have pointed out the size effects affecting the mechanics of heterogeneous materials. In this section, we will revisit the fundamentals of the fractal approach by Carpinteri (1994a,b). For the sake of clarity and conciseness, in this section we will present the one-dimensional approach; for details about the three-dimensional formulation, the reader is referred to Carpinteri et al. (2003).

The behavior of concrete-like materials is well described by the cohesive crack model (Hillerborg et al. 1976). Accordingly, the material is characterized by a stress–strain relationship ($\sigma - \varepsilon$), valid for the undamaged zones, and by a stress-crack opening displacement relationship ($\sigma - w$, the cohesive law), describing how the stress decreases from its maximum value σ_u down to zero as the distance between the crack lips increases from zero to the critical displacement w_c. The area below the cohesive law represents the energy $\mathscr{G}_F$ spent to create the unit crack surface. The cohesive crack model is able to simulate tests where high stress gradients are present, e.g., bending tests on pre-notched specimens, and captures the ductile-brittle transition occurring by increasing the size of the structure. On the other hand, uniaxial tensile tests on dog-bone shaped specimens (Carpinteri and Ferro 1994, 1998; van Mier and van Vliet 1999) have shown that the three material parameters defining the cohesive law are not size independent: particularly, increasing the specimen size, σ_u tends to decrease, whilst $\mathscr{G}_F$ and w_c tend to increase; correspondingly, the tail of the cohesive law raises. These size effects can not be predicted by the cohesive model, but can be explained assuming that damage occurs within a fractal domain.

Experimental results about concrete porous microstructure led Carpinteri (1994a,b) to model damage in concrete by assuming that the rarefied resisting section in correspondence with the critical load is represented by a lacunar fractal set (Fig. 1a) with dimension $2 - d_\sigma$ ($d_\sigma \geq 0$) and measure $A^*_{res} = b^{2-d_\sigma}$, b being the side of a prismatic specimen ($A_0 = b^2$). On the other hand, in material science, it can be noticed that, often, damage is not localized onto a single section but is spread over a finite band where it shows fractal patterns (Kleiser and Bocek 1986). Hence, as representative of the damaged band, we can consider a bar of length b subjected to tension (Fig. 1b), where, at the maximum load, dilation strain tends to concentrate into different softening regions, while the rest of the body undergoes elastic unloading. Assuming that strain is localized onto cross-sections whose projection on the longitudinal axis is given by a lacunar set of dimension

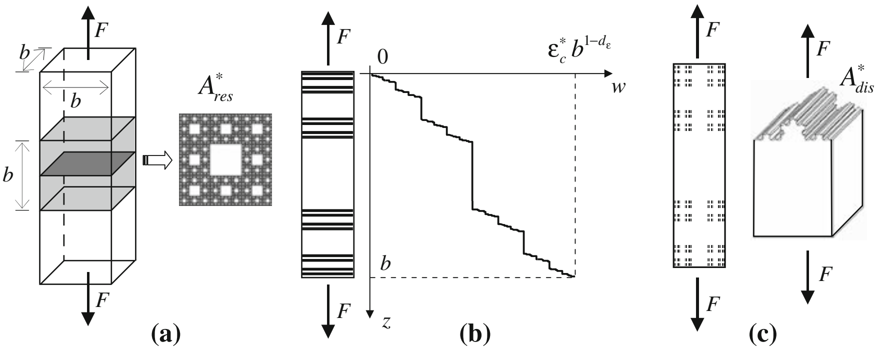

Fig. 1 Fractal damage domains: porous cross section (**a**), strain localization (**b**) energy dissipation zone (**c**)

$1 - d_\varepsilon$ ($d_\varepsilon \geq 0$) and measure b^{1-d_ε}, the displacement function at rupture can be represented by a devil's staircase graph (Fig. 1b). Consequently, in the process of rupture, the total fracture energy W is dissipated over the Cartesian product of the longitudinal and transversal fractal lacunar sets (Fig. 1c). We may indicate the dissipation domain dimension with $2+d_{\mathscr{G}}$ ($d_{\mathscr{G}} \geq 0$), its measure being $A^*_{dis} = b^{2+d_{\mathscr{G}}}$. Since 1994, in order to explain the hypothesis of fractal dissipation domains, the lead Author used the von Koch curve as the archetype of invasive fractal to visualize the energy dissipation domain. It must be taken into account, however, that the dissipation domain for fracture energy is not simply a single crack, but can be any complex three-dimensional arrangement of micro-cracks, as evidenced in more recent papers. Thus, the fractal approach can reflect that in small structures there are many micro-cracks eventually coalescing into a major crack and implicitly accounts for stress redistributions that occur up to the peak load.

Hence we can compute the maximum load F, the critical displacement w_c and the total dissipated energy W as:

$$F = \sigma_u A_0 = \sigma_u^* A^*_{res} \tag{1a}$$

$$w_c = \varepsilon_c b = \varepsilon_c^* b^{1-d_\varepsilon} \tag{1b}$$

$$W = \mathscr{G}_F A_0 = \mathscr{G}_F^* A^*_{d_\sigma} \tag{1c}$$

where σ_u, ε_c and $\mathscr{G}_F$ are respectively the ultimate strength, the critical strain and the fracture energy. These nominal quantities are size-dependent properties. The true scale-independent quantities are the right hand side ones, i.e., the fractal ultimate strength σ_u^*, the fractal critical strain ε_c^* and the fractal fracture energy $\mathscr{G}_F^*$. They show non-integer or anomalous physical dimensions: $[F][L]^{-(2-d_\sigma)}$ for σ_u^*, $[L]^{d_\varepsilon}$ for ε_c^*, and $[F][L]^{-(1+d_{\mathscr{G}})}$ for $\mathscr{G}_F^*$. Because of the measure of A^*_{res} and A^*_{dis}, from Eq. (1) the scaling laws for ultimate strength, critical strain and fracture energy can be obtained:

$$\sigma_u = \sigma_u^* b^{-d_\sigma} \tag{2a}$$

$$\varepsilon_c = \varepsilon_c^* b^{-d_\varepsilon} \tag{2b}$$

$$\mathscr{G}_F = \mathscr{G}_F^* b^{d_{\mathscr{G}}} \tag{2c}$$

Dealing with real fractals (opposite to mathematical fractals), it must be always taken into account that they develop in a finite range between an upper and a lower cut-off. Therefore, these equations hold their validity in the physical range where the fractal domains develop only. The power-laws (2) provide size effects in perfect agreement with the trends arising from experiments. The third scaling law can be derived also from the assumption of invasive fractal crack surface (Fig. 1c). Finally, due to the Cartesian product, we know that the sum of the three scaling exponents is always equal to one:

$$d_\sigma + d_\varepsilon + d_{\mathscr{G}} = 1 \tag{3}$$

The $\sigma^* - \varepsilon^*$ diagram has been called the fractal or scale-independent cohesive law. Contrarily to the classical cohesive law, which is experimentally sensitive to the structural size, this curve is an exclusive property of the material since it is able to capture the fractal nature of the damage process. The area below the softening fractal stress-strain diagram represents now the fractal fracture energy $\mathscr{G}_F^*$.

The model has been applied to two experimental data sets obtained respectively by Carpinteri and Ferro (1998) (direct tensile tests under fixed boundary conditions in a scale range 1:8) and by van Mier and van Vliet (1999) (direct tensile tests with rotating boundary conditions in a scale range 1:32). Fitting the experimental results for σ_u and $\mathscr{G}_F$ provides the values of d_σ and $d_{\mathscr{G}}$;

then, from Eq. (3), the value of the exponent d_ε can be computed. Given these three values, the fractal cohesive laws can be plotted. As expected, all the curves related to each single size tend to merge in a unique, scale-independent cohesive law. The experimental validation is described in detail in (Carpinteri et al. 2002, 2003).

3 The multifractal scaling laws

The aforementioned uniaxial tension experiments have shown that, the fractal scaling of σ_u and $\mathcal{G}_F$ is strictly valid only in a limited scale range, where the fractal dimensions of the supporting domains can be considered to be constant. As the size increases, in fact, the concept of geometrical multifractality (Carpinteri 1994b) implies the progressive vanishing of fractality ($d_\sigma \to 0, d_\mathcal{G} \to 0$) with a corresponding homogenization of the domains. Observe that the *geometrical* multifractality is strictly connected with the characteristics of *self-affine* fractals: its meaning differs from the one traditionally found in the literature (where multifractal denotes sets with a spectrum of fractal dimensions, see Feder 1988).

Intuitively, since the microstructure of a disordered material is the same, independently of the macroscopic specimen size, the influence of disorder on the mechanical properties essentially depends on the ratio between a characteristic material length l_{ch} and the external size b of the specimen. In other words, if the fractality effect disappears, this implies that there is no size effect. This may seem to be in contrast with the widely accepted Weibull theory. As a matter of fact, the explanation of Weibull theory based on Fractal Mechanics is founded on the hypothesis of unbounded maximum defect size. If we remove this hypothesis by assuming that the maximum defect size is bounded, the size effect consequently disappears above a certain value of the structural size. Notice that, this hypothesis is equivalent to introducing a characteristic length l_{ch} into Weibull theory.

At the smallest scales, Carpinteri (1994b) observed that a Brownian disorder seems to be the highest possible, yielding, respectively, for invasive and lacunar morphologies, fractal scaling exponents equal to $+1/2$ or $-1/2$. Stereological analysis of the concrete microstructure performed by Carpinteri and Cornetti (2002a,b) agrees satisfactorily with these values,

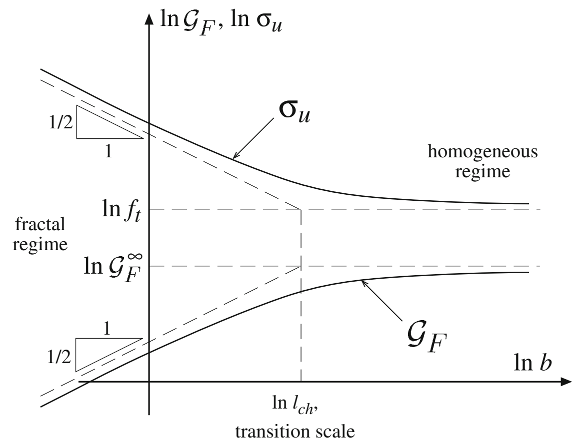

Fig. 2 Multifractal scaling laws for tensile strength and fracture energy

considering them as an upper bound for concrete surface roughness and cross-section lacunarity.

On the basis of these physical and geometrical arguments, two Multifractal Scaling Laws (MFSL) were proposed for tensile strength (Carpinteri et al. 1995, 1998) and fracture energy (Carpinteri and Chiaia 1995; Carpinteri and Chiaia 1996b) respectively:

$$\sigma_u(b) = f_t \left[1 + \frac{l_{ch}}{b}\right]^{1/2} \tag{4a}$$

$$\mathcal{G}_F(b) = \mathcal{G}_F^\infty \left[1 + \frac{l_{ch}}{b}\right]^{-1/2} \tag{4b}$$

These scaling laws are both two-parameters models, where the asymptotic value of the nominal quantity ($\mathcal{G}_F^\infty$ or f_t), corresponding respectively to the highest nominal fracture energy and to the lowest nominal tensile strength, is reached only in the limit of infinite sizes. The dimensionless term within square brackets, which is controlled by the characteristic length l_{ch}, represents the variable influence of disorder on the mechanical behaviour.In the bilogarithmic diagrams, shown in Fig. 2, the transition from the fractal scaling regime to the Euclidean one is evident, the transition scale being represented by the point of abscissa $\ln l_{ch}$. In analogy with the MFSLs for tensile strength and fracture energy—see Eqs. (4a) and (4b)—a further Multifractal Scaling Law can be proposed for the critical strain ε_c (Carpinteri et al. 2002, 2003).

These scaling laws are applicable to type 1 size effect, i.e., only to structures of positive geometry (unnotched), failing at fracture initiation, and not to type 2 size effect, which occurs in notched or unnotched structures that reach the peak load only after extended and stable crack growths (Carpinteri and Chiaia 2002; Bažant and Yavari 2007). In addition, these scaling laws do not explicitly incorporate differences in shape (or geometry), although they are applicable to any body of positive geometry. The effects of geometry are lumped into the best-fit parameters of Eq. (4).

It is also worthwhile recalling that the trend emerging at the smaller scales has found several different confirmations. The slope $-1/2$ for tensile strength has been obtained by at least two different theoretical approaches and was confirmed by numerical Monte Carlo simulations.

The first explanation was provided by Carpinteri (1994a) on the basis of the scaling of ε_c and Eq. (3). The scaling behavior of the kinematical parameters shows that the critical nominal strain ε_c decreases as the bar length increases. As shown experimentally (and also analytically by Carpinteri et al. 2004), by increasing the size, the bar progressively loses its deformation capacity (or ductility) and tends to a more brittle behavior. For a given material, the fractal exponent d_ε increases with the size of the bar, from the value $d_\varepsilon = 0$ (homogeneous deformation) to $d_\varepsilon = 1$ (highly localized deformation). In the limit of very large and very small sizes, the collapse kinematics will be ruled respectively by a pure opening displacement (w_c^∞) of a single crack or by a pure dilation (ε_c^0). In the latter case, Eq. (3) reduces to: $d_\sigma + d_\mathscr{G} = 1$. On the other hand, Carpinteri (1994a) observed that $d_\mathscr{G}$ cannot exceed $1/2$ for kinematical reasons (if $d_\mathscr{G} > 1/2$, crack opening or closing is impossible) and thus the maximum size effect on $\mathscr{G}_F$ corresponds to the maximum size effect on strength, i.e., $d_\sigma = 1/2$.

The second explanation is based on the self-similarity distribution of Griffith cracks (Carpinteri 1986, 1989), recently revisited (Carpinteri and Puzzi 2008). Let us assume a set of similar bodies containing a multitude of imperfections (cracks, voids, etc.) of a given size distribution, distributed uniformly in space and isotropically, i.e., without preferential orientations. For the sake of simplicity, let us assume that the bodies are cubes of side b. As originally shown by Carpinteri (1994b), if the size of the maximum defect size $a_{\max}$ is proportional to the size of the body b, then, by assuming that defects are cracks, and that interactions among them are negligible, as well as material non-linearity, the slope of the size effect on tensile strength is exactly $-1/2$. The point here is: under which conditions is the maximum defect size (which obviously is a random variable) proportional to the body size b, at least in a mean sense? This happens if and only if the defect size distribution is the distribution of self-similarity, i.e., a power-law with the following probability density function (pdf):

$$p(a) = \frac{C}{a^{N+1}} \tag{5}$$

and the exponent $N = 3$. On the other hand, $N < 3$ is not possible, since in this case $a_{\max} \propto b^\beta$ and $\beta > 1$; thus increasing the size, defects become grater than the body itself. Thus, $-1/2$ is the maximum slope of the size effect on tensile strength, corresponding tho the self-similarity distribution of Griffith cracks.

This result has been verified by Monte Carlo numerical simulations based on the adoption of the pdf in Eq. (5). The simulations confirm that the scaling of the maximum defect size with respect to the body size b is a power-law with exponent $\beta = 3/N$. Details can be found in Carpinteri and Puzzi (2008). Here, for the sake of brevity, we report two graphs only; the first one shows the power-law trend of $a_{\max}$ as a function of b in the case of $N = 3$ (see Fig. 3a). The obtained slope, very close to 1, confirms the above summarized treatment. The second graph, see Fig. 3b, reports the values obtained by adopting different values of N, compared with the analytical function $\beta = 3/N$. Once again, the agreement is quite satisfactory.

4 Transition between different collapse mechanisms: diagonal tension failure

As shown in the previous sections, self-similarity plays a relevant role in the determination of size effect on mechanical properties. Several other approaches, not fractal-based, require instead geometrical similitude. This is the case of the well known Bažant's Size Effect Law (SEL) (Bažant 1984), which bridges plasticity and fracture mechanics with this implicit assumption: that a notch (or a crack caused by stable propagation) of length proportional to the specimen size exists at the peak load. In the case of a crack, its shape must scale homotetically in geometrically similar structures (Bažant and Kazemi 1991; Collins and Kuchma 1999; Vecchio 2000; Angelakos et al. 2001; Bentz and Buckley 2005; Bažant and Kazemi 2006; Bažant and Yu 2005; Bažant and Yavari 2005, 2007).

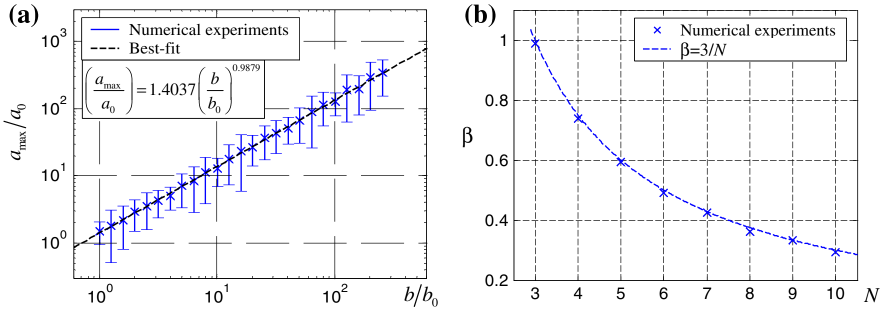

Fig. 3 Monte Carlo simulations: power-law relationship between $a_{\max}$ and b in the case of $N = 3$ **(a)** and scaling exponent β as a function of N **(b)**

Concerning size effects in the failure of RC beams, there exists a wide literature on models involving separately the three possible failure mechanisms: flexure, shear and crushing. However, the study of the transition between these mechanisms inside a consistent theoretical framework is still an open question, especially with reference to the experimentally known size effects. In particular, shear crack propagation and diagonal tension failure have been addressed in the literature by several Authors. The available approaches, just to mention the most popular, include cohesive crack modeling (see, e.g., Gustafsson and Hillerborg 1983), LEFM-based models (Jenq and Shah 1989 and So and Karihaloo 1993 used a superposition technique that is somehow conceptually close to the bridged crack model by Carpinteri (1981)), the modified compression field theory (MCFT, Vecchio and Collins 1986 and its developments) and Bažant's SEL (Bažant 1984).

Recently, Carpinteri et al. (2007a,b, 2008) proposed in the framework of LEFM a conceptual scheme allowing for a direct relation between the failure modes. In this model, which is an extension of the original bridged crack model (Carpinteri 1981, 1984), the crack initiation is found based on the stability of the cracking process. The shape and the initiation point for the critical crack are not assumed *a priori*, as in the majority of models available in the Literature. Although the model is linear elastic, the extension to the case of cohesive cracks is not difficult (Carpinteri et al. 2007a,b).

Let us consider the cracked beam reported in Fig. 4; let the section width be b, the height h, the crack depth a, the depth of the steel bars c, the crack tip horizontal coordinate x, the crack mouth horizontal coordinate x_0 and the shear span l. Concerning the crack trajectory, it is split into two parts: a vertical segment extending from the bottom to the reinforcement layer; and then a power-law with exponent μ, extending from the end of the first part to the loading point. Details can be found

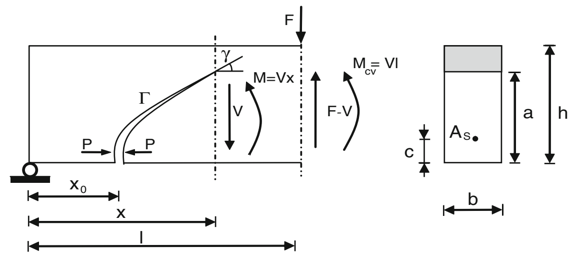

Fig. 4 Geometry of the RC cracked element

in (Carpinteri et al. 2007a,b). Concerning the failure mechanisms, both flexural and shear cracks propagate under a LEFM condition ruled by the comparison of the stress-intensity factors K_I to the concrete toughness K_{IC}. The criterion for concrete crushing is introduced on the basis of previous work by Ventura et al. (2004). A unified general model is obtained, so that the study of the transitional phenomena is naturally accomplished.

Skipping the details, the main result obtained within this unified framework is a clear picture of all the failure mode transitions, which are ruled by three non-dimensional parameters, namely two brittleness numbers: $N_P = \sigma_y h^{1/2}\rho/K_{IC}$ (Carpinteri 1981) (σ_y being the yield strength of steel and ρ the reinforcement percentage) and $N_C = \sigma_{cu} h^{1/2}/K_{IC}$ (Ventura et al. 2004) (σ_{cu} being the compressive strength of concrete) and the span slenderness ratio λ_l. Graphically, this complex picture is summarized in the sketch of Fig. 5.

The scheme illustrates all the failure mode transitions in reinforced concrete elements; an increase of N_P is associated with a transition from flexural to diagonal tension failure. It can be read as: an increase in the reinforcement area (transition d–e); a decrease in the scale with constant reinforcement area (transition a–e); an increase in the scale with a constant reinforcement percentage (transition g–e). The latter case has been addressed by several Authors dealing with the

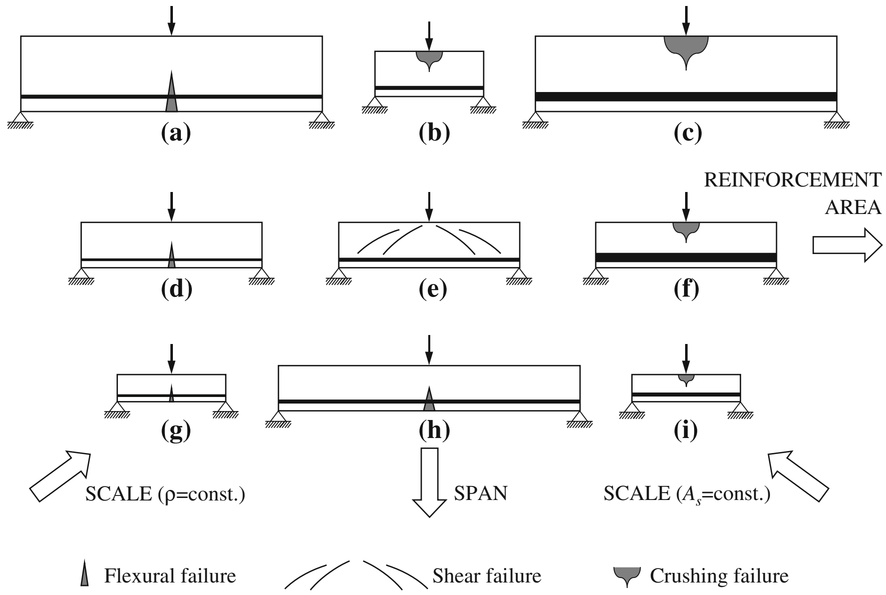

Fig. 5 Global scheme illustrating failure mode transitions

size-scale effects in shear failure, by assuming similarity in the crack pattern leading the beam to collapse. As clearly shown, this similarity is untenable. Moreover, if N_P is further increased (by varying the beam height h and keeping ρ constant, another transition appears (transition e–c). Increasing N_P implies increasing N_C; therefore, both the loads for flexural/shear collapse (ruled by N_P) and the load for crushing collapse (ruled by N_C) increase, but with different rates, so that above a certain value of beam depth h, the collapse by crushing precedes the flexural/shear collapse. A more extensive description and explanation is given in Carpinteri et al. (2007a,b).

Thus, as a major result, it is clearly obtained that the diagonal tension failure is only the intermediate (transitional) failure mode between flexural failure and crushing collapse, although, in some cases, depending on material and geometrical properties, the intermediate transition through (e) may be skipped and a direct transition from yielding to crushing can be observed (Carpinteri et al. 2007a,b).

5 Conclusions

In this paper, we revisited the fundamentals of the fractal modeling of the size-scale effects (Carpinteri 1994a,b), evidencing hypotheses, strengths and the weaknesses of the approach and briefly presenting the two demonstrations, the fractal and the statistical ones, that explain the slopes in the multifractal scaling laws towards the smaller scales. The last part of the paper summarized the major results recently obtained by Carpinteri et al. (2007a,b, 2008) in the modeling of RC beams failing in shear. The commonly adopted hypothesis of homiletical failure crack patterns in beams of different size is discussed in the light of these results. As clearly pointed out, a step forward is needed to model the size-scale effects of diagonal tension failure, since this phenomenon cannot be studied without taking into account transitions towards other collapse mechanisms.

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References

Angelakos D, Bentz E, Collins M (2001) Effect of concrete strength and minimum stirrups on shear strength of large members. ACI Struct J 98:290–300

Baia Y, Lu C, Ke F, Xia M (1994) Evolution induced catastrophe. Phys Lett A 185:196–200

Bažant Z (1984) Size effect in blunt fracture: concrete, rock, metal. J Eng Mech (ASCE) 110:518–535

Bažant Z (2002) Concrete fracture models: testing and practice. Eng Fract Mech 69:165–205

Bažant Z, Kazemi M (1991) Size effect on diagonal shear failure of beams without stirrups. ACI Struct J 88:268–276

Bažant Z, Kazemi M (2006) Discussion on "Repeating a classic set of experiments on size effect in shear of members

without stirrups" paper by E.C. Bentz and S. Buckley. ACI Struct J 105:754–755
Bažant Z, Yavari A (2005) Is the cause of size effect on structural strength fractal or energetic-statistical? Eng Fract Mech 72:1–31
Bažant Z, Yavari A (2007) Response to A. Carpinteri, B. Chiaia, P. Cornetti and S. Puzzi's comments on "Is the cause of size effect on structural strength fractal or energetic-statistical?" Eng Fract Mech 74:2897–2910
Bažant Z, Yu Q (2005) Designing against size effect on shear strength of reinforced concrete beams without stirrups: I. formulation. J Struct Eng (ASCE) 131:1877–1885
Bentz E, Buckley S (2005) Repeating a classic set of experiments on size effect in shear of members without stirrups. ACI Struct J 102:832–838
Bouchaud E (1997) Scaling properties of cracks. J Phys Condens Matter 9:4319–4344
Bouchaud E, Lapasset G, Planés J (1990) Fractal dimension of fractured surfaces: a universal value. Europhys Lett 13: 73–79
Carpinteri A (1981) A fracture mechanics model for reinforced concrete collapse. In: Advanced mechanics of reinforced concrete. Proceedings of a IABSE Colloquium, Delft, the Netherlands, Delft University of Technology Press, Delft, pp 17–30
Carpinteri A (1984) Stability of fracturing process in R.C. beams. J Struct Eng (ASCE) 110:544–558
Carpinteri A (1986) Mechanical damage and crack growth in concrete: plastic collapse to brittle fracture. Martinus Nijhoff Publishers, Dordrecht
Carpinteri A (1989) Decrease of apparent tensile and bending strength with specimen size: two different explanations based on fracture mechanics. Int J Solids Struct 25: 407–429
Carpinteri A (1994) Fractal nature of material microstructure and size effects on apparent mechanical properties. Mech Mater 18:89–101
Carpinteri A (1994) Scaling laws and renormalization groups for strength and toughness of disordered materials. Int J Solids Struct 31:291–302
Carpinteri A, Chiaia B (1995) Multifractal nature of concrete fracture surfaces and size effects on nominal fracture energy. Mater Struct 28:435–443
Carpinteri A, Chiaia B (1996) Power scaling laws and dimensional transitions in solid mechanics. Chaos Solitons Fractals 7:1343–1364
Carpinteri A, Chiaia B (1996) Size effects on concrete fracture energy: dimensional transition from order to disorder. Mater Struct 29:259–266
Carpinteri A, Chiaia B (1997) Multifractal scaling laws in the breaking behavior of disordered materials. Chaos Solitons Fractals 8:135–150
Carpinteri A, Chiaia B (2002) Embrittlement and decrease of apparent strength in large-sized concrete structures. Proceedings of the Indian Academy of Sciences (Sadhana) 27, 425–448
Carpinteri A, Cornetti P (2002) A fractional calculus approach to the description of stress and strain localization in fractal media. Chaos Solitons Fractals 13:85–94
Carpinteri A, Cornetti P (2002) Size effects on concrete tensile fracture properties: an interpretation of the fractal approach based on the aggregate grading. J Mech Behav Biomed Mater 13:233–246
Carpinteri A, Ferro G (1994) Size effects on tensile fracture properties: a unified explanation based on disorder and fractality of concrete microstructure. Mater Struct 28:563–571
Carpinteri A, Ferro G (1998) Scaling behaviour and dual renormalization of experimental tensile softening responses. Mater Struct 31:303–309
Carpinteri A, Puzzi S (2008) The fractal-statistical approach to the size-scale effects on material strength and toughness. Probabilistic Engineering Mechanics. doi:10.1016/j.probengmech.2008.01.003
Carpinteri A, Chiaia B, Ferro G (1995) Size effects on nominal tensile strength of concrete structures: multifractality of material ligaments and dimensional transition from order to disorder. Mater Struct 28:311–317
Carpinteri A, Chiaia B, Ferro G (1998) Scale dependence of tensile strength of concrete specimens: a multifractal approach. Mag Concr Res 50:237–246
Carpinteri A, Chiaia B, Cornetti P (2002) A scale-invariant cohesive crack model for quasi-brittle materials. Eng Fract Mech 69:207–217
Carpinteri A, Chiaia B, Cornetti P (2003) On the mechanics of quasi-brittle materials with a fractal microstructure. Eng Fract Mech 70:2321–2349
Carpinteri A, Cornetti P, Kolwankar KM (2004) Calculation of the tensile and flexural strength of disordered materials using fractional calculus. Chaos Solitons Fractals 21:623–632
Carpinteri A, Cornetti P, Puzzi S (2006) Scaling laws and multiscale approach to the mechanics of heterogeneous and disordered materials. Appl Mech Rev 59:283–305
Carpinteri A, Carmona J, Ventura G (2007) Propagation of flexural and shear cracks through reinforced concrete beams by the bridged crack model. Mag Concr Res 59:743–756
Carpinteri A, Ventura G, Carmona J (2007) Flexural to shear and crushing failure transitions in RC beams by the bridged crack model. In: Carpinteri A, Gambarova P, Ferro G, Plizzari G (eds) Fracture Mechanics of Concrete Structures (Proceedings of the 6th International FraMCoS Conference, Catania, Italy, 2007), vol. 2. Taylor & Francis, London, pp 677–684
Carpinteri A, Carmona J, Ventura G (2008) Flexural, shear and crushing failure transitions in RC beams p. to appear
Collins M, Kuchma D (1999) How safe are our large, lightly reinforced concrete beams, slabs and footings? ACI Struct J 96:482–490
Feder J (1988) Fractals. Plenum Press, New York
Gustafsson P, Hillerborg A (1983) Sensitivity in shear strength of longitudinally reinforced concrete beams to fracture energy of concrete. ACI Struct J 83:286–294
Hillerborg A, Modeer M, Petersson P (1976) Analysis of crack formation and crack growth in concrete by means of fracture mechanics and finite elements. Cem Concr Res 6:773–782
Jenq Y, Shah S (1989) Shear resistance of reinforced concrete beams - a fracture mechanics approach. In: Li V, Bažant Z

(eds) Fracture mechanics: applications to concrete. Concrete Institute, Detroit pp 237–258

Kleiser T, Bocek M (1986) The fractal nature of slip in crystals. Zeitschrift für Metallkunde 77:582–587

Lu C (2007) Some notes on the study of fractals in fracture. In: F.A. et al. (ed) Proceedings of the 5th Australasian Congress on Applied Mechanics, Brisbane, pp 234–239

Lu C, Mai YW (2005) Influence of aspect ratio on barrier properties of polymer-clay nanocomposites. Phys Rev Lett 95(088):303

Mandelbrot B (1982) The fractal geometry of nature. W. H. Freemann & Company, New York

Mandelbrot BB, Passoja DE, Paullay AJ (1984) Fractal character of fracture surfaces of metals. Nature 308:721–722

Panin V, Elsukova T, Angelova G, Kuznetsov P (2002) Mechanism of formation of fractal mesostructure at the surface of polycrystals upon cyclic loading. Phys Met Metallogr 94:402–412

So K, Karihaloo B (1993) Shear capacity of longitudinally reinforced beams—a fracture mechanics approach. ACI Struct J 90:591–600

van Mier J, van Vliet M (1999) Effect of strain gradients on the size effect of concrete in uniaxial tension. Int J Fract 94:195–219

Vecchio F (2000) Analysis of shear-critical reinforced concrete beams. ACI Struct J 97:102–110

Vecchio F, Collins M (1986) The modified compression-field theory for reinforced concrete elements subjected to shear. ACI J 83:219–231

Ventura G, Carpinteri A, Ferro G (2004) Double brittle-to-ductile transition in bending of fibre reinforced concrete beams with rebars. Int J Numer Methods Geomech 28:737–756

Weiss J (2001) Self-affinity of fracture surfaces and implications on a possible size effect on fracture energy. Int J Fract 109:365–381