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One, no one, and one hundred thousand crack propagation laws: A generalized Barenblatt and Botvina dimensional analysis approach to fatigue crack growth

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ABSTRACT

Barenblatt and Botvina with elegant dimensional analysis arguments have elucidated that Paris' power-law is a weak form of scaling, so that the Paris' parameters C and m should not be taken as material constants. On the contrary, they are expected to depend on all the dimensionless parameters of the problem, and are really "constants" only within some specific ranges of all these. In the present paper, the dimensional analysis approach by Barenblatt and Botvina is generalized to explore the functional dependencies of m and C on more dimensionless parameters than the original Barenblatt and Botvina, and experimental results are interpreted for a wider range of materials including both metals and concrete. In particular, we find that the size-scale dependencies of m and C and the resulting correlation between C and m are quite different for metals and for quasi-brittle materials, as it is already suggested from the fact the fatigue crack propagation processes lead to $m = 2-5$ in metals and $m = 10-50$ in quasi-brittle materials. Therefore, according to the concepts of complete and incomplete self-similarities, the experimentally observed breakdowns of the classical Paris' law are discussed and interpreted within a unified theoretical framework. Finally, we show that most attempts to address the deviations from the Paris' law or the empirical correlations between the constants can be explained with this approach. We also suggest that "incomplete similarity" corresponds to the difficulties encountered so far by the "damage tolerant" approach which, after nearly 50 years since the introduction of Paris' law, is still not a reliable calculation of damage, as Paris himself admits in a recent review.

1. Introduction

More than 40 years ago, Paris and Erdogan (1963) suggested using the stress-intensity factor range, ΔK, to obtain the rate of crack advance per cycle, $\mathrm{d}a/\mathrm{d}N$, proposing a very general and simple correlation:

$$\frac{\mathrm{d}a}{\mathrm{d}N} = f(\Delta K) \tag{1}$$

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that was considered so revolutionary that received a strong opposition from the scientific Community (see Paris et al., 1999).

Actually, two years before, Paris et al. (1961) proposed a fatigue crack growth criterion where da/dN was considered to be proportional to K_{max}^4. Only in his doctoral thesis, Paris (1962) analyzed the experimental data by A.J. McEvily and found an impressively good power-law fit for some Al-alloy with an exponent which could not correspond to any of the previous laws. In Paris and Erdogan (1963) it was therefore suggested that Eq. (1) should have a power-law form, with m as a free parameter (see the Paris' own recollection in a recent tribute to Professor A.J. McEvily's contributions, Lados and Paris, 2007):

> One of the present authors made use of McEvily's data in his doctoral thesis plotted on a double logarithmic basis to develop an empirical law of crack growth ... These graphs showed that the data correlated reasonably well with other data... This led to the familiar power law for the crack propagation ($a =$ crack length, $N =$ number of cycles):
>
> $$\frac{da}{dN} = C\Delta K^m, \tag{2}$$
>
> where C depends on load ratio and m[2] is at least in the range of 3–4 for these data.

Actually, the first original "competitor" of his law was the Head's law (Eqs. (2) and (4) in Paris and Erdogan, 1963), which could in retrospective be put in a power-law form (2) with either $m = 3$ or 2, depending on whether plastic zone in the denominator is constant as originally suggested by Head, or is considered to be dependent on Irwin's plastic-zone size, as Paris argued. McEvily's data on 7075-T6 and on 2024-T3 compared well with a power-law fit with $m = 4$ in the Paris' original Thesis plots in 1962. But Paris did not propose a fixed value for m, because other data analyzed in the second paper by Paris and Erdogan (1963) fitted power-laws with $m = 2$ or 3 like in Head's law!

Hence, it was actually Paris' strong belief on the use of the stress-intensity factor, which Paris himself recollects being his idea already in 1957 in a summer internship at NASA (Paris et al., 1999), and the McEvily data suggesting $m = 4$ that really led to propose to free up the exponent in the equation. Curiously, while Paris' law was perceived clear and strong enough to lead to the so-called damage tolerance approach (see, e.g. Suresh, 1998), the progress in subsequent years has seen a proliferation of "generalized laws", mainly to model the various observed deviations from the power-law regime. However, while ambitions gradually decreased to have a single simple law, the enthusiasm had nevertheless been already pervaded industries and research centers, so Paris' law continued to be perceived as a "law" almost with the status of a physics law, and only few authors, including Barenblatt and Botvina (BB), have really returned to warn that this is not the case. Today, it is well known that the power-law holds in an intermediate range of ΔK (region II), where there should be a limited dependence on the material microstructure, loading ratio and environmental conditions, so that C and m appear in this regime essentially as "material constants". On the other hand, it is admitted that a region I exists, where there is a decrease in the crack growth rate until below a threshold stress-intensity factor, ΔK_{th}, long cracks do not propagate anymore. This threshold significantly depends on the material microstructure, environmental aspects, as well as on the loading ratio. Similarly, when $K_I \to K_{IC}$, another deviation from the Paris' law regime is observed, since the Griffith–Irwin crack growth instability is approached. Actually, the condition $\Delta K < \Delta K_{th}$ is not completely sufficient to guarantee crack arrest since short cracks propagate also below this threshold, whereas for very high nominal stresses, the dependence of da/dN is no longer on ΔK but perhaps on ΔJ, where J is the EPFM (elasto-plastic fracture mechanics) parameter. More importantly, the dependence of the phenomenon of fatigue crack growth on the material microstructure in region II can be relevant, so that some results established for metals may not be extrapolated to other materials. From this preliminary introduction, it clearly emerges that the Paris' law in its original form holds only within a very limited range of conditions.

Moreover, while a pure "crack propagation" approach to damage tolerance is possible to control large enough cracks under not too severe loading conditions, for long lives or high frequency loading, it is still not possible to propose inspection intervals safe enough when cracks are essentially in the initiation phase for most of their life. Hence, it is difficult to account for all the deviations from the Paris' simple regime, and no computational model is entirely satisfactory today, even in the opinion of Paris himself (Paris et al., 1999). In parallel to the damage tolerance approach, research is still active on "damage tolerance in HCF" (high cycle fatigue) where most of the design approach is based on threshold and fatigue limits, returning in part to the original SN curves "empirical" approach, and not using Paris' type of laws (see for example a recent US Air Force initiative in the excellent reviews by Nicholas, 1999, 2006).

It is clear that the observation of the strong power-law nature of crack propagation, originally recognized by pure observation and great intuition, comes from the underlying *self-similarity* of crack propagation connected to the self-similarity of the crack geometry, when the crack length a is larger than the microstructural dimensions, yet smaller than any other dimensions. However, this process is not a *single* process, but a series of potentially different processes depending on several potential dimensionless parameters, length scales, and the crack length itself. This explains partial success of the earliest attempt to generalize Paris' law considered different material or fitting constants in addition to the nominal load and the crack length. To cite a few, we mention the models based on perfect plasticity mechanism, those considering damage ahead of the crack tip in terms of low cycle fatigue using the Coffin–Manson relationships, as well as the models

[2] In the original text, n.

taking into account the crack tip cyclic stresses and strains, or using Miner's law to analyze the effect of the increasing amplitude of loading while a material point approaches the propagating crack tip (see e.g. Glinka, 1982; Kaisand and Mowbray, 1979; Majumdar and Morrow, 1969; Weiss, 1968).

Barenblatt and Botvina (1980) (BB in the following, see also Barenblatt, 1996, 2006), considered the problem of scaling of fatigue crack growth in the general context of scaling processes and other authors have more recently re-examined the idea (see e.g. Ritchie, 2005; Spagnoli, 2005; Carpinteri and Paggi, 2007). BB in particular noticed that *complete similarity* would imply $m = 2$ in Eq. (2), which is not observed if not as nearly a limit case. They introduced the concept of *incomplete similarity*, analyzing the dependence of the Paris' law parameter m on the dimensionless number $Z = \sigma_y\sqrt{h}/K_{IC}$, where σ_y is the tensile strength, K_{IC} is the fracture toughness and h is the specimen thickness. They were very careful to "provisionally" suggest a certain special form of the dependence for metals: namely, that m should be constant for Z less than about unity and then linearly increase with Z. Following Ritchie and Knott (1973), BB proposed a possible interpretation by observing that large specimens imply more "static" modes of failure, as it is well known that constraint at the crack tip is only highest for large enough width and thickness of the specimen (see also the prescription in ASTM E399-90, (2002)[3] for toughness measurement, as further remarked by Ritchie, 2005).

Ritchie (2005) also made interesting further comparisons of data points using this approach. However, Ritchie's plot seems to imply a much wider scatter than the original plots in the BB's paper, which of course were for different materials, and this suggested us a generalization of the BB's approach to look for more general dependencies on dimensionless quantities, and also analyzing the constant C rather than just m. We also consider not only the original data points of the BB's and Ritchie's papers, but also other fatigue data for concrete obtained by Bažant and Shell (1993) and Bažant and Xu (1991) and recently reexamined by Spagnoli (2005).

In the present paper, therefore, by revisiting and generalizing the BB's dimensional analysis approach, we will show an anomalous relationship between m and Z in concrete as compared to metals. A novel interpretation is made by noting that the slope of the linear m vs. Z relationship is strongly correlated to another dimensionless parameters (apart from R, as well known), namely the ratio between the elastic modulus and the material tensile strength, $E/\sigma_y = \varepsilon_y$. Moreover, by looking at the corresponding relationship between C and Z, we find an inverse relationship between these two parameters, which can be mathematically treated according to either complete or incomplete self-similarity, depending on the material being considered. Eliminating Z between the two relationships, one between m and Z and the other between C and Z, a correlation between C and m is found. As far as metals are concerned, we will show that such a relationship is very close to the correlations proposed in the past by several authors on empirical basis. On the contrary, we will demonstrate that concrete behaves quite differently from metals, emphasizing the important role played by the material microstructure even in the Paris' law regime (region II), leading to incomplete self-similarity in Z.

As suggested by BB, Paris' equation can be applied only within certain ranges of variations of the dimensionless parameters governing the problem, and we give some further hints in particular on more parameters than those considered by BB. The drawback in not recognizing these aspects is therefore the dangerous risk of *extrapolation*. This is even more evident when considering that also the initiation process (or processes) leads to fatigue power-laws, such as those by Basquin and by Coffin–Manson, which evidently are perceived as more empirical, but actually depend on other dimensionless numbers, as recently shown in the context of scaling phenomena (Brechet et al., 1992; Chan, 1993). Certainly, a challenging task will be to interpret all these power-laws within a unified framework, comprising also the well-known Hall–Petch relationship, which relates the material yield strength to microstructural quantities.

2. BB's approach

Early attempts to interpret the Paris' law with a fully plastic "striation" mechanism suggested that the Paris' law parameter m should be equal to 2, which, however, is only the lower limit in most experiments (and sometimes, surprisingly, is found less than 2). This limit can be elegantly associated to the concept of "complete similarity" for the limit $\Delta K/K_{IC} \to 0$[4] (Barenblatt and Botvina, 1980), since in this case the Paris' law is obtained from pure dimensional analysis arguments considering only one of the possible material parameters having the physical dimensions of stress. However, it should be immediately remarked that, in presenting their approach, BB assumed only the first of the many possibilities depending on which material constant with physical dimensions of stress we consider in the denominator:

$$\frac{da}{dN} = C_1\left(\frac{\Delta K}{\sigma_y}\right)^2, \quad (3)$$

$$\frac{da}{dN} = C_2\left(\frac{\Delta K}{E}\right)^2, \quad (4)$$

[3] A state of small-scale yielding, i.e. a valid K-field at the crack tip, is reached when r_y is small (typically $\frac{1}{15}$) compared to the specimen in-plane dimensions.

[4] Strictly speaking, a first difficulty already emerges here: How can we reach the limit $\Delta K/K_{IC} \to 0$ without encountering the problem of the ΔK_{th} deviation? It is clear that, at most, a intermediate scaling can occur.

$$\frac{da}{dN} = \beta \Delta \mathrm{CTOD} \simeq C_3 \left(\frac{\Delta K}{\sqrt{\sigma_y E}} \right)^2, \tag{5}$$

where σ_y is the yield strength and E is the elastic modulus.

Eq. (3) is the possibility considered by Barenblatt and Botvina (1980), Eq. (4) is sometimes proposed with $C_2 \cong 3$, or 5.4, or 8 (see e.g. Chan, 1993; Fleck et al., 1994), and finally Eq. (5) is spurious in this context since we have chosen the geometric mean of two material properties instead of a single constant with dimension of stress. However, Eq. (5) comes from the striation mechanism perfect plasticity model of Neumann (1969) and Pelloux (1970) (see Ritchie, 2005), where $\beta < 0.35$. Notice that even further possibilities may exist, and clearly the number of possible parameters may include fatigue material constants (fatigue limit, or the Coffin–Manson constants), and it is impractical to list them all at this stage. In this approach $m = 2$, as governed by dimensional analysis, and the other Paris' parameter, C, even in this case, should depend at least on the loading ratio R and on the parameter Z which, according to the choice in the equations above, is, respectively:

$$Z_1 = \frac{\sigma_y}{K_{IC}} \sqrt{h}, \quad Z_2 = \frac{E}{K_{IC}} \sqrt{h}, \quad Z_3 = \frac{\sqrt{\sigma_y E}}{K_{IC}} \sqrt{h}, \tag{6}$$

where h can be either the width or the thickness of the specimen (but could also be the size of the crack, or any other characteristic length). Note that Z_1 is equal to the inverse of the brittleness number s introduced by Carpinteri in 1980. The BB choice, Z_1, is also the square root of the ratio between h and the crack tip plastic zone, a ratio that had been already used to characterize the transition from ductile failure, when the plastic zone has less constraint and plastic dissipation acts at its highest degree, to brittle collapse, when the specimen is thick and wide enough for the toughness to reach its minimum value, K_{IC}. In any case, it should be pointed out that, in the simplest Paris' law, the constant C is the first quantity which is supposed to vary according to the other dimensionless parameters, while m is constant and equal to 2.

One step further in complication consists in taking *incomplete similarity* which admits dependence on other dimensionless quantities, and hence allows m to vary, but this warns that the Paris' constants depend in principle on all the other possible dimensionless parameters of the problem. Even limiting the additional dimensional properties to the static ones, as done by Ritchie (2005), Spagnoli (2005) and Carpinteri and Paggi (2007), one could see for example the dependence of the exponent m on the numbers Z_i, as well as of C, in terms of dimensionless functions $\Phi_i(R, Z_i)$ and the chosen material properties as

$$m = 2 + \alpha(R, Z_i),$$
$$C_1 = \frac{\Phi_1(R, Z_1)}{\sigma_y^2 K_{IC}^{\alpha}}, \quad C_2 = \frac{\Phi_2(R, Z_2)}{E^2 K_{IC}^{\alpha}}, \quad C_3 = \frac{\Phi_3(R, Z_2)}{\sigma_y E K_{IC}^{\alpha}},$$

which again suggests alternatives to Ritchie's plot, since the correlation between m and Z_1 in Ritchie's (2005) Fig. 1 presents a significant scatter, also perhaps due to the fact that the specimens are not self-similar, so that a true scaling is not observed. The correlation should be checked also against the corresponding scaling suggested for C, which in all cases seems to decrease with K_{IC}^{α}, where α is the surplus of the Paris' exponent with respect to 2, and additionally there could be dependence on σ_y and E according to the equations above. It would be worth trying to see which of the Z_i gives the cleanest correlation, but this is a very difficult task, and we shall prefer in the present paper to continue with the original BB choice.

The collection of data that we shall include represents steels with almost a fivefold variation in yield strength, from 433 to 2035 MPa (Ritchie, 2005), the original data in the BB's paper (Barenblatt and Botvina, 1980) and the data about high strength and normal strength concrete in Bažant and Shell (1993), Bažant and Xu (1991) and Spagnoli (2005).

One way to approach the problem, which results to be extremely useful, is to look at the change of constants C and m in terms of transition of the mechanisms, as noticed by Masuda et al. (1980), who developed "fracture mechanism maps" (including striation, intergranular fracture, void coalescence, etc.) showing the wide range of possible Paris' curves obtained for different steels, just changing heat treatment. For high yield strength and low toughness (hence, high Z), no striation mechanism occurs and m is high, while in the opposite case of low yield strength and high toughness (hence, low Z), there is a broad range where striation is the dominant mechanism. In the intermediate cases, it is clear that both mechanisms can occur, and m is not well defined. For more recent review of mechanisms and maps, see Sadananda and Vasudevan (2003).

However, these failure maps should be probably considered not only as functions of the material properties, but also on the dimensionless number Z. More specifically, should scale effect also involve such a change of mechanisms, and hence be an alternative source of change for C and m? And are the transitions in C and m based on characteristic length scales, and if so, how?

There are innumerable ways to generate models for fatigue crack growth, since the process itself is very rich, corresponding to a very high strain of the material near the crack tip, yet not easy to evaluate with precision for a number of intrinsic and extrinsic mechanism driving the growth ahead or behind the crack tip (see Ritchie, 1999). According to the model, material constants are introduced, but also microscopically related length scales or properties. Here, we shall not attempt to review all these models, but simply collect some of them and point to known correlations or observed facts, hoping to make a step towards a more "unified" vision.

3. BB's generalized

According to dimensional analysis, the physical phenomenon under observation can be regarded as a *black box* connecting the external variables (called input or governing parameters) with the mechanical response (output parameters). In case of fatigue crack growth in region II, we assume that the mechanical response of the system is fully represented by the crack growth rate, $q_0 = \mathrm{d}a/\mathrm{d}N$, which is the parameter to be determined. This output parameter is a function of a number of variables:

$$q_0 = F(q_1, q_2, \ldots, q_n; s_1, s_2, \ldots, s_m; r_1, r_2, \ldots, r_k), \tag{7}$$

where q_i are quantities with independent physical dimensions, i.e. none of these quantities has a dimension that can be represented in terms of a product of powers of the dimensions of the remaining quantities. Parameters s_i are such that their dimensions can be expressed as products of powers of the dimensions of the parameters q_i. Finally, parameters r_i are dimensionless quantities.

As regards the phenomenon of fatigue crack growth, it is possible to consider the following functional dependence, by extending a little the BB choice to include other fatigue material constants but still omitting for simplicity other possible choices, such as the Coffin–Manson constants σ'_f, ε'_f (fatigue strength and ductility factors, and the corresponding coefficients b and c) as well as any microstructural length scale, such as those related to dislocations or to the grain size:

$$\frac{\mathrm{d}a}{\mathrm{d}N} = F(\sigma_y, K_\mathrm{IC}, \omega; \Delta K, \Delta K_\mathrm{th}, \Delta\sigma_\mathrm{fl}, E, h, a; R), \tag{8}$$

where the governing variables are summarized in Table 1, along with their physical dimensions expressed in the length-force-time (LFT) class.

From this list it is possible to distinguish between three main categories of parameters. The first category regards the static and cyclic material properties, such as the yield stress, σ_y, the fracture toughness, K_IC, the threshold stress-intensity factor range, ΔK_th, the fatigue limit, $\Delta\sigma_\mathrm{fl}$, and Young's modulus, E. The second category comprises the variables governing the testing conditions, such as the stress-intensity factor range, ΔK, the loading ratio, R, and the frequency of the loading cycle, ω. Finally, the last category includes geometric parameters related to the tested geometry, such as the characteristic structural size, h, and the initial crack length, a.

Considering a state with no explicit time dependence and assuming K_IC and σ_y as independent variables, then Buckingham's Π Theorem gives

$$\frac{\mathrm{d}a}{\mathrm{d}N} = \left(\frac{K_\mathrm{IC}}{\sigma_y}\right)^2 \Phi\left(\frac{\Delta K}{K_\mathrm{IC}}, \frac{\Delta K_\mathrm{th}}{K_\mathrm{IC}}, \frac{\Delta\sigma_\mathrm{fl}}{\sigma_y}, \frac{E}{\sigma_y}, \frac{\sigma_y^2}{K_\mathrm{IC}^2}h, \frac{\sigma_\mathrm{fl}^2}{\Delta K_\mathrm{th}^2}a; 1-R\right), \tag{9}$$

where the dimensionless parameters are

$$\begin{aligned}
&\Pi_1 = \frac{\Delta K}{K_\mathrm{IC}}, \quad \Pi_2 = \frac{\Delta K_\mathrm{th}}{K_\mathrm{IC}}, \quad \Pi_3 = \frac{\Delta\sigma_\mathrm{fl}}{\sigma_y},\\
&\Pi_4 = \frac{E}{\sigma_y}, \quad \Pi_5 = \frac{\sigma_y^2}{K_\mathrm{IC}^2}h, \quad \Pi_6 = \frac{\sigma_\mathrm{fl}^2}{\Delta K_\mathrm{th}^2}a,\\
&\Pi_7 = r_1 = 1 - R.
\end{aligned}$$

It has to be noticed that Π_5 takes into account the effect of the specimen size and it corresponds to the square of the dimensionless number Z defined by Barenblatt and Botvina (1980), and to the inverse of the square of the *brittleness number*

Table 1
Governing variables of the fatigue crack growth phenomenon

Variable	Definition	Symbol	Dimensions
q_1	Tensile yield stress of the material	σ_y	FL^{-2}
q_2	Material fracture toughness	K_IC	$\mathrm{FL}^{-3/2}$
q_3	Frequency of the loading cycle	ω	T^{-1}
s_1	Stress-intensity range	$\Delta K = K_\mathrm{max} - K_\mathrm{min}$	$\mathrm{FL}^{-3/2}$
s_2	Threshold stress-intensity factor	ΔK_th	$\mathrm{FL}^{-3/2}$
s_3	Fatigue limit	$\Delta\sigma_\mathrm{fl}$	FL^{-2}
s_3	Elastic modulus	E	FL^{-2}
s_4	Characteristic structural size	h	L
s_5	Initial crack length	a	L
r_1	Loading ratio	$R = \dfrac{K_\mathrm{min}}{K_\mathrm{max}}$	–

s introduced in Carpinteri (1981a, b, 1982, 1983, 1994). Since the plastic-zone size, r_p, scales with K_{IC}^2/σ_y^2 according to Irwin, it follows that $\Pi_5 \sim h/r_p$. Therefore, this dimensionless parameter rules the transition from small-scale yielding, when $\Pi_5 \to \infty$, to large-scale yielding, when $\Pi_5 \to 0$ (see also Ritchie, 2005).

The parameter Π_6 is responsible for the dependence of the fatigue phenomenon on the initial crack length, as recently pointed out by Spagnoli (2005). In fact, if we introduce the El Haddad et al. (1979) length scale:

$$a_0 = \frac{1}{\pi}\left(\frac{\Delta K_{th}}{\Delta \sigma_{fl}}\right)^2, \tag{10}$$

then it follows that $\Pi_6 \sim a/a_0$ and we can define a dimensionless number $Z_{SC} = \Pi_6^2$ which is analogous to Z and governs the transition from short-cracks, when $Z_{SC} \to 0$, to long-cracks, when $Z_{SC} \gg 1$. Here it has to be remarked that in general the El Haddad length scale a_0 is also a function of the loading ratio.

At this point, we want to see if the number of the quantities involved in the relationship (9) can be reduced further from five. For example, starting from ΔK, this parameter can be considered as non-essential when, for very large or very small values of the corresponding dimensionless parameter Π_1, a finite non-zero limit of the function Φ exists:

$$\lim_{\Pi_1 \to 0 \text{ or } \Pi_1 \to \infty} \Phi(\Pi_1, \Pi_2, \Pi_3, \Pi_4, \Pi_5, \Pi_6, \Pi_7) = \Phi_1(\Pi_2, \Pi_3, \Pi_4, \Pi_5, \Pi_6, \Pi_7). \tag{11}$$

In this case we speak about *complete self-similarity*, or *self-similarity of the first kind* (Barenblatt, 1996), in the parameter Π_1. On the other hand, if the limit of the function Φ tends to zero or infinity, the quantity Π_1 remains essential no matter how small or large it becomes. However, in some cases, the limit of the function Φ tends to zero or infinity, but the function Φ has a power-type asymptotic representation:

$$\lim_{\Pi_1 \to 0 \text{ or } \Pi_1 \to \infty} \Phi(\Pi_1, \Pi_2, \Pi_3, \Pi_4, \Pi_5, \Pi_6, \Pi_7) = \Pi_1^{\beta_1} \Phi_1(\Pi_2, \Pi_3, \Pi_4, \Pi_5, \Pi_6, \Pi_7), \tag{12}$$

where the exponent β_1 and, consequently, the dimensionless parameter Φ_1, cannot be determined from considerations of dimensional analysis alone. Moreover, the exponent β_1 may depend on the dimensionless parameters Π_i. In such cases, we speak about *incomplete similarity*, or *self-similarity of the second kind* in the parameter Π_1 (Barenblatt, 1996). It is remarkable to notice that the parameter β_1 can only be obtained either from a best-fitting procedure on experimental results or according to numerical simulations.

As regards the parameter ΔK, the corresponding dimensionless parameter Π_1 is usually small in region II of fatigue crack growth. However, since it is well known that the fatigue crack growth phenomenon is strongly dependent on this variable, a complete self-similarity in Π_1 cannot be accepted. Hence, assuming an incomplete self-similarity in Π_1, we have

$$\begin{aligned}\frac{da}{dN} &= \left(\frac{K_{IC}}{\sigma_y}\right)^2 \left(\frac{\Delta K}{K_{IC}}\right)^{\beta_1} \Phi_1(\Pi_2, \Pi_3, \Pi_4, \Pi_5, \Pi_6, \Pi_7) \\ &= \left(\frac{K_{IC}^{2-\beta_1}}{\sigma_y^2}\right) \Delta K^{\beta_1} \Phi_1(\Pi_2, \Pi_3, \Pi_4, \Pi_5, \Pi_6, \Pi_7),\end{aligned} \tag{13}$$

where β_1 may depend on Π_i.

Repeating this reasoning for the parameters Π_5, Π_6 and Π_7, we find the following generalized representation:

$$\frac{da}{dN} = \left(\frac{K_{IC}^{2-\beta_1}}{\sigma_y^2}\right) \Delta K^{\beta_1} \left(\pi \frac{h}{r_p}\right)^{\beta_2} \left(\pi \frac{a}{a_0}\right)^{\beta_3} (1-R)^{\beta_4} \Phi_2(\Pi_2, \Pi_3, \Pi_4), \tag{14}$$

where, again, the exponents β_i may depend on Π_i. Comparing Eq. (14) with the expression of the Paris' law, we find that our proposed formulation encompasses the classical Paris' equation as a limit case when the Paris' law parameters m and C are given by

$$m = \beta_1, \tag{15a}$$

$$C = \left(\frac{K_{IC}^{2-m}}{\sigma_y^2}\right) \left(\pi \frac{h}{r_p}\right)^{\beta_2} \left(\pi \frac{a}{a_0}\right)^{\beta_3} (1-R)^{\beta_4} \Phi_2(\Pi_2, \Pi_3, \Pi_4). \tag{15b}$$

As a consequence, from Eq. (15b) it is possible to note that the parameter C is dependent on two material parameters, such as the fracture toughness, K_{IC}, and the yield stress, σ_y, as well as on the loading ratio, R, and on the length scales h and a.

Hence, the phenomenon of fatigue crack growth presents different length scales, i.e. the specimen size, h, the plastic-zone size, r_p, and the transition crack length corresponding to the breakdown of LEFM concepts and to the activation of short crack effects, a_0. The crack length interacts with such length scales and the fatigue response is influenced by them in the different stages of crack growth.

In general, depending on the values assumed by h/r_p and a/a_0, the following situations may occur.

(1) Incomplete self-similarity only in Π_5: this may happen when Π_5 is neither too small nor too large, i.e. the size of the process zone is comparable with the structural size and therefore we have a transition from small-scale to large-scale

yielding. On the other hand, the crack length is long enough such that a long-crack regime can be considered. In this case, the scaling law in Eq. (15b) gives $C \sim (h/r_p)^{\beta_2}$, i.e. a Paris' law parameter dependent on the structural size-scale.

(2) Incomplete self-similarity only in Π_6: this may occur when the crack length is comparable with the El Haddad transition crack length, a_0. This usually occurs in the short-crack regime. In this case, the scaling law in Eq. (15b) gives $C \sim (a/a_0)^{\beta_3}$, i.e. a Paris' law parameter dependent on the initial crack length.

(3) Incomplete self-similarity both in Π_5 and in Π_6: this is an intermediate situation where both the transitional sizes are comparable. Therefore, in this situation, both microstructural and structural aspects could affect the fatigue response.

In all of these situations, the parameter $\beta_1 = m$ can depend on Π_5 and Π_6.

Notice that our choice of dimensionless ratios is not unique, and more distinctions and microstructural length scales would be needed to include all the possible categories of crack in well-known classifications (Suresh and Ritchie, 1984; Ritchie and Lankford, 1986; Miller, 1999). For example, if we choose instead of h/r_p the ratio a/r_p (or, equivalently, we consider the ratio between two of our dimensionless ratios), we would have another form of transition, where the Irwin parameter is no longer very useful, and EPFM should be used, introducing the J-integral in the crack growth equation. Notice that this can occur both for long and for short cracks (in the sense of a/a_0), and indeed it is better to talk of physically or mechanically small cracks (Miller, 1999). Obviously, very large nominal loads are required to have a very large plastic-zone size at the tip (close to the full-yielding condition) when the crack size is in its turn also of the order of a_0. The final case is when cracks are *microscopically short* (microstructurally small) for which continuum mechanics breaks down and *microstructural fracture mechanics* is needed (Hobson et al., 1986; Navarro and de los Rios, 1988); this is perhaps the most complex category, since crack deceleration or self-arrest is dependent on the grains size and orientations, and possible decelerations or "minima" in da/dN and multiple small-crack curves can be found (Ritchie and Lankford, 1986).

As regards the scaling laws for C in concrete, it is interesting to note that Carpinteri and Spagnoli (2004) considered incomplete self-similarity in Π_5 to prove that C is structural-size dependent. Afterwards, Spagnoli (2005) reinterpreted the same fatigue data in terms of incomplete self-similarity in Π_6, simply noting that the initial crack length was proportional to the structural size in the tested specimens (geometric similarity). However, this is rigorously true if and only if the situation (3) occurs. In such a case, we have either $C \sim h^{\beta_2+\beta_3}$ or, equivalently, $C \sim a^{\beta_2+\beta_3}$.

Another example of anomalous scaling is represented by the fatigue crack growth representation by Frost (1966):

$$a = \bar{a} \exp(\lambda \Delta\sigma^3 N), \tag{16}$$

where $\bar{a}$ is the initial crack length and λ is a material constant. Differentiating this law with respect to N, we see that this parametric representation of fatigue has $\beta_1 = m = 3$ and $C(a) \sim a^{-1/2}$, i.e. $\beta_3 = -\frac{1}{2}$. Yet, this is still used apparently by the Australian Air Force (Molent et al., 2005) with some success.

The anomalous crack-size dependence for the parameter C has also important consequences for the scaling of the fatigue threshold (Paggi and Carpinteri, 2008). In fact, if we determine the value of ΔK_{th} by inverting the Paris' law in correspondence of a conventional value of the crack growth rate, v_{th}, then we have

$$\Delta K_{th} = \left(\frac{v_{th}}{C}\right)^{1/m}. \tag{17}$$

This implies that, since $C(a) \sim a^{\beta_3}$, we find $\Delta K_{th} \sim a^{-\beta_3/m}$. For $\beta_3 = -\frac{1}{2}$ and $m = 3$, we have $\Delta K_{th} \sim a^{1/6}$, which corresponds to the scaling law for ΔK_{th} proposed by Frost (1966) and Murakami and Endo (1986).

Notice that another form of anomalous scaling of C is due to microstructure, which we have not included in the present treatment. Indeed, Chan (1995) derived a crack growth equation which depends on the dimensionless number

$$\Pi_8 = \frac{Es}{4\sigma_y \varepsilon'_f d}, \tag{18}$$

which is derived from using the Coffin–Manson equation (the plastic term only, hence the appearing of the parameter ε'_f and the exponent b) with a strain range derived from the crack-tip opening displacement (CTOD) and the dislocation barrier spacing d, and assuming the propagation is for the dislocation cell element of size s, also giving the striation spacing and a more precise basis to the similar but more empirical approach in Glinka (1982), Kaisand and Mowbray (1979), Majumdar and Morrow (1969) and Weiss (1968). The resulting crack growth equation is

$$\frac{da}{dN} = \Pi_8^{1/b} (2s)^{1-1/b} \left(\frac{\Delta K}{E}\right)^{2/b}, \tag{19}$$

which, for $b = \frac{1}{2}$ as often approximately observed in many metals, leads to $m = 4$ and the equations derived by Rice-Weerman and Mura (see also Chan, 1995). Chan (1995) argues that the reason why this law is not observed in most cases—from which the common belief that C should not depend on microstructure in region II of propagation—is due to the fact that, for decreasing dislocation barrier spacing, yield stress and fatigue ductility usually increase, so that Π_8 only spans a limited range and its dependence is not seen. This is, however, not always the case, as proved by Chan (1995) for special types of steels (HSLA, high-strength low-alloy steels).

4. Analysis of the functional dependencies of the Paris' law parameters

The original data of the Paris' law in Paris et al. (1961) and Paris and Erdogan (1963) only showed the intermediate range of da/dN vs. ΔK curve in a bi-logarithmic diagram, where m and C were sufficient to characterize the whole curve. However, immediately afterwards, it was recognized that the slope is changing when ΔK is in the near-threshold region (region I) or in the rapid-crack propagation region (region III), as experimentally evidenced by Radhakrishnan (1979, 1980). This well-known result can be reinterpreted in the framework of BB's dimensional analysis stating that m is dependent on $\Pi_1 = \Delta K/K_{IC}$. More specifically, the parameter m tends to infinity when either $\Pi_1 \to \Delta K_{th}/K_{IC}$ or when $\Pi_1 \to 1$. This trend is shown in Fig. 1, where the effective slope of a typical fatigue crack growth curve is computed as a function of Π_1. Clearly, the Paris' law applies only in region II, where m is approximately constant.

The slope m of the da/dN vs. ΔK relationship in a bi-logarithmic plane is also dependent on the dimensionless parameter $\Pi_6 = Z_{SC}^2 \sim a/a_0$. The crack growth rate depends on the crack length regime in region I, as schematically shown in Fig. 2. Short cracks are characterized by $\Pi_6 < 1$ and the da/dN vs. ΔK curve may have a negative slope in region I. On the contrary, the classical positive slope m is found for long cracks having $\Pi_6 \gg 1$.

The main point raised by BB was, however, that, because of incomplete similarity, the Paris' law parameter m may depend on Π_5, which corresponds to the square of the brittleness number Z. Analyzing aluminium alloys, 4340 steel and low-carbon steels, Barenblatt and Botvina (1980) firstly found that m is a linear function of Z, being the slope of such a relationship different from a material to another. For very low values of Z, they found that m turns out to be almost constant and independent of Z. To explain such a trend, Barenblatt and Botvina (1980) supposed that the relationship between m and Z has three regimes: $m \cong$ const for small Z, m linear with Z for $1 < Z < 2$ and again $m \cong$ const for large Z.

The BB's data concerning aluminium alloys, 4340 steel and low-carbon steels are herein reanalyzed in Fig. 3, along with the data for ASTM steels and for normal and high strength concretes. All the data refer to a loading ratio $R = 0$. As can be seen, the slope of the linear relationship between m and Z progressively decreases from aluminium alloys to steels. For low-carbon steels, m becomes nearly independent of Z and the slope becomes negative valued for normal and high strength concretes. Clearly, the BB's interpretation of the slope variability is not consistent with the analyzed data. In fact,

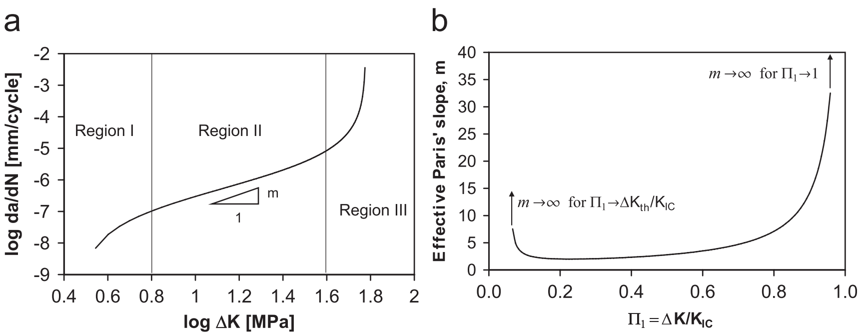

Fig. 1. Π_1-dependence of the Paris' law parameter m. (a) A typical da/dN curve for steel ($R = 0$, $K_{IC} = 60\,\text{MPa}\sqrt{\text{m}}$, $\Delta K_{th} = 3\,\text{MPa}\sqrt{\text{m}}$). (b) Effective Paris' slope m vs. $\Pi_1 = \Delta K/K_{IC}$ computed from (a).

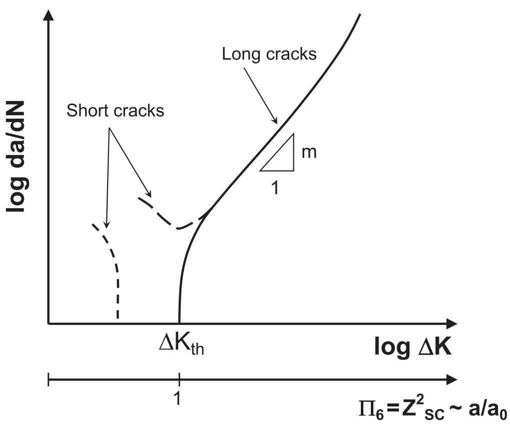

Fig. 2. Dependence of the crack growth rate on $\Pi_6 = Z_{SC}^2$.

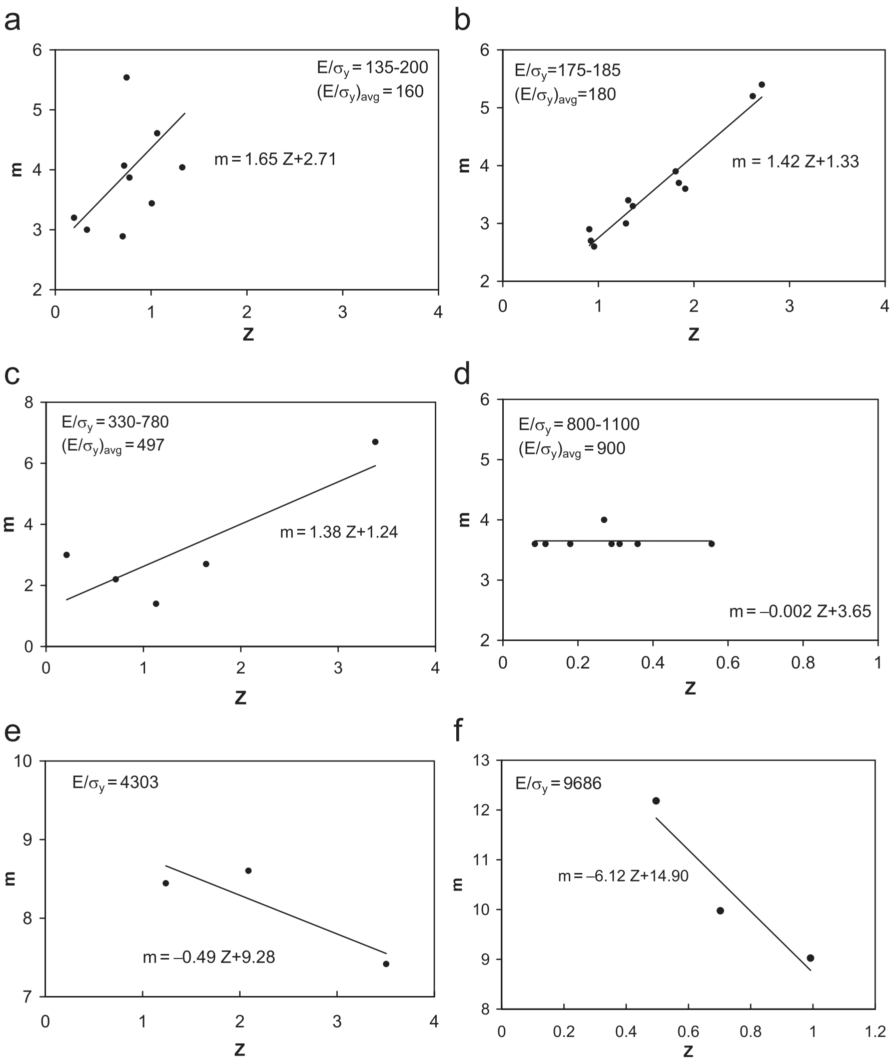

Fig. 3. *Z*-dependence of the Paris' law parameter *m*. (a) Aluminium alloys (Yarema and Ostash, 1975; Ostash et al., 1977). (b) 4340 steel (Heiser and Mortimer, 1972). (c) ASTM steels (Clark and Wessel, 1970). (d) Low carbon steel (Ritchie and Knott, 1974; Ritchie et al., 1975). (e) High strength concrete (data from Bažant and Shell, 1993 reinterpreted by Spagnoli, 2005). (f) Normal strength concrete (data from Bažant and Xu, 1991 reinterpreted by Spagnoli, 2005).

although the range of variation of Z is almost the same for high strength concrete, 4340 steel and ASTM steels, their slopes are significantly different.

An alternative explanation can be put forward by considering the key role played by the dimensionless parameter $\Pi_4 = E/\sigma_y$. In fact, the slope of the m vs. Z relationship is a decreasing function of E/σ_y, being the slope positive valued for $E/\sigma_y \lesssim 1000$ and negative valued for $E/\sigma_y \gtrsim 1000$. This trend is shown in Fig. 4, where a best-fitting linear equation is superimposed to the experimental data points obtained from Fig. 3.

The Z-dependence of the Paris' law parameter C can also be examined. This functional dependence has received a minor attention in the past as compared to that for m, although the variability of the parameter C is extremely important from the engineering point of view and its size-scale dependence may have strong consequences for the damage tolerance design of large-scale structures. Dimensional analysis suggests two different scaling laws for C, depending on whether or not the incomplete self-similarity in Π_5 takes place. If this is not the case, then we have $\beta_2 = 0$ in Eq. (15b) and the functional dependence of C on Z is no longer a power-law. On the other hand, if we have incomplete self-similarity in Π_5, then $\beta_2 \neq 0$ and a power-law dependence of C on Z can be obtained, i.e. $C \propto \Pi_5^{\beta_2} \propto Z^{2\beta_2}$ or, equivalently, $\log C \propto 2\beta_2 \log Z$.

To assess which of such conditions takes place, we can simply plot the available experimental data in the $\log C$ vs. Z and $\log C$ vs. $\log Z$ diagrams. The former plot would correspond to $C \propto 10^Z$ and therefore no power-law dependence between C and Z, whereas the latter gives the scaling $C \propto Z^{2\beta_2}$ characteristic of incomplete self-similarity. A best fitting linear equation can be determined in both situations and the corresponding linear regression coefficient r^2 will provide their goodness of fit. The situation having r^2 closer to unity gives the best correlation. To do so, the Paris' law parameter C has not been

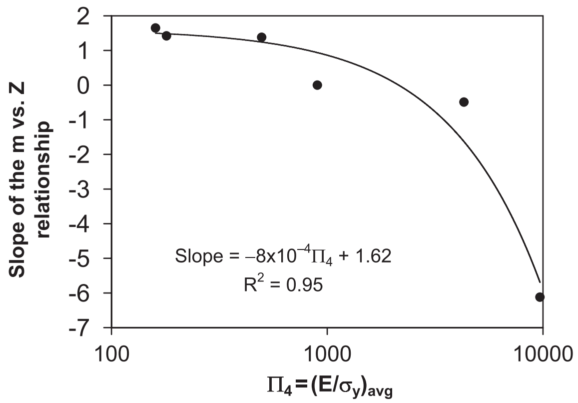

Fig. 4. Slope of the m vs. Z relationship as a function of $\Pi_4 = (E/\sigma_y)_{avg}$.

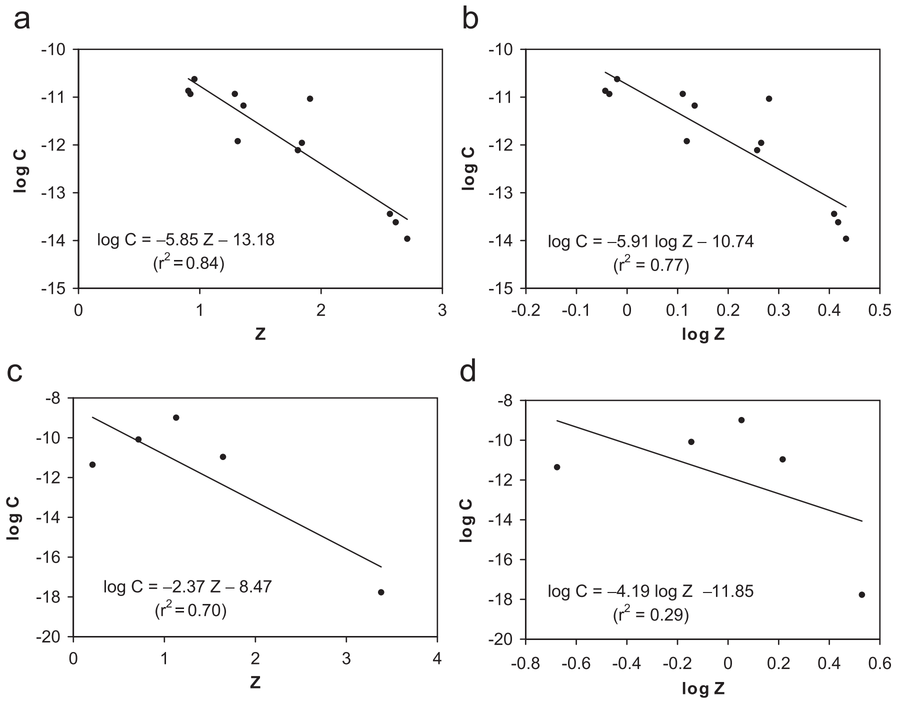

Fig. 5. Assessment of incomplete self-similarity in Π_5 in metals (C evaluated using ΔK in MPa $\sqrt{m}$ and da/dN in m/cycle). (a) and (b) 4340 steel (Heiser and Mortimer, 1972). (c) and (d) ASTM steels (Clark and Wessel, 1970).

computed by the present authors starting from experimental fatigue curves, but the values have been directly taken from the Literature, where C was determined considering only the experimental data falling within regime II.

This procedure is performed in Fig. 5 for 4340 steel and ASTM steels. In this case, as can be seen, the $\log C$ vs. Z relationships present correlation coefficients higher than those for the $\log C$ vs. $\log Z$ representation. Therefore, this seems to exclude an incomplete self-similarity in Π_5 in metals. As will be shown in the sequel, this result has also important implications on the correlation between the Paris' law parameters C and m. On the other hand, this cannot be considered as a universal result. In fact, for concrete, the situation is the opposite and the assumption of incomplete self-similarity in Π_5 gives the best correlation (see Fig. 6).

Finally, it is interesting to note that the $m-Z$ and $C-Z$ relationships are also dependent on the dimensionless parameter Π_7, i.e. on loading ratio R. The effect of R can be experimentally assessed by considering the fatigue data of tempered steels (at $T = 200$ and $405\,^\circ C$) related to two different loading ratios, $R = 0.35$ and 0.68. As can be seen from Fig. 7, the slope of the m vs. Z relationship is an increasing function of R, whereas exactly an opposite trend is observed for C.

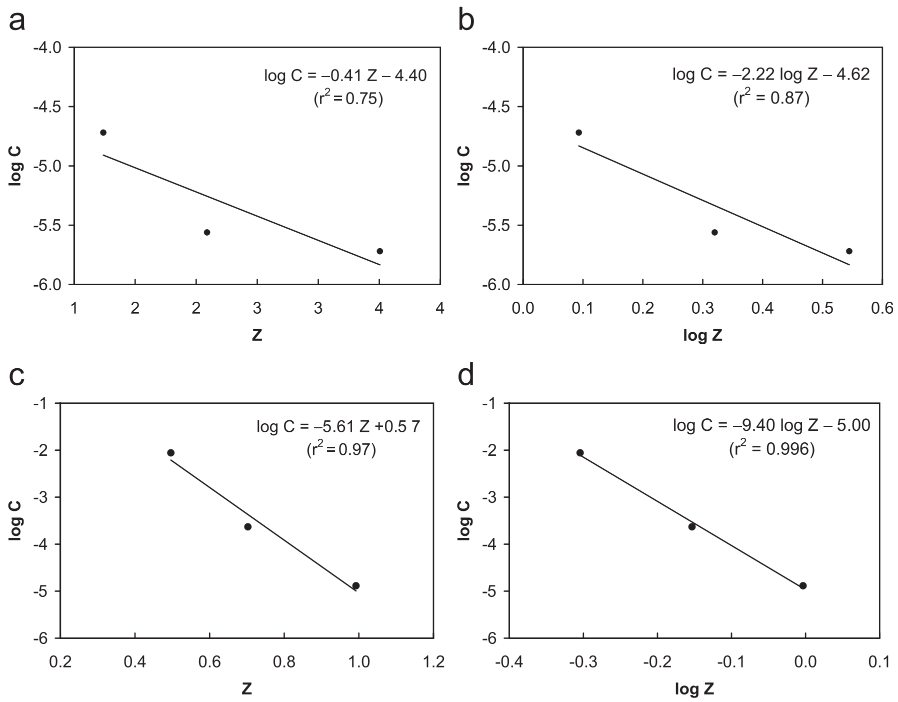

Fig. 6. Assessment of incomplete self-similarity in Π_5 in concrete (C evaluated using ΔK in MPa $\sqrt{\mathrm{m}}$ and da/dN in m/cycle). (a) and (b) High strength concrete (data from Bažant and Shell, 1993 reinterpreted by Spagnoli, 2005). (c) and (d) Normal strength concrete (data from Bažant and Xu, 1991 reinterpreted by Spagnoli, 2005).

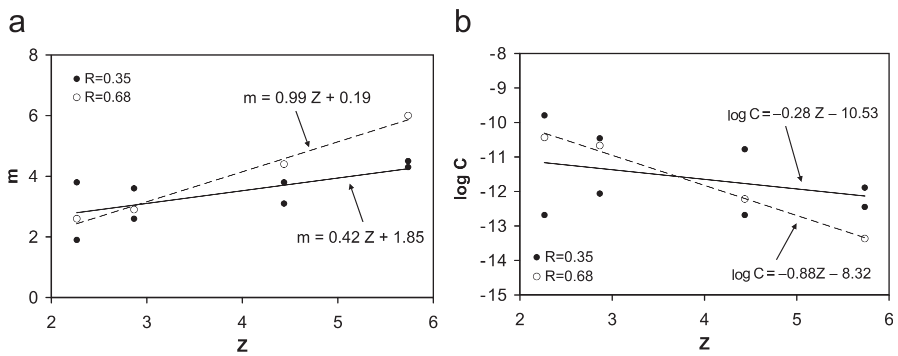

Fig. 7. The effect of R on the $m-Z$ and $\log C-Z$ relationships for tempered steels (experimental data from Evans et al., 1971, C evaluated using ΔK in MPa $\sqrt{\mathrm{m}}$ and da/dN in m/cycle).

5. Correlations between the Paris' law parameters

It is useful at this point to step back and review the existing observed correlations between C and m. In fact, in view of the previous findings about the $m(Z)$ and $C(Z)$ relationships, it would be useful to see if eliminating Z between the two relations, the resulting correlation is more or less of the form obtained by previous authors, who evidently were not looking specifically at size-scale effects. In fact, empirical correlations between C and m are usually obtained by considering a large series of samples with slightly different mechanical properties but with almost the same specimen size (see e.g. Cavallini and Iacoviello, 1991; Bergner and Zouhar, 2000, among others).

We can for example start from the overview by Fleck et al. (1994), which makes some suggestions and also checks various simplified equations using a wide database. About Paris' law, the central idea is that the coordinates of the points at threshold and at critical propagation are about the same for each class of materials, with crack growth rates of around 1×10^{-6} and 1×10^{-2} mm/cycles, respectively. This permits, assuming the Paris' law to hold in the range between these two points, to find a simple correlation between m, the threshold stress-intensity factor range and the

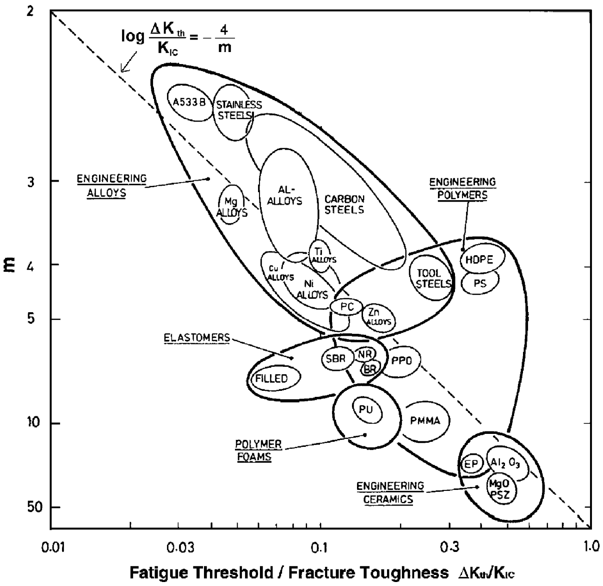

Fig. 8. Paris' law parameter m vs. $\Delta K_{\mathrm{th}}/K_{\mathrm{IC}}$ (reprinted from Fleck et al., 1994 with permission).

fracture toughness:

$$\log_{10}\left(\frac{\Delta K_{\mathrm{th}}}{K_{\mathrm{IC}}}\right) = -\frac{4}{m} \tag{20}$$

and Fig. 8 seems to confirm this qualitative scaling, with steels having m a little lower than such a predicted value, very ductile steels having m ranging between 2 and 5, as well as the most brittle steels (tool steels) having m ranging from 4 to 5. Ceramic materials, such as concrete, have m ranging between 10 and 50.

Evidently, the special steels having higher values of m were not included. Fleck et al. (1994) also suggest that the 4 orders of magnitude spanned in the crack growth rate suggest that the geometric mean between threshold and toughness

$$\Delta K_{-4} = \sqrt{\Delta K_{\mathrm{th}} K_{\mathrm{IC}}} \tag{21}$$

should correspond to a crack growth rate of 1×10^{-4} mm/cycles. This also permits an interesting integration of the Paris' law suggesting that ΔK_{-4} is almost constant for several classes of materials, including steels, being a kind of "pivot point". Obviously, by constant we mean here order of magnitude, since for steels it still varies between about 20 and 80 MPa $\sqrt{\mathrm{m}}$, while for Ni alloys it is on the highest side of this range, whereas lower values of ΔK_{-4} correspond to materials such as ceramics and polymers, and it is even lower for elastomers and foams. Notice that the Paris' law, when combining the assumptions on ΔK_{-4}, is in the form ($\mathrm{d}a/\mathrm{d}N$ in mm/cycle)

$$\frac{\mathrm{d}a}{\mathrm{d}N} = 1 \times 10^{-2}\left(\frac{\Delta K}{K_{\mathrm{IC}}}\right)^m, \tag{22}$$

which in fact simply states that the crack growth rate at critical conditions ($\Delta K/K_{\mathrm{IC}} = 1$) is 100 times higher than that at ΔK_{-4}, as originally assumed.

One could also see this as an equation for C:

$$C = 1 \times 10^{-2} K_{\mathrm{IC}}^{-m} \quad \text{or} \quad \log C = \log(1 \times 10^{-2}) - m \log K_{\mathrm{IC}}. \tag{23}$$

Similar equations have been experimentally found by many authors, see e.g. Radhakrishnan (1979) and Tanaka (1979). For instance, Tanaka (1979) proposed the following empirical correlation for brittle and ductile steels in the case of $R = 0$ (ΔK in MPa $\sqrt{\mathrm{m}}$ and $\mathrm{d}a/\mathrm{d}N$ in m/cycle):

$$\log C = \log(2.893 \times 10^{-8}) - m \log(15.49) \quad \text{for brittle steels,} \tag{24}$$

$$\log C = \log(1.700 \times 10^{-7}) - m \log(32.12) \quad \text{for ductile steels.} \tag{25}$$

According to theoretical arguments and dimensional analysis, Carpinteri and Paggi (2007) determined the following correlation between C and m:

$$\log C = \log v_{\mathrm{cr}} - m \log[(1 - R)K_{\mathrm{IC}}], \tag{26}$$

where v_{cr} is the crack growth rate corresponding to the transition from regions II to III. As compared to Tanaka's correlation, this formulation has the main advantage that can be applied not only to steels, but, in principle, to any other material. Moreover, the effect of the loading ratio is taken into account, whereas the correlations by Fleck et al. (1994) and Tanaka (1979) apply only to the special case of $R = 0$. In any case, all these correlations suggest that $\log C$ is a decreasing function of m.

As far as the previous findings about the $m(Z)$ and $C(Z)$ relationships are concerned, we have found the following functional dependencies in metals, where incomplete self-similarity in Π_5 does not apply:

$$m = k_1 + k_2 Z, \tag{27}$$

$$\log C = k_3 + k_4 Z, \tag{28}$$

where k_i are best-fitting parameters. Determining Z from the first equation and introducing it in the second one, we find:

$$\log C = \left(k_3 - \frac{k_4 k_1}{k_2}\right) + \frac{k_4}{k_2} m, \tag{29}$$

which suggests a relationship between C and m very similar to that provided by the correlations previously discussed. In fact, since the ratio k_4/k_2 is negative valued according to experimental data (see Figs. 3 and 5), $\log C$ is a decreasing function of m.

On the other hand, when the condition of incomplete self-similarity in Π_5 applies, like in concrete, then we have

$$m = k_1 + k_2 Z, \tag{30}$$

$$\log C = k_5 + k_6 \log Z. \tag{31}$$

Determining Z from the first equation and introducing it in the second one, we find:

$$\log C = k_5 + k_6 \log\left(\frac{m - k_1}{k_2}\right). \tag{32}$$

Considering that the argument of the logarithm in the right-hand side of the equation is a small number close to unity, we can make a power-series expansion and obtain the following approximate correlation:

$$\log C \cong \left[k_5 + k_6 \log\left(-\frac{k_1}{k_2}\right)\right] - \frac{k_6}{k_1} m. \tag{33}$$

In this case, we note that the ratio k_6/k_1 is negative valued according to experimental data (see Figs. 3 and 6) and therefore $\log C$ should be an increasing function of m.

The experimental data regarding $\log C$ for 4340 steel, ASTM steels, high strength and low strength concretes previously analyzed are herein reported in Fig. 9 in terms of m. A best-fitting linear curve is shown with solid line in the same diagrams and the corresponding equations are written. Moreover, the correlations by Tanaka (1979) and by Carpinteri and Paggi (2007) are superimposed to the same diagrams whenever possible with dashed-dotted or dashed lines, respectively.

As far as steels are concerned, direct correlations between $\log C$ and m is found and are close to those obtained by Tanaka (1979) and Carpinteri and Paggi (2007), although some quantitative deviations are clearly seen, specifically in the case of 4340 steels. However, it is a clear indication that, in metals, we have the following approximate correspondence:

$$\log[(1 - R)K_{IC}] \sim -\frac{k_4}{k_2}, \tag{34}$$

$$\log v_{cr} \sim k_3 - \frac{k_4 k_1}{k_2}, \tag{35}$$

as otherwise the similar trends would not have been obtained.

As argued before, the case of concrete appears quite singular in this respect, since incomplete self-similarity in Π_5 changes the slope of the $\log C$ vs. m relationship from negative to positive valued. In this case, the correlation by Carpinteri and Paggi (2007) would predict $\log C$ almost independent of m and is not able to capture the actual experimental trend.

6. Other complete and incomplete similarity laws

The findings about the strong relevance of both the elastic modulus and the yield stress, or, in dimensionless form, of the ratio E/σ_y, require additional considerations. In this section, we will show that other fatigue crack growth representations where the role of Young's modulus is put into evidence can be derived according to dimensional analysis. Moreover, other variants of the classical Paris' law are discussed, always in the context of complete and incomplete self-similarities.

6.1. Representation based on Young's modulus

The original choice of BB to look at σ_y as the "basic" quantity with dimension of stress was a good initial choice, but as it has appeared also in our findings, Young's modulus E is also extremely important as hence the choice of σ_y is an alternative independent variable for the derivation of the dimensionless formulation. Indeed, as already noticed by the Boeing team of

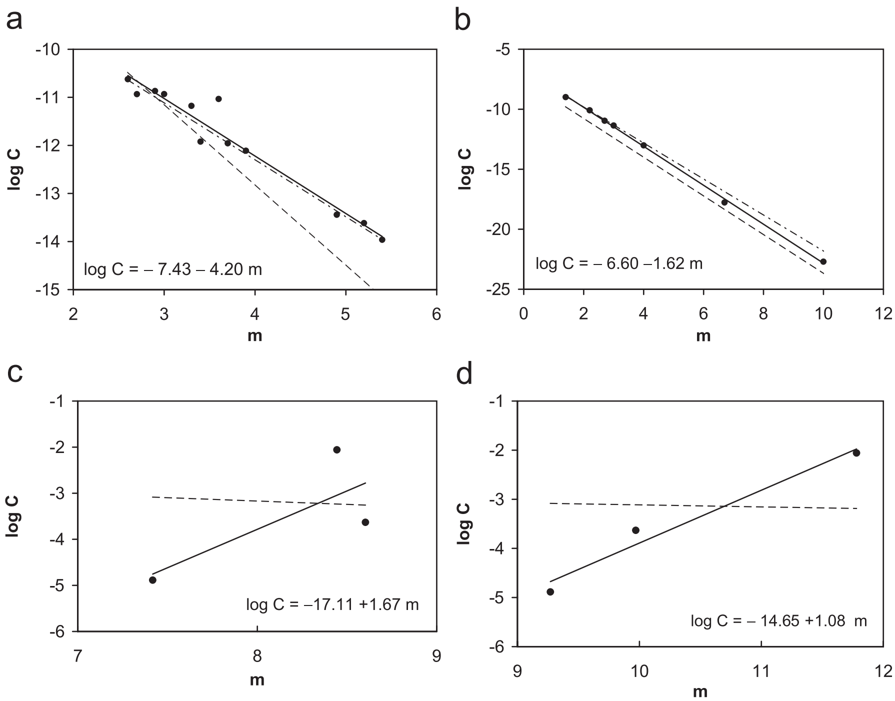

Fig. 9. Correlations between C and m (C evaluated using ΔK in MPa $\sqrt{\text{m}}$ and da/dN in m/cycle). Dashed lines refer to the correlation by Carpinteri and Paggi (2007), whereas dashed-dotted lines refer to the correlation by Tanaka (1979). (a) 4340 steel (Heiser and Mortimer, 1972). (b) ASTM steels (Clark and Wessel, 1970). (c) High strength concrete (data from Bažant and Shell, 1993 reinterpreted by Spagnoli, 2005). (d) Normal strength concrete (data from Bažant and Xu, 1991 reinterpreted by Spagnoli, 2005).

Anderson with whom Paris was working in the late 1950s, the crack growth curves seem to collapse to almost a single curve when scaled according to E. This is also recollected by Paris himself in Lados and Paris (2007).

In fact, the complete similarity fatigue crack growth representation derived by many authors (see e.g. Chan, 1993; Fleck et al., 1994):

$$\frac{da}{dN} = C\left(\frac{\Delta K}{E}\right)^2 \tag{36}$$

can therefore be recovered as a special case of the representation if we set complete self-similarity in Π_5, Π_6 and Π_7.

Notice this law is close to that proposed by Hertzberg (1995) which has exponent 3, and apparently a quite universal value when crack closure effects are removed (see Paris et al., 1999):

$$\frac{da}{dN} = b\left(\frac{\Delta K}{E\sqrt{b}}\right)^3, \tag{37}$$

where b is the burger vector which has the further advantage of being nearly constant for many materials.

Paris himself seems quite fascinated by this form of the law, also perhaps as a result of his first Ph.D. student, as he clearly states in Lados and Paris (2007). Indeed, Hertzberg, 30 years after the original observation of the correlation of data by Boeing's group, added in 1993 the essential ingredient which at that time was not available, i.e. the closure-free conditions, to derive an equation which seems universal for all metals, at least steels and Al-alloys. Certainly not for ceramic and concrete, which have m much higher, due to completely different process of crack growth. Paris et al. (1999) make some more comparison with a detailed crack closure model implemented (similar to the original Hertzberg comparisons of 1993), finding a satisfactory agreement between the data and the prediction line. Moreover, since the two main effects affecting crack closure (the load ratio R, and the length of the crack) are correctly taken into account, indicate that: *the 'Hertzberg (Paris/McClintock) Law' stands as an example of the power of simple continuum models to describe fatigue crack growth behavior.* In principle, it would be possible to include the crack closure length scale as a separate parameter, to derive another form of dimensional analysis, but this is not attempted here.

6.2. Representation based on the stress ratio or on the maximum stress-intensity factor

That Paris' law is of the "incomplete similarity" form, was implicitly recognized a long time ago, when the loading ratio R was noticed. Indeed, instead of including this effect in empirical equations for C (or in alternative forms of the Paris' law), one way to modify the classical one-parameter Paris' law is the use of two parameters for the description of the phenomenon of fatigue crack growth, which some authors suggest is more apt to include deviations for short cracks, crack closure, effect of underload and overloads, environmental effects, in a more effective way (see "unified approach" by the Naval Research Lab, Sadananda and Vasudevan, 2003, or McEvily et al., 2007). This is hardly surprising, since by separating the two effects of ΔK and of the stress ratio, $R = K_{\min}/K_{\max}$, or equivalently, the maximum stress-intensity factor, $K_{\max}$, it seems that we obtain a more general form

$$\frac{da}{dN} = f(\Delta K, R) = g(\Delta K, K_{\max}). \tag{38}$$

However, most authors limit the functional form to power laws, and hence the "two-parameter forms" are still "incomplete similarity" forms of Paris' law, as it can be easily shown. For instance, for different types of materials, Roberts and Erdogan (1967), Klesnil and Lukas (1972), Hojo et al. (1987), Liu and Chen (1991), Walker (1970) and Radhakrishnan (1979) proposed to include in the formulation of both R and ΔK:

$$\frac{da}{dN} = C(1-R)^p(\Delta K)^m. \tag{39}$$

Since the exponent p is negative valued (for instance, $p = -1.38$ in Radhakrishnan, 1979), Eq. (39) implies that the crack propagation rate in region II is an increasing function of R.

Alternatively, the use of $K_{\max}$ instead of R was adopted in Liu and Chen (1991) and Dauskardt et al. (1992) and was mechanistically explained in Ritchie (1999):

$$\frac{da}{dN} = C\Delta K^m K_{\max}^n. \tag{40}$$

In this case, recalling that $K_{\max} = \Delta K/(1-R)$, Eq. (40) can be rewritten as follows:

$$\frac{da}{dN} = C(1-R)^{-n}\Delta K^{m+n}, \tag{41}$$

which perfectly coincides with Eq. (39) if we set $n = -p$ and $m + n = m'$. Therefore, these parametric representations can be recovered as special cases of Eq. (14) obtained according to dimensional analysis arguments if we set $\beta_1 = m' = m + n$, $\beta_2 = \beta_3 = 0$ and $\beta_4 = p = -n$.

Finally, note that the parameter $\beta_1 = m$ may depend on the loading ratio R. For instance, the following correlation has been experimentally determined by Radhakrishnan (1979):

$$m = \frac{m_0}{(1-R)^{0.25}}, \tag{42}$$

where m_0 is the slope of $\log da/dN$ vs. $\log \Delta K$ for $R = 0$.

7. Discussion and conclusions

Before Paris, engineers had studied crack propagation in the 1950s for about 10 years, and had generally used $a-N$ plots showing the size of the crack vs. number of cycles. This clearly showed many different curves, depending on the initial crack length, on the stress amplitude, but they had come to the conclusion, mainly through plastic models at the crack tip, that crack propagation should depend on some combination of stress amplitude powers and crack length power. The powers, however, were fixed, depending on the details of the model being considered. Hence, engineers had attempted to fit experimental evidence based on some simplified models, which could not capture details of the extremely complex process at the crack tip. Paris' main early contribution was to suggest that da/dN plots would collapse all the data permitting a much cleaner comparison, and that da/dN should correlate with the stress-intensity factor range ΔK, already when no data were available in the late 1950s as he recalls in Lados and Paris (2007).

It is partly coincidence that the data originally available to Paris were so nicely in a "power-law" form. As further and further deviations were found, theories for crack propagation sometimes return to "non-Paris" forms, including in some cases combinations of crack length and stress range not obtained by the Irwin stress-intensity factor, or even data which would violate the 1961 original functional form $da/dN = f(\Delta K)$ since they are multivalued (short cracks), and indeed return to the form where $\Delta\sigma$ and a are separate variables. For example, some of the crack propagation laws for short cracks suggest a power-law dependence on the stress range, and a linear dependence on the crack size

$$\frac{da}{dN} = H\Delta\sigma^h a, \tag{43}$$

where H is a constant and $h = 8$ (Nisitani, 1981; Nisitani and Goto, 1987; Nisitani et al., 1992; Murakami and Miller, 2005). This law is remarkably close to a standard Basquin law for uncracked steels, when integrated between initial and final crack lengths, apart from a mild (logarithmic) dependence on the crack size which perhaps is not observed in general.

More generally, it has been shown that if the differential equation describing the crack growth process has separate variables, like

$$\frac{da}{dN} \propto \Delta\sigma^{\alpha} a^{\beta}, \tag{44}$$

then the Palmgren–Miner damage accumulation rule is true (Svensson and deMare, 1999; Todinov, 2001). Obviously, Paris' law is a special case of this condition, and hence it does not take account of the load sequence effect, which could be significant in some situations. So, the more we understand of crack propagation, the weaker the original Paris form appears to be, and the harder it is to return to a form general enough to be prone to enough fitting parameters to adjust the experimental data.

Hence, if engineers have partly learned how to use deviations of Paris' regime, or to "adjust" C and m, for example by crack closure and effect of constraint, power-laws seem a guide, and no more than this. We should not expect them to be "laws", in the stricter and more specific sense of physical laws.

Barenblatt and Botvina (1980) concluded their paper with the hope that similar dimensional analysis arguments should be applied to the other power laws in fatigue, namely Basquin and Coffin–Manson, also to provide a unified framework, including also Hall–Petch relationships. While some of this work has been already done (see Chan, 1993), it appears that much more study is needed along these lines. So, while crack propagation criteria have for some time been perceived as less "empirical" than previous laws of fatigue (like those by Basquin and Coffin–Manson), it may appear that the contrary may well be true, since Basquin and Coffin–Manson's exponents tend to be less dependent on size effects. It should be in principle easy to check if this is indeed the case, since size effects on either fatigue limit or static strength are relatively well known. Hence, it appears that the lesson of Barenblatt and Botvina should be reconsidered, and further extended.

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