# On the Stress Singularities at Multimaterial Interfaces and Related Analogies With Fluid Dynamics and Diffusion

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*Joining of different materials is a situation frequently observed in mechanical engineering and in materials science. Due to the difference in the elastic properties of the constituent materials, the junction points can be the origin of stress singularities and a possible source of damage. Hence, a full appreciation of these critical situations is of fundamental importance both from the mathematical and the engineering standpoints. In this paper, an overview of interface mechanical problems leading to stress singularities is proposed to show their relevance in engineering. The mathematical methods for the asymptotic analysis of stress singularities in multimaterial junctions and wedges composed of isotropic linear-elastic materials are reviewed and compared, with special attention to in-plane and out-of-plane loadings. This analysis mathematically demonstrates in a historical retrospective the equivalence of the eigenfunction expansion method, of the complex function representation, and of the Mellin transform technique for the determination of the order of the stress singularity in such problems. The analogies between linear elasticity and the Stokes flow of dissimilar immiscible fluids, the steady-state heat transfer across different materials, and the St. Venant torsion of composite bars are also discussed. Finally, advanced issues for the stress singularities due to joining of angularly nonhomogeneous elastic wedges are presented. This review article contains 147 references.* [DOI: 10.1115/1.2885134]

## 1 Introduction

Interfaces between two phases are defined as bounding surfaces where a discontinuity of some kind occurs. The property variation may be sharp or gradual. In general, the interface is an essentially bidimensional region through which material characteristics, such as concentration of an element, crystal structure, elastic coefficients, density, and coefficient of thermal expansion, change from one side to another. This mismatch in the material properties is the reason for the occurrence of stress singularities.

Interfaces must typically sustain mechanical and thermoelastic stresses without failure. Consequently, they exert an important, and sometimes controlling, influence on the performance of the material. The interest in this research field is clearly demonstrated by several workshops and special sessions in recent international conferences devoted to interface problems and interface modeling [1–3]. Moreover, this topic is characterized by interdisciplinary aspects since interface problems are one of the main concern in civil, mechanical, and electronic engineering, as well as in biomechanics and in materials science (see Ref. [4] for a wide overview).

In several boundary value problems of linear elasticity where different materials are present, the stress field is found to have singularities [5,6]. Physically, these singular points correspond to regions of high stress in which some type of nonlinear behavior, e.g., plastic flow or even fracture, is expected to relieve them. Hence, from the mathematical point of view, they violate the basic assumptions of linear elasticity, although they do comply with all the field equations. As a consequence, as stated by Dempsey and Sinclair [7], stress singularities are a real fact of the theory of linear elasticity and must be taken into account in any elastic analysis.

The wide literature in this field shows that different mathematical techniques have been proposed and used for the characterization of the singular stress field. A long debate on their equivalence has clearly produced a lack of standardization. Some problems have been tackled using different mathematical methods, obtaining almost the same conclusions. Therefore, the main aim of this article is to review, compare, and unify the main mathematical methods used for the analysis of stress singularities occurring in interface problems. To keep the scope of the article within reasonable limits, the study is restricted to two-dimensional elasticity and to isotropic materials with or without homogeneous elastic properties.

Compared to previous review articles on biharmonic problems and stress singularities [8–10], the present study focuses on the mathematical aspects and on the physical analogies of singularities related to multimaterial interfaces. The paper is organized as follows: In Sec. 2, an overview of some important interface mechanical problems leading to stress singularities is presented. In Sec. 3 the problem of multimaterial junctions subjected to in-plane loading is addressed. The mathematical methods are then compared, and a unified matrix formulation is proposed. The physical analogy existing between elasticity problems and the flow of viscous fluids is also discussed. In Sec. 4, the problem of multimaterial junctions subjected to out-of-plane loading is analyzed. Also in this case, the mathematical methods for the analysis of such problems are compared and unified. Physical analogies with the steady-state heat transfer in wedges composed of different materials and with the St. Venant torsion of composite bars are put into evidence. In Sec. 5, the hypothesis of homogeneity of the constituent materials is removed and the advanced issues concerning the singular stress field in nonhomogeneous materials are reviewed. Finally, concluding remarks and future perspectives complete the manuscript.

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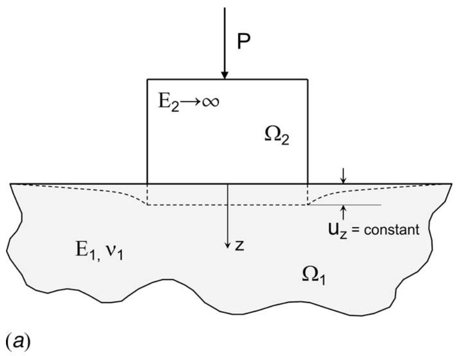


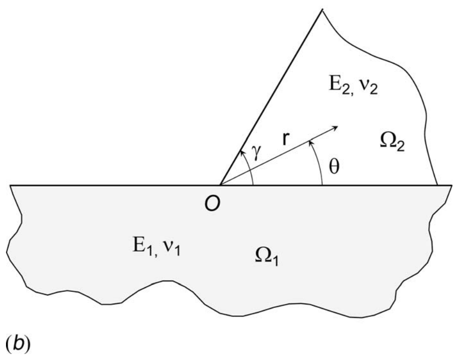


**Fig. 1 Indentation of an elastic half-plane (*a*) by a frictionless rigid flat punch and (*b*) by an elastic punch with corner angles other than $\pi/2$**

## 2 Overview of Interface Mechanical Problems Leading to Stress Singularities

An important class of mechanical problems leading to stress singularities are those related to two-dimensional contact mechanics in which displacements are imposed within the contact region of the boundary of an elastic half-space. In these cases, the unknown contact tractions can be computed by solving two coupled *singular integral equations* (see Ref. [11] for a fully comprehensive description of the mathematical formulation).

In this framework, firstly Muskhelishvili (Ref. [12], Sec. 115) and then Nadai [13] proposed the solution to the problem of the indentation of an elastic half-space by a frictionless rigid flat punch. They demonstrated that the stresses within the solid are singular near the edges of the punch with an order of stress singularity equal to $-1/2$; i.e., the stress components $\sigma_{ij}$ vary with the inverse square root of the radial distance from the singular point (see Fig. 1(*a*)),

$$\sigma_{ij} \sim Kr^{-0.5} \tag{1}$$

where $K$ is the stress-intensity factor.

On the basis of the fact that the order of the stress singularity for the indentation of an elastic half-space by a frictionless rigid flat punch is the same as that for a crack inside a homogeneous material, Giannakopoulos et al. [14] and Giannakopoulos and Suresh [15] proposed a quantitative equivalence between contact mechanics and fracture mechanics via asymptotic matching. This analogy, strictly valid for small-scale yielding and for either incompressible elastic substrates or rigid orthogonal punches with frictionless interfaces only, was then applied to life prediction in the case of fretting fatigue.

Muskhelishvili (Ref. [12], Sec. 114) also gave a solution to the problem of normal indentation of the half-plane by a flat-ended rough rigid punch acting over a finite contact region. This problem is a typical mixed boundary value problem in which both components of displacements are specified over the contact region and stress-free conditions are specified over the remaining free surface. The stress singularity at the edges of the punch has a more involved character than the previous one since a complex singularity was found. Denoting by $\lambda$ the complex eigenvalue defining the order of the stress singularity, the singular stress field is written as

$$\sigma_{ij} \sim \widetilde{K} r^{\mathrm{Re}\,\lambda-1} \tag{2}$$

where $\widetilde{K}$ is referred to as the *generalized stress-intensity factor* [16].

Starting from the principle appreciated by Boussinesq [17], which states that the pressure distribution between two elastic bodies, whose profiles are continuous through the boundary of the contact area, falls continuously to zero at the boundary, it is possible to generalize the above results concerning singularities in contact problems to other geometric scenarios. In fact, the main consequence of the above principle is that when one or both of the bodies has a discontinuous profile at the edge of the contact region, i.e., the contact is said to be *nonconforming*, then stress concentrations or even intensifications would be expected at these singular points.

Several researchers extended Muskhelishvili's results to contact problems involving punches that are also elastic as the half-plane substrate and with a corner angle other than $\pi/2$ (see Fig. 1(*b*)). Different boundary conditions along the bimaterial interface were also analyzed: Dundurs and Lee [18] treated the frictionless problems; Gdoutos and Theocaris [19] and Comninou [20] took into account friction in the formulation; Bogy [21] investigated the case of a fully bonded interface.

This last case where two different elastic wedges are joined together with a total wedge angle less than $2\pi$ is usually referred to as a bimaterial wedge. On the contrary, the terminology bimaterial junction implies that the total wedge angle formed by the two-material regions equals $2\pi$; i.e., the whole plane is occupied by the elastic materials without any voids. Wedges and junctions are very commonly observed in composite materials and received a great deal of attention in the past. The problem of bimaterial junctions was firstly analyzed by Bogy and Wang [22] in 1971, and multimaterial junctions were addressed by Theocaris [23] in 1974. Several configurations involving trimaterial junctions with perfectly bonded or debonded interfaces were also investigated by Pageau et al. [24] in 1994. Bimaterial wedges firstly treated by Bogy [21] were also addressed by Hein and Erdogan [25] in 1971, whereas a remarkable study on trimaterial wedges was due to Inoue and Koguchi [26] in 1996.

In the field of composite materials, interfaces play a crucial role during both the fabrication process and the service conditions [27,28]. The geometry of the contact between material components during fabrication is, in fact, an important issue, which is usually studied in terms of the wettability concept [27]. The term wettability has its origin in the fluid-mechanics literature, where it is used to describe the angle at which an isolated liquid island meets the solid substrate on which it rests (see Fig. 2). The liquid drop will spread and wet the surface completely only if this results in a net reduction of the system free energy. The same approach is routinely used to describe morphologies where intersections of solid-solid interfaces occur. When the solids behave linear elastically, the stress field diverges at the corner of the wedge formed near the triple line.

In 2001, Srolovits and Davis [29] investigated the effect of the singular stress field on the wetting angle $\theta$. In the absence of elastic stresses, interfacial thermodynamics defines the equilibrium angle $\theta$ in the configuration of Fig. 2 to satisfy the well-known Young's equation [30],

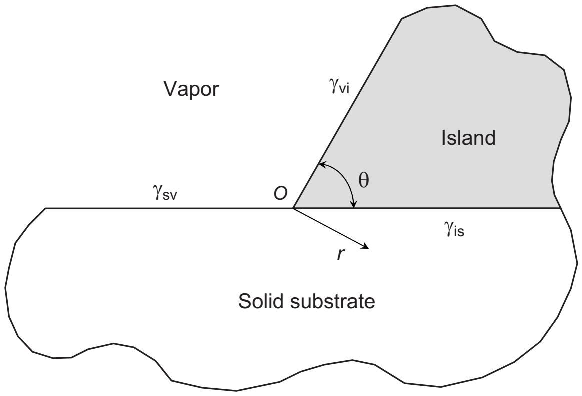


**Fig. 2 Intersection of a solid or liquid island with a solid substrate. $\theta$ is the wetting angle and $\gamma_{sv}$, $\gamma_{is}$, and $\gamma_{vi}$ are, respectively, the interfacial tensions associated with substrate-vapor, island-substrate, and vapor-island interfaces.**

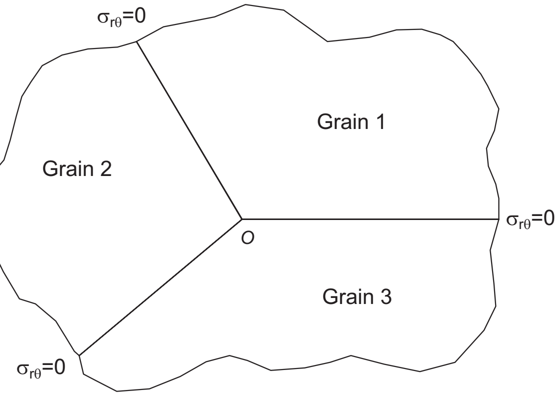


**Fig. 3 Scheme of the geometry of a grain triple junction. The grains are composed of the same material, and grain boundaries are assumed to be freely sliding.**

$$\cos\theta = \frac{\gamma_{si} - \gamma_{is}}{\gamma_{vi}} \tag{3}$$

where $\gamma_{vi}$, $\gamma_{sv}$ and $\gamma_{is}$ are, respectively, the vapor-island, substrate-vapor, and island-substrate interfacial energies. When elastic deformation is present, the singular stress field at the trimaterial junction has the form reported in Eq. (2).

In order to determine whether the elastic field may change the equilibrium wetting angle, Srolovits and Davis [29] analyzed the asymptotic behavior of the interface tension and strain energy singularity. Calculating these two contributions within a circle of radius $R$ centered at the singular point, they found that the behavior at the trimaterial junction is dominated by the interfacial energy for $\mathrm{Re}\,\lambda > 1/2$, and, in this case, the contribution of the elastic energy singularity is negligible. The two contributions are equally important when $\mathrm{Re}\,\lambda = 1/2$, whereas the elastic energy is dominant when $\mathrm{Re}\,\lambda < 1/2$, with a possible modification of the wetting angle predicted according to Young's equation.

Recently, experimental and theoretical studies have emphasized the role of interfaces and grain boundaries in understanding the mechanical properties of nanocrystalline materials [31,32]. In such materials, having grain sizes less than 30 nm, a considerable fraction of atoms reside in interfacial regions. As a consequence, nanocrystalline materials often exhibit quite different properties than their conventional polycrystalline counterparts.

While for some properties it may be sufficient to consider only the total volume fraction of the material associated with the interfaces, it was recognized that other phenomena require a detailed analysis of the various contributing factors such as grain boundaries, trimaterial junctions, and quadruple nodes. As a result, with decreasing grain size, an unexpected reduction of the material strength and of Young's modulus was observed. This phenomenon was attributed to grain boundary and trimaterial junction effects.

In 1996, Picu and Gupta [33] investigated stress singularities at trimaterial junctions in single-phase polycrystals due to freely sliding grain boundaries. This study was motivated by the need of modeling the nucleation of cracks and cavities in polycrystalline materials. They found that the exponent of the singular stress field depends on the grain boundary positions only, i.e., on the amplitudes of the wedge angles formed by the grains at the singular point (see Fig. 3). Furthermore, supersingularities, i.e., singularities with $\mathrm{Re}\,\lambda < 0.5$ in Eq. (2), were found, indicating critical geometrical configurations of the microstructure, which can give rise to crack nucleation.

In 2001, Costantini et al. [34] performed a theoretical study based on molecular-dynamics atomistic simulations of the structure and energetics of a multiple-twin triple junction in silicon (see Fig. 4). By analyzing the calculated excess line and volume energies, as well as the atomic-level stress, they showed that a triple junction can be considered as a true line defect. These critical points can be a source of residual stresses that cumulate and concentrate at the junction vertex.

In order to overcome the drawbacks due to an abrupt variation of the mechanical properties occurring at interfaces and material junctions, functionally graded materials (FGMs) have been recently designed. They are multicomponent materials with a continuous inhomogeneity of composition or structure. They consist in one material on one side and in a second material on the other. The composition of the intermediate layer changes smoothly from one material at one surface to the other material at the opposite surface. FGMs arose from a unique idea for realization of innovative properties that cannot be achieved by conventional materials. The FGM concept was proposed in 1984 as a way of preparing super-heat-resistant materials for spacecraft. Since the structural components of the spacecraft are exposed to high heat load, the material has to withstand severe thermomechanical loading. To solve this problem, a FGM was produced by using heat-resistant ceramics on the high-temperature side and tough metals with high thermal conductivity on the low-temperature side.

The gradient composition at the interface can effectively relax thermal stress concentrations, which may occur between the two materials [35,36]. A collection of technical papers that represents current research interests with regard to the fracture behavior of FGMs has been recently proposed in Ref. [37]. The possibility to

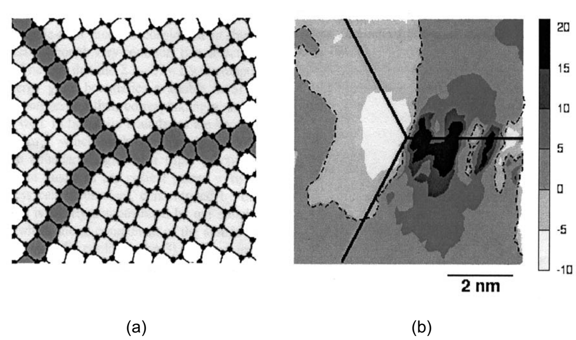


**Fig. 4 Atomic structure of the multiple-twin junction obtained after MD simulation (*a*). Shaded rings indicate the grain boundaries. Contour plot of the stress field in the vertical direction around the triple junction (*b*). The right scale stress are in units of kbar (reprinted from Ref. [34]).**

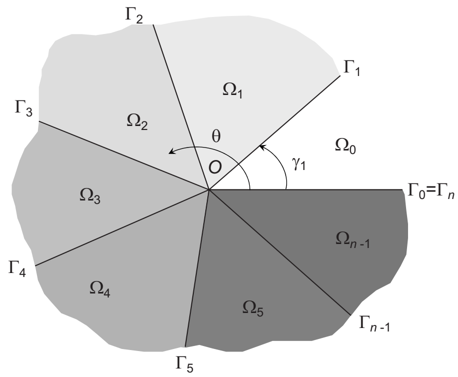


**Fig. 5 Scheme of a multimaterial junction**

use such nonhomogeneous materials in multimaterial junctions was recently explored by Carpinteri and Paggi [38], and the main results are summarized in Sec. 5.

The characterization of stress singularities in anisotropic materials has been pursued in the literature with reference to bi- and trimaterial wedges and junctions with either perfectly bonded or debonded interfaces [39–48]. The majority of these works, assuming plane strain or plane stress conditions, uses the powerful Lekhnitskii–Eshelby-Stroh [49–52] complex variable formalism of anisotropic elasticity. Recent results on this topic have contributed to a renewed interest in singularity problems involving isotropic materials since it has been demonstrated that the solutions to these problems may also be gained by using the mathematical formalism of anisotropic elasticity applied to degenerate anisotropic materials (see Refs. [47,53,54], among others). Hence, solutions for isotropic materials can be particularly useful when we are interested in checking the effectiveness of new approaches for anisotropic multimaterial junctions by comparison with limit isotropic solutions [47].

Finally, a number of studies on the order of stress singularities at three-dimensional bimaterial interface corners have to be mentioned. In these cases, the basic assumptions of plane elasticity do not hold anymore [55–61]. The motivation, especially in earlier works, concerned the study of the intersection of a crack with a free surface where a singularity different from the classical for a crack occurs. In most approaches, the stress singularities are obtained as eigenvalues of a quadratic eigenvalue problem [60]. They depend on the elastic mismatch and on the asymptotic corner geometry. While several efforts have been directed toward computing the order of the stress singularity for various corner geometries, only a few studies [57–59,62] have focused on the complete determination of the stress field at these singular points.

## 3 In-Plane Loading

**3.1 Statement of the Biharmonic Problem.** The geometry of a plane elastostatic problem consisting of $n$ dissimilar isotropic, homogeneous sectors of arbitrary angles perfectly bonded along their interfaces converging at the same vertex $O$ is shown in Fig. 5. Each of the material regions is denoted by $\Omega_i$, with $i=0,\ldots,n-1$, and it is comprised between the interfaces $\Gamma_i$ and $\Gamma_{i+1}$. The first and last interfaces, defined by $\theta=0$ and $\theta=2\pi$, coincide and are referred to as $\Gamma_0$.

In plane elasticity, by neglecting body forces, stress and displacement fields can be represented in terms of the Airy stress function $\Phi_i(r,\theta)$. In this case, in polar coordinates we have

$$\sigma_r = \frac{1}{r}\frac{\partial \Phi_i}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi_i}{\partial \theta^2} \tag{4a}$$

$$\sigma_\theta = \frac{\partial^2 \Phi_i}{\partial r^2} \tag{4b}$$

$$\tau_{r\theta} = -\frac{1}{r}\frac{\partial^2 \Phi_i}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial \Phi_i}{\partial \theta} \tag{4c}$$

$$\frac{\partial u_r}{\partial r} = \frac{1}{2G_i}\left[\frac{1}{r}\frac{\partial \Phi_i}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi_i}{\partial \theta^2} - \left(1 - \frac{1}{m_i}\right)\nabla^2 \Phi_i\right] \tag{4d}$$

$$\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} = \frac{1}{G_i}\left[-\frac{1}{r}\frac{\partial^2 \Phi_i}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial \Phi_i}{\partial \theta}\right] \tag{4e}$$

where the constant $m_i$ depends on the Poisson ratio,

$$m_i = \begin{cases} 1+\nu_i & \text{for plane stress} \\ 1/(1-\nu_i) & \text{for plane strain} \end{cases} \tag{5}$$

The strain compatibility equation expressed in terms of cylindrical coordinates is

$$\frac{1}{r^2}\frac{\partial^2 \epsilon_r}{\partial \theta^2} + \frac{\partial^2 \epsilon_\theta}{\partial r^2} - \frac{1}{r}\frac{\partial \epsilon_r}{\partial r} + \frac{2}{r}\frac{\partial \epsilon_\theta}{\partial r} = \frac{2}{r}\frac{\partial^2 \gamma_{r\theta}}{\partial r \partial \theta} + \frac{2}{r^2}\frac{\partial \gamma_{r\theta}}{\partial \theta} \tag{6}$$

When the stress-strain relations and Eqs. (4*a*)–(4*c*) are introduced into Eq. (6), the well-known biharmonic condition for the airy stress function is derived [63],

$$\nabla^4 \Phi_i = 0, \quad \forall\,(r,\theta) \in \Omega_i \tag{7}$$

**3.2 Boundary Conditions.** A unified treatment of elasticity problems concerning stress singularities was provided by Rao [64] in 1971. In his study, a general procedure for the identification of stress singularities and concentrations at the intersection of two or more interfaces in domains governed by harmonic and biharmonic types of equations was presented (see Fig. 6 for a sketch of some problems involving interfaces investigated in Ref. 64). He listed six types of boundary conditions for interfaces and five distinct homogeneous edge conditions along with their significance to problems of plane stress extension, plane strain extension, and thin plate flexure.

Afterward, Dempsey and Sinclair [7] specialized on Rao's results for the single-material wedge and the $n$ material wedge problems under either plane strain or plane stress conditions (see Fig. 7). By denoting, respectively, with $u_r$ and $u_\theta$, and with $\sigma_r$, $\sigma_\theta$, and $\tau_{r,\theta}$ the displacement and stress components in polar coordinates with respect to point $O$, four types of homogeneous stress and/or displacement boundary conditions can be considered on a wedge face [7]. They can also be interpreted in the context of contact problems and are summarized in Table 1. It has to be remarked that when the coefficient of friction $f$ is equal to zero, Case 4 in Table 1 corresponds to the boundary condition associated with the rigid lubricated punch.

Analogously, five types of *interface conditions* can be considered along an interface between two dissimilar wedges and are summarized in Table 2. The parameter $f_i$ denotes the coefficient of friction of the $i$th interface. From the engineering point of view, Cases B, C, D, and E differ from Case A in that one boundary condition is required in place of a fourth matching condition. Cases C, D, and E are, in general, less frequent than Cases A and B.

Boundary conditions in the case of a perfect bonding are represented by stress and displacement continuity conditions along the interfaces. For the $i$th interface, they are as follows:

$$\sigma_\theta^i(r,\gamma_{i+1}) = \sigma_\theta^{i+1}(r,\gamma_{i+1})$$

$$\tau_{r\theta}^i(r,\gamma_{i+1}) = \tau_{r\theta}^{i+1}(r,\gamma_{i+1})$$

$$u_r^i(r,\gamma_{i+1}) = u_r^{i+1}(r,\gamma_{i+1})$$

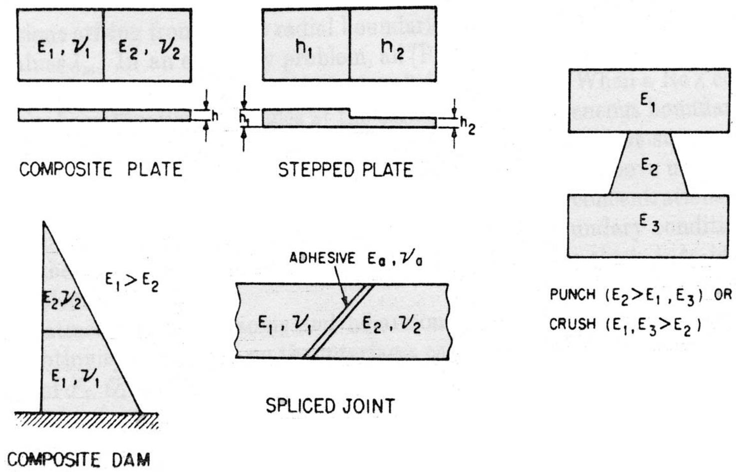


**Fig. 6 Some interface problems addressed by Rao (reprinted from Ref. [64])**

$$u_\theta^i(r,\gamma_{i+1}) = u_\theta^{i+1}(r,\gamma_{i+1}) \tag{8}$$

paying attention to the fact that for multimaterial junctions, the interface $\Gamma_0$ has to be defined by $\gamma_0=0$ for region 1 and by

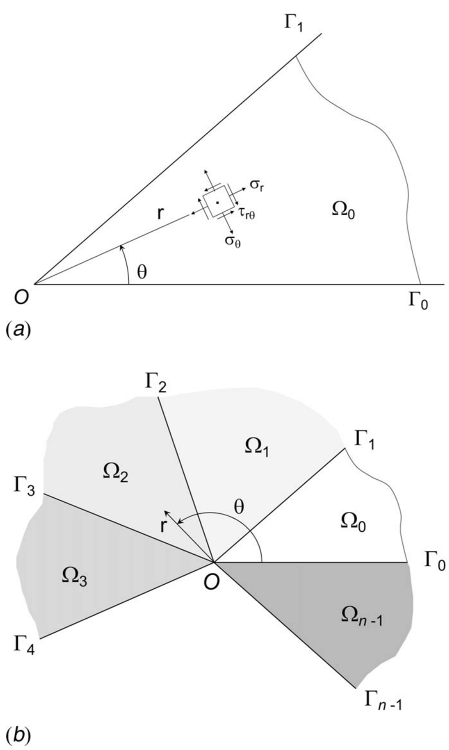

**Fig. 7 Schemes of a two-dimensional single-material wedge (*a*) and of a two-dimensional composite wedge (*b*)**

**Table 1 Boundary conditions in a wedge face from Ref. 7**

| Case | Prescribed quantities | Physical interpretation | Contact interpretation |
|---|---|---|---|
| 1 | $\sigma_\theta = \tau_{r\theta} = 0$ | Stress-free | Flexible lubricated punch |
| 2 | $\sigma_\theta = 0,\ u_r = 0$ | Antisymmetry | Flexible adhesive punch |
| 3 | $u_r = u_\theta = 0$ | Clamped | Rigid adhesive punch |
| 4 | $u_\theta = 0,\ \tau_{r\theta} - f_{\sigma_\theta = 0}$ | — | Rigid punch with friction |

$\gamma_{n+1} = 2\pi$ for region $n$.

Furthermore, as argued by Chen and Nisitani [65], when the problem is characterized by a geometric symmetry, it is possible to subdivide the elastic field into a symmetric part and a skew-symmetric part, namely, into a part due to the Mode I deformation and a part due to the Mode II deformation (see Fig. 8).

In the former case, the following symmetric conditions are applied at $\bar{\theta} = 0, \pi$:

$$\tau_{r\theta}^j(r,\bar{\theta}) = 0$$

**Table 2 Conditions on an interface from Ref. 7**

| Case | Matched quantities | Boundary condition | Contact interpretation |
|---|---|---|---|
| A | $\sigma_\theta,\ \tau_{r\theta},\ u_r,\ u_\theta$ | — | Perfectly bonded |
| B | $\sigma_\theta,\ \tau_{r\theta},\ u_\theta$ | $\tau_{r\theta} - \mu_i \sigma_\theta = 0$ | No separation, slip in the pres ence of friction |
| C | $\sigma_\theta,\ \tau_{r\theta},\ u_r$ | $\sigma_\theta = 0$ | Rough with separation, surfaces just resisting one another |
| D | $\sigma_\theta,\ u_r,\ u_\theta$ | $u_r = 0$ | No separation, no displacement in the radial direction |
| E | $\sigma_\theta,\ u_r,\ u_\theta$ | $u_\theta = 0$ | No separation, no displacement across the contact surface |

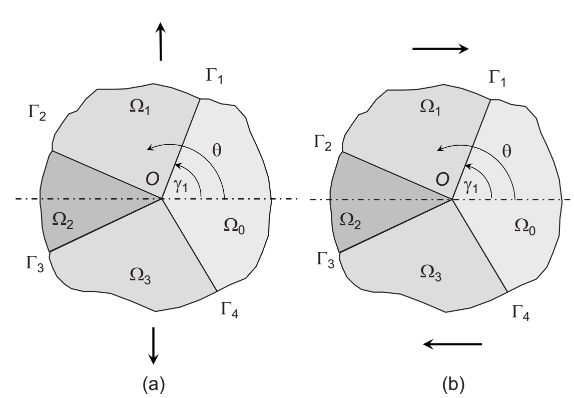

**Fig. 8 Scheme of a multimaterial junction with a geometric symmetry and subjected to symmetric (Mode I) (*a*) and to skew-symmetric (Mode II) (*b*) deformation**

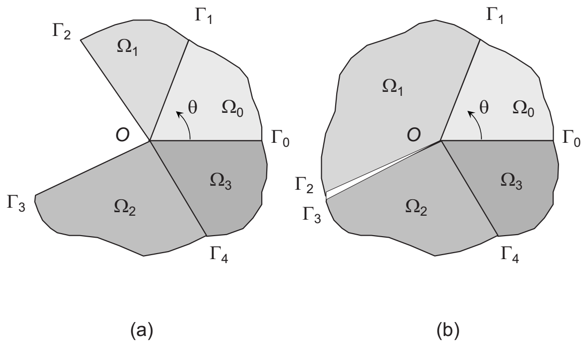


**Fig. 9 Scheme of a multimaterial junction with a reentrant corner (*a*) and with a crack (*b*). Case (*a*) is usually referred to as a multimaterial wedge.**

$$u_\theta^i(r,\bar{\theta}) = 0 \tag{9}$$

In the latter, skew-symmetric conditions are imposed,

$$\sigma_\theta^i(r,\bar{\theta}) = 0$$

$$u_r^i(r,\bar{\theta}) = 0 \tag{10}$$

In this way, it is possible to study half of the problem only, and the total eigenequation is reduced to the product of two factors: the former determines the eigenvalues corresponding to the Mode I deformation, whereas the latter determines the eigenvalues corresponding to the Mode II deformation.

In the case of either a reentrant corner or a crack due to an imperfect bonding (see Fig. 9), traction-free conditions along the faces have to be considered in addition to continuity conditions of stresses and displacements Eq. (8) at interfaces,

$$\sigma_\theta^i(r,\gamma_{i+1}) = 0$$

$$\tau_{r\theta}^i(r,\gamma_{i+1}) = 0$$

$$\sigma_\theta^{i+1}(r,\gamma_{i+1}) = 0$$

$$\tau_{r\theta}^{i+1}(r,\gamma_{i+1}) = 0 \tag{11}$$

Summarizing, in the case of multimaterial junctions with a perfect bonding without a geometric symmetry in the problem, boundary conditions are represented by Eq. (8) only. On the other hand, if a geometric symmetry exists, then Eqs. (8)–(10) can be considered. In the case of either a reentrant corner or a crack, boundary conditions are given by Eq. (8) for interfaces and by Eq. (11) for stress-free surfaces.

The solution to this boundary value problem can be gained by using one of the mathematical methods reviewed and compared in the next subsection. For the sake of simplicity and without any loss of generality, the comparison among the formulations will be carried out on the basis of the boundary conditions for a perfectly bonded multimaterial junction.

**3.3 Mathematical Methods.** Three different methods for the asymptotic analysis of stress singularities due to interface problems in composite bodies consisting of dissimilar isotropic, homogeneous, and elastic wedges were proposed in the literature, namely, the *eigenfunction expansion method*, the *complex function representation*, and the *Mellin transform technique*. All of these methods herein reviewed permit us to formulate an eigenproblem whose eigenvalues are directly related to the order of the stress singularity.

*3.3.1 Eigenfunction Expansion Method*. The eigenfunction expansion method was independently proposed by Wieghardt [66] in 1907 and by Williams [67] in 1952 to the analysis of stress singularities due to reentrant corners in plates in extension. They demonstrated that the solution can be generally written as a series of power functions in the radial coordinate originating from the singular point. The eigenvalues obtained from the solution procedure determine the exponents of the radial coordinate. The eigenfunctions describe the angular variation of stresses. Clearly, the eigenvalues and eigenfunctions for a composite wedge or junction depend on its geometry and material properties.

Numerous analytical solutions of the order of stress singularity and the stress fields for various material and geometric combinations have been derived in the literature. For example, Williams [67] addressed the problem of stress singularities resulting from various boundary conditions in angular corners of plates in extension, Zak and Williams [68] considered the problem of a crack perpendicular to a bimaterial interface, and Williams [69] and Rice and Sih [70] considered a crack along the interface between two isotropic materials. Afterward, Fenner [71] extended the solutions by Zak and Williams [68] by considering different inclinations of the crack meeting the bimaterial interface. This method was also used by Dempsey and Sinclair [72] in 1981 for the study of the singular behavior at the vertex of a bimaterial wedge, with the aim of unifying some previous findings obtained according to other mathematical techniques. In the sequel, the application of this technique to multimaterial junctions was mainly limited to the study of problems involving real eigenvalues, e.g., the antiplane stress field in trimaterial junctions [73].

Munz and Yang [74] applied this method to the computation of real eigenvalues resulting from singularities at the interface in bonded dissimilar materials under mechanical and thermal loadings. Only very recently, Carpinteri and Paggi [75,76] applied the eigenfunction expansion method to the study of Mode I and Mode II stress singularities in bi- and trimaterial junctions with either perfectly bonded interfaces or interface cracks. Size-scale effects on the material strength due to dominant stress singularities were also discussed in Ref. [77].

According to this method, it is possible to assume, for the $i$th material region, the following separable form for the biharmonic stress function $\Phi_i$:

$$\Phi_i(r,\theta) = \sum_j r^{\lambda_j+1} f_{i,j}(\theta,\lambda_j) \tag{12}$$

where $\lambda_j$ and $f_{i,j}$ are referred to as eigenvalues and eigenfunctions, respectively. Special cases where the separable form (12) requires a modification are discussed in Sec. 3.5. The summation with respect to the subscript $j$ is introduced in Eq. (12), since it is possible to have more than one eigenvalue for each problem, and the superposition principle can be applied due to linear elasticity.

By substituting Eq. (12) into Eqs. (4*a*)–(4*e*), stress and displacement fields can be expressed in terms of the eigenfunction,

$$\sigma_r^i = r^{\lambda-1}[f_i'' + (\lambda+1)f_i]$$

$$\sigma_\theta^i = r^{\lambda-1}[\lambda(\lambda+1)f_i]$$

$$\tau_{r\theta}^i = r^{\lambda-1}[-\lambda f_i']$$

$$u_r^i = \frac{r^\lambda}{2G_i}\left\{-(\lambda+1)f_i + \frac{1}{\lambda m_i}[f_i'' + (\lambda+1)^2 f_i]\right\}$$

$$u_\theta^i = \frac{r^\lambda}{2G_i}\left\{-f_i' - \frac{1}{\lambda(\lambda-1)m_i}[f_i''' + (\lambda+1)^2 f_i']\right\} \tag{13}$$

Furthermore, the biharmonic condition requires $f_{i,j}$ to be of the form

$$f_{i,j}(\theta,\lambda_j) = A_{i,j}\sin[(\lambda_j+1)\theta] + B_{i,j}\cos[(\lambda_j+1)]\theta + C_{i,j}\sin[(\lambda_j - 1)\theta] + D_{i,j}\cos[(\lambda_j-1)\theta] \quad (14)$$

where $A_{i,j}$, $B_{i,j}$, $C_{i,j}$, and $D_{i,j}$ are undetermined constants. To simplify the mathematical notation, the subscript $j$ of the eigenvalue in Eq. (14) will be removed in the sequel.

Introducing Eq. (14) into Eq. (13), stresses and displacements in polar coordinates can be explicitly written in terms of the unknowns $A_{i,j}$, $B_{i,j}$, $C_{i,j}$, $D_{i,j}$, and $\lambda$,

$$\sigma_r^i = \lambda r^{\lambda-1}\{-A_{i,j}(\lambda+1)\sin[(\lambda+1)\theta] - B_{i,j}(\lambda+1)\cos[(\lambda+1)\theta] + C_{i,j}(3-\lambda)\sin[(\lambda-1)\theta] + D_{i,j}(3-\lambda)\cos[(\lambda-1)\theta]\}$$

$$\sigma_\theta^i = \lambda(\lambda+1)r^{\lambda-1}\{A_{i,j}\sin[(\lambda+1)\theta] + B_{i,j}\cos[(\lambda+1)\theta] + C_{i,j}\sin[(\lambda-1)\theta] + D_{i,j}\cos[(\lambda-1)\theta]\}$$

$$\tau_{r\theta}^i = \lambda r^{\lambda-1}\{-A_{i,j}(\lambda+1)\cos[(\lambda+1)\theta] + B_{i,j}(\lambda+1)\sin[(\lambda+1)\theta] - C_{i,j}(\lambda-1)\cos[(\lambda-1)\theta] + D_{i,j}(\lambda-1)\sin[(\lambda-1)\theta]\}$$

$$u_r^i = \frac{r^\lambda}{2G_i}\{-A_{i,j}(\lambda+1)\sin[(\lambda+1)\theta] - B_{i,j}(\lambda+1)\cos[(\lambda+1)\theta] + C_{i,j}(4/m_i - \lambda - 1)\sin[(\lambda-1)\theta] + D_{i,j}(4/m_i - \lambda - 1)\cos[(\lambda - 1)\theta]\}$$

$$u_\theta^i = \frac{r^\lambda}{2G_i}\{-A_{i,j}(\lambda+1)\cos[(\lambda+1)\theta] + B_{i,j}(\lambda+1)\sin[(\lambda+1)\theta] - C_{i,j}(4/m_i + \lambda - 1)\cos[(\lambda-1)\theta] + D_{i,j}(4/m_i + \lambda - 1)\sin[(\lambda - 1)\theta]\} \quad (15)$$

Introducing Eq. (15) into the interface matching conditions represented by Eq. (8), a set of $4n$ equations in $4n+1$ unknowns $A_{i,j}$, $B_{i,j}$, $C_{i,j}$, $D_{i,j}$, and $\lambda_j$ can be symbolically written as

$$\mathbf{\Lambda v} = \mathbf{0} \quad (16)$$

where $\mathbf{\Lambda}$ denotes the coefficient matrix that depends on the eigenvalue and $\mathbf{v}$ represents the vector that collects the unknowns $A_{i,j}$, $B_{i,j}$, $C_{i,j}$, and $D_{i,j}$.

The coefficient matrix in Eq. (16) is characterized by a sparse structure,

$$\mathbf{\Lambda} = \begin{bmatrix} \mathbf{M}_{\gamma_1}^0 & -\mathbf{M}_{\gamma_1}^1 & & & & & & \\ & \mathbf{M}_{\gamma_2}^1 & -\mathbf{M}_{\gamma_2}^2 & & & & & \\ & & \cdots & \cdots & & & & \\ & & & \mathbf{M}_{\gamma_i}^{i-1} & -\mathbf{M}_{\gamma_i}^{i} & & & \\ & & & & \cdots & \cdots & & \\ & & & & & \mathbf{M}_{\gamma_{n-1}}^{n-2} & -\mathbf{M}_{\gamma_{n-1}}^{n-1} \\ -\mathbf{M}_{\gamma_0=0}^0 & & & & & & \mathbf{M}_{\gamma_n=2\pi}^{n-1} \end{bmatrix} \quad (17)$$

By letting $\lambda^+=(\lambda+1)$ and $\lambda^-=(\lambda-1)$, the elementary matrix $\mathbf{M}_\theta^i$ can be written as follows:

$$\mathbf{M}_\theta^i = \begin{bmatrix} \lambda\lambda^+\sin(\lambda^+\theta) & \lambda\lambda^+\cos(\lambda^+\theta) & \lambda\lambda^+\sin(\lambda^-\theta) & \lambda\lambda^+\cos(\lambda^-\theta) \\ -\lambda\lambda^+\cos(\lambda^+\theta) & \lambda\lambda^+\sin(\lambda^+\theta) & -\lambda\lambda^-\cos(\lambda^-\theta) & \lambda\lambda^-\sin(\lambda^-\theta) \\ -\frac{\lambda^+}{2G_i}\sin(\lambda^+\theta) & -\frac{\lambda^+}{2G_i}\cos(\lambda^+\theta) & \left(\frac{4}{m_i}-\lambda^+\right)\frac{\sin(\lambda^-\theta)}{2G_i} & \left(\frac{4}{m_i}-\lambda^+\right)\frac{\cos(\lambda^-\theta)}{2G_i} \\ -\frac{\lambda^+}{2G_i}\cos(\lambda^+\theta) & \frac{\lambda^+}{2G_i}\sin(\lambda^+\theta) & -\left(\frac{4}{m_i}+\lambda^-\right)\frac{\cos(\lambda^-\theta)}{2G_i} & \left(\frac{4}{m_i}+\lambda^-\right)\frac{\sin(\lambda^-\theta)}{2G_i} \end{bmatrix} \quad (18)$$

and the components of the vector $\mathbf{v}$ are

$$\mathbf{v} = \{\mathbf{v}^0, \mathbf{v}^1, \mathbf{v}^2, \ldots, \mathbf{v}^i, \ldots, \mathbf{v}^{n-2}, \mathbf{v}^{n-1}\}^T \quad (19)$$

where

$$\mathbf{v}^i = \{A_{i,j}, B_{i,j}, C_{i,j}, D_{i,j}\}^T \quad (20)$$

A nontrivial solution of the equation system (16) exists if and only if the determinant of the coefficient matrix vanishes. This condition yields an eigenequation, which has to be solved for eigenvalues that are, in general, complex. For the present purposes, we are concerned only with those values of $\lambda_j$, which may lead to singularities in the stress field. This fact, together with the condition of continuity of the displacement field at the vertex where regions meet, implies that we are looking for eigenvalues in the range $0 < \mathrm{Re}\,\lambda_j < 1$. Then, to find solutions to the eigenequations, numerical techniques are usually employed [7,44,47,71,78–82].

*3.3.2 Complex Function Representation*. The complex function representation was mainly developed by Muskhelishvili [12] and applied to several problems of plane elasticity. Williams [83], following Huth [84], presented in 1956 a method based on the complex potentials for the study of a single-material wedge, duplicating some of the findings previously obtained according to the eigenfunction expansion method [67].

Then, with no claims of completeness, some noticeable studies about stress singularities carried out with this method have concerned reentrant corners of plates in extension [85,86], perfectly bonded bimaterial wedges, [78,87–89], and junctions [65]. The problem of a crack between dissimilar materials [90,91] was also addressed with this method. Moreover, Wang and Chen [92] applied the complex function representation to the problem of a crack perpendicular to a bimaterial interface, reobtaining some of the results previously derived by Zak and Williams [68] using the eigenfunction expansion method.

The mathematical formulation for multimaterial junctions was set up by Theocaris [23] in 1974 and was subsequently adopted by Gdoutos and Theocaris [19] for the analysis of stress concentrations at the apex of a plane indenter acting on an elastic half-plane. This formulation was then revised by Pageau et al. [24] and applied to trimaterial junctions with either perfectly bonded or debonded interfaces. This mathematical technique was also recently adopted by Carpinteri et al. [93] for the analysis of reentrant corners symmetrically meeting a bimaterial interface.

According to Muskhelishvili [12] and England [5,94], the stress and displacement fields in the $i$th material can be expressed in terms of the complex potentials $\Omega_i$ and $\omega_i$

$$\sigma_r^i + i\tau_{r\theta}^j = \Omega_i'(z) + \overline{\Omega_i'(z)} - z\overline{\Omega_i''(z)} - \bar{z}z^{-1}\overline{\omega_i'(z)}$$

$$\sigma_\theta^i - i\tau_{r\theta}^j = \Omega_i'(z) + \overline{\Omega_i'(z)} + z\overline{\Omega_i''(z)} + \bar{z}z^{-1}\overline{\varpi_i'(z)}$$

$$u_r^i + iu_\theta^i = \frac{1}{2G_i}e^{-i\theta}[\kappa_i\Omega_i(z) - z\overline{\Omega_i'(z)} - \overline{\omega_i(z)}] \tag{21}$$

where $z = re^{i\theta}$ is a complex variable, $G_i$ is the shear modulus of the $i$th material, and $\kappa_i$ is the Kolosoff constant related to the Poisson ratio $\nu_i$ by

$$\kappa_i = \begin{cases} 3 - 4\nu_i & \text{for plane strain} \\ 3 - \dfrac{4\nu_i}{1+\nu_i} & \text{for plane stress} \end{cases} \tag{22}$$

Symbols ( ′) and (−) in Eq. (21) denote, respectively, a derivative with respect to $z$ and the complex conjugate of the variable. Following Refs. [85,89,95], the complex potentials are assumed to have the following form as $z \to 0$:

$$\Omega_i = (E_{i,j} + iF_{i,j})z^{\lambda_j}$$

$$\omega_i = (G_{i,j} + iH_{i,j})z^{\lambda_j} \tag{23}$$

where $E_{i,j}$, $F_{i,j}$, $G_{i,j}$, and $H_{i,j}$ are undetermined constants. By substituting Eq. (23) into Eq. (21), stress and displacement fields in the materials can be expressed in terms of the unknowns $E_{i,j}$, $F_{i,j}$, $G_{i,j}$, $H_{i,j}$, and λ,

$$\sigma_r^i = \lambda r^{\lambda-1}\{E_{i,j}(3-\lambda)\cos[(\lambda-1)\theta] - F_{i,j}(3-\lambda)\sin[(\lambda-1)\theta] - G_{i,j}\cos[(\lambda+1)\theta] + H_{i,j}\sin[(\lambda+1)\theta]\}$$

$$\sigma_\theta^i = \lambda r^{\lambda-1}\{E_{i,j}(\lambda+1)\cos[(\lambda-1)\theta] - F_{i,j}(\lambda+1)\sin[(\lambda-1)\theta] + G_{i,j}\cos[(\lambda+1)\theta] - H_{i,j}\sin[(\lambda+1)\theta]\}$$

$$\tau_{r\theta}^j = \lambda r^{\lambda-1}\{E_{i,j}(\lambda-1)\sin[(\lambda-1)\theta] + F_{i,j}(\lambda-1)\cos[(\lambda-1)\theta] + G_{i,j}\sin[(\lambda+1)\theta] + H_{i,j}\cos[(\lambda+1)\theta]\}$$

$$u_r^i = \frac{1}{2G_i}r^\lambda\{E_{i,j}(\kappa_i-\lambda)\cos[(\lambda-1)\theta] - F_{i,j}(\kappa_i-\lambda)\sin[(\lambda-1)\theta] - G_{i,j}\cos[(\lambda+1)\theta] + H_{i,j}\sin[(\lambda+1)\theta]\}$$

$$u_\theta^i = \frac{1}{2G_i}r^\lambda\{E_{i,j}(\kappa_i+\lambda)\sin[(\lambda-1)\theta] + F_{i,j}(\kappa_i+\lambda)\cos[(\lambda-1)\theta] + G_{i,j}\sin[(\lambda+1)\theta] + H_{i,j}\cos[(\lambda+1)\theta]\} \tag{24}$$

Analogous to the eigenfunction expansion method, by considering Eq. (24) along with the interface continuity conditions (Eq. (8)), the matrix form of the equation system (Eq. (16)) can be obtained. By noting that the parameter $\kappa_i$ is related to $m_i$ by the following relation:

$$\kappa_i = \frac{4}{m_i} - 1 \tag{25}$$

the elementary matrix $\mathbf{M}_\theta^i$ can be written as

$$\mathbf{M}_\theta^i = \begin{bmatrix} -\sin(\lambda^+\theta) & \cos(\lambda^+\theta) & -\lambda^+\sin(\lambda^-\theta) & \lambda^+\cos(\lambda^-\theta) \\ \cos(\lambda^+\theta) & \sin(\lambda^+\theta) & \lambda^-\cos(\lambda^-\theta) & \lambda^-\sin(\lambda^-\theta) \\ \dfrac{\sin(\lambda^+\theta)}{2G_i} & -\dfrac{\cos(\lambda^+\theta)}{2G_i} & -\left(\dfrac{4}{m_i} - \lambda^+\right)\dfrac{\sin(\lambda^-\theta)}{2G_i} & \left(\dfrac{4}{m_i} - \lambda^+\right)\dfrac{\cos(\lambda^-\theta)}{2G_i} \\ \dfrac{\cos(\lambda^+\theta)}{2G_i} & \dfrac{\sin(\lambda^+\theta)}{2G_i} & \left(\dfrac{4}{m_i} + \lambda^-\right)\dfrac{\cos(\lambda^-\theta)}{2G_i} & \left(\dfrac{4}{m_i} + \lambda^-\right)\dfrac{\sin(\lambda^-\theta)}{2G_i} \end{bmatrix} \tag{26}$$

where $\lambda^+$ and $\lambda^-$ denote, respectively, $(\lambda+1)$ and $(\lambda-1)$.

According to this formulation, the components of the vector **v** are

$$\mathbf{v}^i = \{H_{i,j}, G_{i,j}, F_{i,j}, E_{i,j}\}^T \tag{27}$$

It is possible to show that the elementary matrix (Eq. (26)) obtained according to the Muskhelishvili complex function representation coincides with the matrix derived according to the eigenfunction expansion method in Eq. (18), once the following change of variables is considered:

$$H_{i,j} = -\frac{A_{i,j}}{\lambda(\lambda+1)}$$

$$G_{i,j} = \frac{B_{i,j}}{\lambda(\lambda+1)}$$

$$F_{i,j} = -C_{i,j}$$

$$E_{i,j} = D_{i,j} \tag{28}$$

The expressions of the eigenequations obtained by the computation of det $\mathbf{\Lambda}$ are influenced by this change of variables, whereas the zeros of the eigenequations, which allow us to compute the eigenvalues, do not change. This result demonstrates that both mathematical formulations can be used for the computation of the order of the stress singularity in multimaterial junctions and wedges.

In 1974, Theocaris [23] observed that the form of the complex potentials in Eq. (23) could not be used for determining real stresses and displacements when λ is complex. Noting that if $\lambda \in \mathbb{C}$ is one root of the eigenequation, then also its conjugate number, $\bar{\lambda}$, is a root of the same equation, Theocaris assumed the following alternative expression for the complex potentials [23]:

$$\Omega_i = I_{i,j}z^{\lambda_j} + L_{i,j}z^{\bar{\lambda}_j}$$

$$\omega_i = M_{i,j}z^{\lambda_j} + N_{i,j}z^{\bar{\lambda}_j} \tag{29}$$

where $I_{i,j}$, $L_{i,j}$, $M_{i,j}$, and $N_{i,j}$ are unknown complex coefficients.

After some algebra, by introducing Eq. (29) into Eq. (21), we obtain

$$\sigma_r^i + i\tau_{r\theta}^i = r^{\lambda-1}[I_{i,j}\lambda e^{i\theta(\lambda-1)} - \bar{L}_{i,j}\lambda^2 e^{-i\theta(\lambda-1)} - \bar{N}_{i,j}\lambda e^{-i\theta(\lambda+1)} + 2\bar{L}_{i,j}\lambda e^{-i\theta(\lambda-1)}] + r^{\bar{\lambda}-1}[L_{i,j}\bar{\lambda}e^{i\theta(\bar{\lambda}-1)} - \bar{I}_{i,j}\bar{\lambda}^2 e^{-i\theta(\bar{\lambda}-1)} - \bar{M}_{i,j}\bar{\lambda}e^{-i\theta(\bar{\lambda}+1)} + 2\bar{L}_{i,j}\bar{\lambda}e^{-i\theta(\bar{\lambda}-1)}] \quad (30a)$$

$$\sigma_\theta^i - i\tau_{r\theta}^i = r^{\lambda-1}[I_{i,j}\lambda e^{i\theta(\lambda-1)} + \bar{L}_{i,j}\lambda^2 e^{-i\theta(\lambda-1)} + \bar{N}_{i,j}\lambda e^{-i\theta(\lambda+1)}] + r^{\bar{\lambda}-1}[L_{i,j}\bar{\lambda}e^{i\theta(\bar{\lambda}-1)} + \bar{I}_{i,j}\bar{\lambda}^2 e^{-i\theta(\bar{\lambda}-1)} + \bar{M}_{i,j}\bar{\lambda}e^{-i\theta(\bar{\lambda}+1)}] \quad (30b)$$

$$u_r^i + iu_\theta^i = \frac{r^\lambda}{2G_i}[\kappa_i I_{i,j}e^{i\theta(\lambda-1)} - \bar{L}_{i,j}\lambda e^{-i\theta(\lambda-1)} - \bar{N}_{i,j}e^{-i\theta(\lambda+1)}] + \frac{r^{\bar{\lambda}}}{2G_i}[\kappa_i L_{i,j}e^{i\theta(\bar{\lambda}-1)} - \bar{I}_{i,j}\bar{\lambda}e^{-i\theta(\bar{\lambda}-1)} - \bar{M}_{i,j}e^{-i\theta(\bar{\lambda}+1)}] \quad (30c)$$

The expression for $\sigma_\theta^i$ can be simply obtained by taking one-half of Eq. (30*b*) plus the conjugate of the same equation. Analogously, $\tau_{r\theta}^i$ can be computed as the difference of the conjugate of Eq. (30*b*) and Eq. (30*b*) itself divided by $2i$. The same procedure can also be applied to Eq. (30*c*) to obtain the expressions of $u_r^i$ and $u_\theta^i$

$$\sigma_\theta^i = \frac{1}{2}\{r^{\lambda-1}[M_{i,j}e^{i\theta(\lambda+1)} + \bar{N}_{i,j}\lambda e^{-i\theta(\lambda+1)} + I_{i,j}\lambda(\lambda+1)e^{i\theta(\lambda-1)} + \bar{L}_{i,j}\lambda(\lambda+1)e^{-i\theta(\lambda-1)}] + r^{\bar{\lambda}-1}[\bar{M}_{i,j}\bar{\lambda}e^{-i\theta(\bar{\lambda}+1)} + N_{i,j}\bar{\lambda}e^{i\theta(\bar{\lambda}+1)} + \bar{I}_{i,j}\bar{\lambda}(\bar{\lambda}+1)e^{-i\theta(\bar{\lambda}-1)} + L_{i,j}\bar{\lambda}(\bar{\lambda}+1)e^{i\theta(\bar{\lambda}-1)}]\} \quad (31a)$$

$$\tau_{r\theta}^i = \frac{1}{2i}\{r^{\lambda-1}[M_{i,j}\lambda e^{i\theta(\lambda+1)} + \bar{N}_{i,j}\lambda e^{-i\theta(\lambda+1)} + I_{i,j}\lambda(\lambda-1)e^{i\theta(\lambda-1)} - \bar{L}_{i,j}\lambda(\lambda-1)e^{-i\theta(\lambda-1)}] + r^{\bar{\lambda}-1}[-\bar{M}_{i,j}\bar{\lambda}e^{-i\theta(\bar{\lambda}+1)} - N_{i,j}\bar{\lambda}e^{i\theta(\bar{\lambda}+1)} - \bar{I}_{i,j}\bar{\lambda}(\bar{\lambda}-1)e^{-i\theta(\bar{\lambda}-1)} + L_{i,j}\bar{\lambda}(\bar{\lambda}-1)e^{-i\theta(\bar{\lambda}-1)}]\} \quad (31b)$$

$$u_r^i = \frac{1}{4G_i}\{r^\lambda[-M_{i,j}e^{i\theta(\lambda+1)} - \bar{N}_{i,j}\lambda e^{-i\theta(\lambda+1)} + I_{i,j}(\kappa_i-\lambda)e^{i\theta(\lambda-1)} + \bar{L}_{i,j}(\kappa_i-\lambda)e^{-i\theta(\lambda-1)}] + r^{\bar{\lambda}}[-\bar{M}_{i,j}e^{-i\theta(\bar{\lambda}+1)} - N_{i,j}\bar{\lambda}e^{i\theta(\bar{\lambda}+1)} + \bar{I}_{i,j}(\kappa_i-\bar{\lambda})e^{-i\theta(\bar{\lambda}-1)} + L_{i,j}(\kappa_i-\bar{\lambda})e^{i\theta(\bar{\lambda}-1)}]\} \quad (31c)$$

$$u_\theta^i = \frac{1}{4G_i i}\{r^\lambda[M_{i,j}e^{i\theta(\lambda+1)} - \bar{N}_{i,j}e^{-i\theta(\lambda+1)} + I_{i,j}(\kappa_i+\lambda)e^{i\theta(\lambda-1)} - \bar{L}_{i,j}(\kappa_i+\lambda)e^{-i\theta(\lambda-1)}] + r^{\bar{\lambda}}[-\bar{M}_{i,j}e^{-i\theta(\bar{\lambda}+1)} + N_{i,j}e^{i\theta(\bar{\lambda}+1)} - \bar{I}_{i,j}(\kappa_i+\bar{\lambda})e^{-i\theta(\bar{\lambda}-1)} + L_{i,j}(\kappa_i+\bar{\lambda})e^{i\theta(\bar{\lambda}-1)}]\} \quad (31d)$$

Now, by substituting Eqs. (31*a*)–(31*d*) into the interface matching conditions represented by Eq. (8), we observe that such relations must hold for every value of the variable $r$. Hence, the coefficients multiplying $r^{\lambda-1}$ and $r^{\bar{\lambda}-1}$ in Eqs. (31*a*) and (31*b*) and those multiplying $r^\lambda$ and $r^{\bar{\lambda}}$ in Eqs. (31*c*) and (31*d*) should be separately equated.

However, by recognizing that the expressions multiplying $r^{\bar{\lambda}}$ and $r^{\bar{\lambda}-1}$ are simply the complex conjugates of those multiplying $r^\lambda$ and $r^{\lambda-1}$, only the coefficients multiplying $r^\lambda$ and $r^{\lambda-1}$ need be considered and have to be retained in the equation system of the type of Eq. (16). After some algebra, the elementary matrix $\mathbf{M}_\theta^i$ is then found to be

$$\mathbf{M}_\theta^i = \begin{bmatrix} \frac{\lambda e^{i\theta\lambda^+}}{2} & \frac{\lambda e^{-i\theta\lambda^+}}{2} & \frac{\lambda\lambda^+ e^{i\theta\lambda^-}}{2} & \frac{\lambda\lambda^+ e^{-i\theta\lambda^-}}{2} \\ \frac{\lambda e^{i\theta\lambda^+}}{2i} & \frac{\lambda e^{-i\theta\lambda^+}}{2i} & \frac{\lambda\lambda^- e^{i\theta\lambda^-}}{2i} & -\frac{\lambda\lambda^- e^{-i\theta\lambda^-}}{2i} \\ -\frac{e^{i\theta\lambda^+}}{4G_i} & -\frac{e^{-i\theta\lambda^+}}{4G_i} & \frac{\left(\frac{4}{m_i}-\lambda^+\right)e^{i\theta\lambda^-}}{4G_i} & \frac{\left(\frac{4}{m_i}-\lambda^+\right)e^{-i\theta\lambda^-}}{4G_i} \\ \frac{e^{i\theta\lambda^+}}{4iG_i} & -\frac{e^{-i\theta\lambda^+}}{4iG_i} & \frac{\left(\frac{4}{m_i}+\lambda^-\right)e^{i\theta\lambda^-}}{4iG_i} & -\frac{\left(\frac{4}{m_i}+\lambda^-\right)e^{-i\theta\lambda^-}}{4iG_i} \end{bmatrix} \quad (32)$$

with the following unknown vector:

$$\mathbf{v}^i = \{M_{i,j}, \bar{N}_{i,j}, I_{i,j}, \bar{L}_{i,j}\}^T \quad (33)$$

To demonstrate that the equation system characterized by the complex elementary matrix (Eq. (32)) has the same determinant as that obtained according to the eigenfunction expansion method, we will show that $\mathbf{M}_\theta^i$ in Eq. (32) can be transformed into a real form, which matches exactly the elementary matrix in Eq. (18).

First of all, we have to remember that the governing equation of the problem is represented by the biharmonic condition (Eq. (7)) for the Airy stress function. In the case of the eigenfunction expansion method, this implies that the general expression of the eigenfunction must satisfy the following fourth order ordinary differential equation (ODE):

$$f^{\mathrm{IV}} + 2(\lambda^2+1)f^{\mathrm{II}} + (\lambda^2-1)^2 = 0 \quad (34)$$

It is easy to demonstrate that the general solution to Eq. (34) is

$$f(\theta) = A_{i,j}e^{i\theta(\lambda+1)} + B_{i,j}e^{-i\theta(\lambda+1)} + C_{i,j}e^{i\theta(\lambda-1)} + D_{i,j}e^{-i\theta(\lambda-1)} \quad (35)$$

and it is also well known that Eq. (35) can be rewritten in an equivalent real form, which coincides with Eq. (14). Then, by substituting the complex expression (Eq. (35)) of $f$ into Eq. (13), stress and displacement components can be computed, and it is easy to verify that these expressions coincide with those in Eqs. (31*a*)–(31*d*) if the following change of variables is performed:

$$M_{i,j} = 2(\lambda+1)A_{i,j}$$

$$\bar{N}_{i,j} = 2(\lambda+1)B_{i,j}$$

$$I_{i,j} = 2C_{i,j}$$

$$\bar{L}_{i,j} = 2D_{i,j} \quad (36)$$

Keeping this result in mind, it is now clear that the elementary matrix (Eq. (32)) just represents the complex form of the elementary matrix obtained according to the eigenfunction expansion method. Thanks to the properties of the solution of the ODE in Eq. (35), we have demonstrated that the complex function representation by Theocaris and the eigenfunction expansion method by Williams are perfectly equivalent for the computation of the eigenvalues. This implies that both formulations can be transformed into the other without changing the zeros of the eigenequation of the problem.

*3.3.3 Mellin Transform Technique*. The Mellin transform technique was introduced to the analysis of stress-singularities in plane elasticity by Tranter [96] in 1948. The potentials of this technique were also discussed by Godfrey [97] and Sneddon [98] in their valuable articles and textbooks.

This mathematical method was mainly applied to the singularity analysis of bimaterial wedges subjected to a variety of boundary conditions by Bogy [21,99,100] and Hein and Erdogan [25]. Stresses in bonded materials with a crack perpendicular to the interface were obtained by Cook and Erdogan [101] in 1972. Approximately in the same years, Bogy and Wang [22] tackled the problem of stress singularities due to bimaterial junctions [22], whereas stress singularities due to trimaterial wedges were analyzed with this technique only in 1996 [26,102].

According to this approach, the boundary value problem represented by Eq. (8) can be solved in a transformed plane by applying the Mellin transform, which is defined as

$$\mathcal{M}[f,s] = \int_0^\infty f(r) r^{s-1} dr \tag{37}$$

where $s$ is a complex transform parameter.

Consequently, the application of the Mellin transform to the Airy stress function reads

$$\mathcal{M}[\Phi_i,s] = \Phi_i^*(s,\theta) = \int_0^\infty \Phi_i(r,\theta) r^{s-1} dr \tag{38}$$

Furthermore, the biharmonic condition requires $\Phi_i^*$ to be of the form [96]

$$\Phi_i^*(s,\theta) = a_{i,j}(s)\sin(s\theta) + b_{i,j}(s)\cos(s\theta) + c_{i,j}(s)\sin[(s+2)\theta] + d_{i,j}(s)\cos[(s+2)\theta] \tag{39}$$

Eventually, taking the Mellin transform of $r^2$ and $r$ times the stresses and displacements in Eqs. (4*a*)–(4*e*), we obtain such quantities in the transformed domain,

$$\mathcal{M}[r^2\sigma_r^i,s] = -a_{i,j}s(s+1)\sin(s\theta) - b_{i,j}s(s+1)\cos(s\theta) - c_{i,j}[s+(s+2)^2]\sin[(s+2)\theta] - d_{i,j}[s+(s+2)^2]\cos[(s+2)\theta]$$

$$\mathcal{M}[r^2\sigma_\theta^i,s] = s(s+1)\{a_{i,j}\sin(s\theta) + b_{i,j}\cos(s\theta) + c_{i,j}\sin[(s+2)\theta] + d_{i,j}\cos[(s+2)\theta]\}$$

$$\mathcal{M}[r^2 r_{r\theta}^i,s] = (s+1)\{a_{i,j}s\cos(s\theta) - b_{i,j}s\sin(s\theta) + c_{i,j}(s+2)\cos[(s+2)\theta] - d_{i,j}(s+2)\sin[(s+2)\theta]\}$$

$$\mathcal{M}[ru_r^i,s] = \frac{1}{2G_i}\{a_{i,j}s\sin(s\theta) + b_{i,j}s\cos(s\theta) + c_{i,j}(s+4/m_i)\sin[(s+2)\theta] + d_{i,j}(s+4/m_i)\cos[(s+2)\theta]\}$$

$$\mathcal{M}[ru_\theta^i,s] = \frac{1}{2G_i}\{-a_{i,j}s\cos(s\theta) + b_{i,j}s\sin(s\theta) - c_{i,j}(s+2-4/m_i)\cos[(s+2)\theta] + d_{i,j}(s+2-4/m_i)\sin[(s+2)\theta]\} \tag{40}$$

Applying the Mellin transform to the boundary conditions (Eq. (8)) and then substituting Eq. (40) into them, a matrix form analogous to Eq. (16) can be determined. To be more specific, by observing that the complex parameter $s$ is related to λ by

$$s = -\lambda - 1 \tag{41}$$

we obtain the following expression for the elementary matrix $\mathbf{M}_\theta^i$:

$$\mathbf{M}_\theta^i = \begin{bmatrix} -\lambda\lambda^+\sin(\lambda^+\theta) & \lambda\lambda^+\cos(\lambda^+\theta) & -\lambda\lambda^+\sin(\lambda^-\theta) & \lambda\lambda^+\cos(\lambda^-\theta) \\ \lambda\lambda^+\cos(\lambda^+\theta) & \lambda\lambda^+\sin(\lambda^+\theta) & \lambda\lambda^-\cos(\lambda^-\theta) & \lambda\lambda^-\sin(\lambda^-\theta) \\ \frac{\lambda^+}{2G_i}\sin(\lambda^+\theta) & -\frac{\lambda^+}{2G_i}\cos(\lambda^+\theta) & -\left(\frac{4}{m_i}-\lambda^+\right)\frac{\sin(\lambda^-\theta)}{2G_i} & \left(\frac{4}{m_i}-\lambda^+\right)\frac{\cos(\lambda^-\theta)}{2G_i} \\ \frac{\lambda^+}{2G_i}\cos(\lambda^+\theta) & \frac{\lambda^+}{2G_i}\sin(\lambda^+\theta) & \left(\frac{4}{m_i}+\lambda^-\right)\frac{\cos(\lambda^-\theta)}{2G_i} & \left(\frac{4}{m_i}+\lambda^-\right)\frac{\sin(\lambda^-\theta)}{2G_i} \end{bmatrix} \tag{42}$$

where $\lambda^+$ and $\lambda^-$ denote, respectively, $\lambda+1$ and $\lambda-1$.

In this formulation, the components of the vector **v** are

$$\mathbf{v}^i = \{a_{i,j}, b_{i,j}, c_{i,j}, d_{i,j}\}^T \tag{43}$$

The elementary matrix in Eq. (42) coincides with that in Eq. (18) obtained by applying the eigenfunction expansion method if the following change of variables is made:

$$a_{i,j} = -A_{i,j}$$

$$b_{i,j} = B_{i,j}$$

$$c_{i,j} = -C_{i,j}$$

$$d_{i,j} = D_{i,j} \tag{44}$$

Also in this case, eigenequations derived from the condition of a vanishing determinant of the matrix $\boldsymbol{\Lambda}$ are not influenced by this change of variables. The eigenvalues computed as the zeros of the eigenequations are then the same in all the reviewed formulations.

**3.4 Eigenequations for Limit Cases.** Closed-form solutions to the determinant of the coefficient matrix, $D=\det\boldsymbol{\Lambda}$, allow us to analytically investigate on the influence of both the junction geometry and the mechanical parameters of the constituent materials on the shape of the eigenfunctions and their roots. Nonetheless, such closed-form solutions represent an important test for establishing the accuracy of numerical procedures used for computing the eigenvalues. Unfortunately, due to the involved character of the mathematical formulations, which yield nonlinear eigenvalue problems, closed-form solutions of $D$ for trimaterial junctions and wedges are available in the literature only for particular cases [102].

As far as perfectly bonded bimaterial junctions are concerned, a closed-form solution to the eigenfunction for any given modular ratio $E_2/E_1$ was provided by Bogy and Wang [22]. Furthermore, they observed that for a limit material configuration involving an infinitely stiff material region, the determinant $D$ of the coefficient matrix is given by (see Fig. 10)

$$D(E_1\to\infty, E_2, \theta_1, \theta_2) = D^{\text{cl-cl}}(\theta_2) \times D^{\text{fr-fr}}(\theta_1)$$

$$D(E_1, E_2\to\infty, \theta_1, \theta_2) = D^{\text{cl-cl}}(\theta_1) \times D^{\text{fr-fr}}(\theta_2) \tag{45}$$

where $D^{\text{cl-cl}}(\theta_i)$ and $D^{\text{fr-fr}}(\theta_i)$ denote, respectively, the eigenequations due to an angular corner of a plate with amplitude $2\pi-\theta_i$

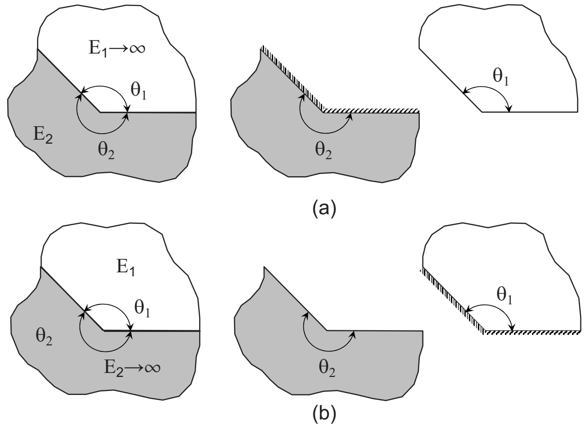


**Fig. 10 Subproblems for the evaluation of the eigenequation for a bimaterial junction with (*a*) $E_1 \to \infty$ and (*b*) $E_2 \to \infty$**

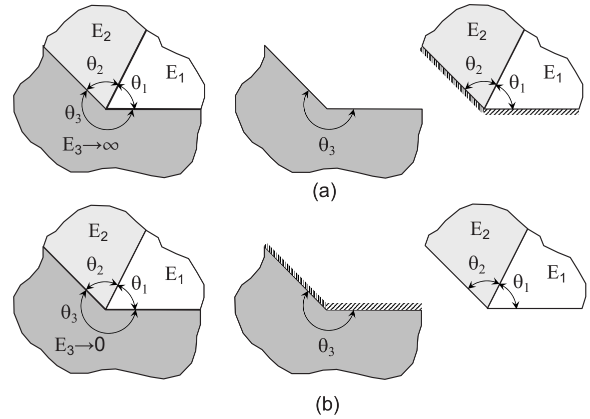


**Fig. 11 Subproblems for the evaluation of the eigenequation for a trimaterial junction with (*a*) $E_3 \to \infty$ and (*b*) $E_3 \to 0$**

with either clamped or free edges (see the scheme in Fig. 10). Closed-form solutions to these problems were provided by Williams [67] and are functions of the corner angle only.

Therefore, the search of the zeros of the eigenequation for the limit bimaterial junction problem reduces to the search of the zeros of the eigenequations of two simpler subproblems whose analytical solutions are well known. Furthermore, by observing that the singular stress fields for the above subproblems exist only in the case of reentrant corners [67], i.e., only for $\theta_i > \pi$, the eigenvalues are given by

$$\text{zeros } D(E_1 \to \infty, E_2, \theta_1, \theta_2) = \begin{cases} \text{zeros } D^{\text{cl-cl}}(\theta_2) & \text{if } \theta_2 > \pi \\ \text{zeros } D^{\text{fr-fr}}(\theta_1) & \text{if } \theta_1 > \pi \\ \text{no singularities} & \text{if } \theta_1 = \theta_2 = \pi \end{cases}$$

$$\text{zeros } D(E_1, E_2 \to \infty, \theta_1, \theta_2) = \begin{cases} \text{zeros } D^{\text{fr-fr}}(\theta_2) & \text{if } \theta_2 > \pi \\ \text{zeros } D^{\text{cl-cl}}(\theta_1) & \text{if } \theta_1 > \pi \\ \text{no singularities} & \text{if } \theta_1 = \theta_2 = \pi \end{cases} \tag{46}$$

On the other hand, the limit problem of a bimaterial wedge with $E_1 \to 0$ (correspondingly $E_2 \to 0$) can be simply treated as the limit case of a bimaterial junction with $E_2 \to \infty$ (correspondingly, $E_1 \to \infty$).

Pageau et al. [24] proposed in 1994 a method for obtaining the eigenequation for a trimaterial junction when the elastic modulus of the third material tends to infinity. To be more specific, they assumed that $D$ for a trimaterial junction with $E_3 \to \infty$ is equal to the eigenequation due to a bimaterial wedge composed of material regions 1 and 2 with clamped edges,

$$D(E_1, E_2, E_3 \to \infty, \theta_1, \theta_2, \theta_3) = D^{\text{cl-cl}}(E_1, E_2, \theta_1, \theta_2) \tag{47}$$

Actually, this solution holds if and only if the wedge angle of the third material is less than $\pi$, i.e., for $\theta_3 < \pi$. In addition, the eigenequation corresponding to the limit trimaterial junction characterized by $E_3 \to 0$ was not addressed in the literature.

A complete solution can be obtained as an extension to trimaterial junctions of the formulation (Eq. (46)) provided by Bogy and Wang [22] for limit bimaterial junctions. Going into the details, when the third material becomes infinitely stiff, the eigenfunction of this limit problem can be reduced to two factors. The former corresponds to the eigenequation of the bimaterial wedge composed of materials 1 and 2 with clamped edges. The latter is due to the eigenequation due to a reentrant corner with free edges and amplitude equal to $2\pi - \theta_3$ (see the scheme in Fig. 11(*a*)).

Analogously, when the third material becomes infinitely soft, the two factors correspond, respectively, to the eigenequation of the bimaterial wedge formed by materials 1 and 2 with free edges and the eigenequation due to a reentrant corner with clamped edges and amplitude $2\pi - \theta_3$ (see the scheme in Fig. 11(*b*)).

Noting that the eigenequations due to an angular corner in a plate can contribute to the singular stress field if and only if $\theta_3$ is greater than $\pi$, we end up with the following compact expressions:

$$D(E_1, E_2, E_3 \to \infty, \theta_1, \theta_2, \theta_3) = D^{\text{cl-cl}}(E_1, E_2, \theta_1, \theta_2) \times D^{\text{fr-fr}}(\theta_3)$$

$$D(E_1, E_2, E_3 \to 0, \theta_1, \theta_2, \theta_3) = D^{\text{fr-fr}}(E_1, E_2, \theta_1, \theta_2) \times D^{\text{cl-cl}}(\theta_3) \tag{48}$$

and

$$\text{zeros } D(E_1, E_2, E_3 \to \infty, \theta_1, \theta_2, \theta_3) = \begin{cases} \text{zeros } D^{\text{cl-cl}}(E_1, E_2, \theta_1, \theta_2) & \text{if } \theta_3 \leqslant \pi \\ \text{zeros } D^{\text{cl-cl}}(E_1, E_2, \theta_1, \theta_2) & \\ \quad + \text{zeros } D^{\text{fr-fr}}(\theta_3) & \text{if } \theta_3 > \pi \end{cases}$$

$$\text{zeros } D(E_1, E_2, E_3 \to 0, \theta_1, \theta_2, \theta_3) = \begin{cases} \text{zeros } D^{\text{fr-fr}}(E_1, E_2, \theta_1, \theta_2) & \text{if } \theta_3 \leqslant \pi \\ \text{zeros } D^{\text{fr-fr}}(E_1, E_2, \theta_1, \theta_2) & \\ \quad + \text{zeros } D^{\text{cl-cl}}(\theta_3) & \text{if } \theta_3 > \pi \end{cases} \tag{49}$$

where $D^{\text{cl-cl}}(E_1, E_2, \theta_1, \theta_2)$ corresponds to the eigenequation for a bimaterial wedge with clamped edges provided by Pageau et al. [24]. $D^{\text{fr-fr}}(E_1, E_2, \theta_1, \theta_2)$ is the eigenequation for a bimaterial wedge with free edges provided by Bogy [21]. $D^{\text{fr-fr}}(\theta_3)$ and $D^{\text{cl-cl}}(\theta_3)$ correspond to the closed-form solutions to the eigenequations investigated by Williams [67] for a reentrant corner in a plate with either free or clamped edges.

Some examples of application of these formulae to limit cases involving trimaterial junctions are illustrated in Table 3 (see also Fig. 12).

**3.5 Eigenfunctions.** Once the eigenvalues are computed, the singular analysis is completed by the evaluation of the stress and displacement fields in the material regions. For this purpose, three methods can be applied.

In the case of the eigenfunction expansion method, for each $\lambda_j$ we can express the unknowns in the vector $\mathbf{v}$ in terms of the first one, $A_{i,j}$. Then, the unknown constant $A_{i,j}$ is normalized such that the following expression for the stress field can be written:

**Table 3 Limit eigenequations and eigenvalues for three different tri material junctions sketched in Fig. 12**

| Figure | $E_1/E_2$ | $E_3/E_2$ | λ limit | $D$ limit |
|---|---|---|---|---|
| 12(*a*) | 5 | 0 | 0.7389 | $D^{\text{cl-cl}}\ (\theta_3)$ |
| | | | 0.8724 | $D^{\text{cl-cl}}\ (\theta_3)$ |
| | 5 | ∞ | 0.6736 | $D^{\text{fr-fr}}\ (\theta_3)$ |
| | | | 0.9345 | $D^{\text{cl-cl}}\ (E_1,E_2,\theta_1,\theta_2)$ |
| 12(*b*) | 10 | 0 | 0.8589 | $D^{\text{fr-fr}}\ (E_1,E_2,\theta_1,\theta_2)$ |
| | | | 1.0000 | $D^{\text{fr-fr}}\ (E_1,E_2,\theta_1,\theta_2)$ |
| | 10 | ∞ | 0.8147 | $D^{\text{cl-cl}}\ (E_1,E_2,\theta_1,\theta_2)$ |
| 12(*c*) | 10 | 0 | 0.6818 | $D^{\text{fr-fr}}\ (E_1,E_2,\theta_1,\theta_2)$ |
| | | | 0.8413 | $D^{\text{fr-fr}}\ (E_1,E_2,\theta_1,\theta_2)$ |
| | 10 | ∞ | 0.7723 | $D^{\text{cl-cl}}\ (E_1,E_2,\theta_1,\theta_2)$ |

$$\sigma_j^i = \widetilde{K}_j r^{\lambda_j - 1} f_{i,j}(\theta) + O(1) \tag{50}$$

where $O(1)$ indicates nonsingular terms. This procedure can be repeated for each $j$th eigenvalue, allowing us to write an expression where all the eigenvalues contribute to the singular stress field near the vertex $O$,

$$\sigma^i = \sum_j \widetilde{K}_j r^{\lambda_j - 1} f_{i,j}(\theta) + O(1) \tag{51}$$

According to this method, stresses and displacements are determined to within multiplicative constants, $\widetilde{K}_j$, which are referred to as generalized stress-intensity factors [16,103].

When the eigenvalues are complex, also the corresponding constants in the vector **v** are complex. Therefore, as pointed out by Pageau et al. [103], the direct application of Eqs. (4*a*)–(4*e*) leads to stresses and displacements that are complex valued which, of course, are not valid. In these cases, a valid solution can be obtained by considering the actual stresses and displacements, computed as a linear combination of the corresponding complex stresses and displacements [103].

Eventually, particular attention has to be paid to materials and geometric configurations that correspond to the transition from two real roots to a complex conjugate pair. When the eigenvalues change from real to complex at some characteristic opening angles of the junction geometry or in correspondence of certain material combinations, multiple eigenvalues corresponding to the same independent eigenfunction may occur. The analytical solution of such a special case cannot be expressed as a single function of the radial coordinate multiplied by a single function of the angular coordinate. These nonseparable solutions include terms not only with power functions, but also with logarithmic functions of the radial coordinate.

The power-logarithmic stress singularities have been identified by Dempsey and Sinclair [7] for multimaterial wedges subjected to homogeneous boundary conditions on the surfaces forming the wedge vertex. Dempsey [104] examined the eigenvalues in the asymptotic solutions for isotropic composite wedges under homogeneous boundary conditions and pointed out that the power-logarithmic function is more singular than the power function at the vertex. Pageau et al. [24] investigated the power-logarithmic stress singularities in trimaterial junctions. Joseph and Zhang [105] reviewed the studies on power-logarithmic stress singularities for multiple material wedges. Angular variations of the displacement and stress fields are presented in Refs. [105–107] for wedges composed of isotropic materials.

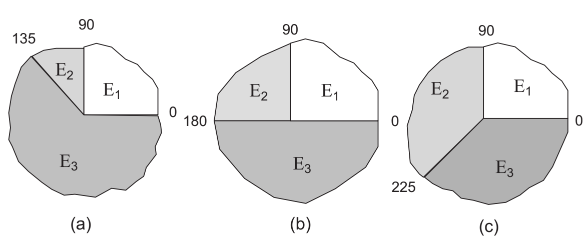


**Fig. 12 Three examples of trimaterial junctions whose limit eigenvalues are reported in Table 3**

Additional logarithmic singularities can be induced by inhomogeneous boundary conditions on the surfaces forming the wedge vertex and by body forces [108]. A related problem is a wedge under a concentrated couple studied by Sternberg and Koiter [109]. Dempsey [110] also obtained a solution for a single-material wedge at its critical angle, which includes a power-logarithmic stress term.

On the other hand, if the complex function representation is used, then the complex formulation outlined by Theocaris [23] has to be adopted for the computation of stress and displacement fields in the case of real and complex eigenvalues. Also in this case, for each $\lambda_j$ we can express the unknowns in the vector **v** in terms of the first, say, $I_{i,j}$. As a result, the eigenfunction can then be expressed in terms of the real and imaginary parts of this complex constant (see Carpenter and Byers [78] for a detailed derivation of the stress field in the case of a bimaterial wedge).

For the application of the Mellin transform, stresses and displacements can be computed by applying the inversion formula of the integral transform,

$$\sigma_r^i = \frac{1}{2\pi i}\int_{c_i - i\infty}^{c_i + i\infty} \left(\frac{d^2}{d\theta^2} - s\right)\Phi_i^* r^{-(s+2)} ds$$

$$\tau_{r\theta}^i + i\sigma_\theta^i = \frac{1}{2\pi i}\int_{c_i - i\infty}^{c_i + i\infty} (s+1)\left(\frac{d}{d\theta} + is\right)\Phi_i^* r^{-(s+2)} ds$$

$$u_r^i + iu_\theta^i = \frac{-1}{4\pi G_i}\int_{c_i - i\infty}^{c_i + i\infty} \left(\frac{d}{d\theta} + is\right) \times \left\{1 + \frac{(1-\nu_i)\left(\frac{d}{d\theta} - is\right)\left[\frac{d}{d\theta} - i(s+2)\right]}{(s+1)(s+2)}\right\} \times \Phi_i^* r^{-(s+1)} ds$$

$$\frac{\partial u_r^i}{\partial r} + i\frac{\partial u_\theta^i}{\partial r} = \frac{1}{4\pi G_i}\int_{c_i - i\infty}^{c_i + i\infty} (s+1)\left(\frac{d}{d\theta} + is\right) \times \left\{1 + \frac{(1-\nu_i)\left(\frac{d}{d\theta} - is\right)\left[\frac{d}{d\theta} - i(s+2)\right]}{(s+1)(s+2)}\right\} \times \Phi_i^* r^{-(s+2)} ds \tag{52}$$

where the parameters $c_i$ are such that $r^{c_i - 1}\Phi_i^*$ are absolutely integrable on $(0,\infty)$. The criterion for the choice of $c_i$ is then provided by the following regularity condition:

$$r^{s+m-1}\frac{d^{m-1}\Phi_i^*}{dr^{m-1}} \to 0 \quad \text{as } r \to (0,\infty), \quad m = 1, \ldots, n \tag{53}$$

**3.6 Stokes Flow Analogy.** Two-dimensional Stokes flow of a viscous incompressible fluid can also be described as a problem governed by a biharmonic function. Stokes flow represents the flow of a fluid in the limit of vanishingly small Reynolds number, where the viscous forces are much larger than the inertial forces.

As a consequence, the equations describing Stokes flow can be deduced from the Navier–Stokes equations by setting the inertial and body force terms equal to zero,

$$\nabla \cdot \mathbf{v} = 0$$

$$-\nabla p + \mu \nabla^2 \mathbf{v} = \mathbf{0} \tag{54}$$

where $p$ is the pressure, $\mathbf{v}=v_r\mathbf{n}+v_\theta\mathbf{t}$ denotes the velocity field in polar coordinates, and $\mu=\rho v$ is the product between the density $\rho$ and the kinematic viscosity $v$ of the fluid.

For two-dimensional incompressible flows, it is possible to recast the Navier–Stokes equations in terms of the stream function $\Psi$ and the vorticity $\omega$. Recalling the identity

$$\nabla^2 \mathbf{v} = \nabla(\nabla \cdot \mathbf{v}) - \nabla \times (\nabla \times \mathbf{v}) \tag{55}$$

the Stokes equations (Eq. (54)) can be rewritten as

$$\nabla \cdot \mathbf{v} = 0$$

$$\nabla p = -\mu \nabla \times \omega \tag{56}$$

where $\omega=\nabla\times\mathbf{v}$ is the vorticity. Taking the curl of the second of Eq. (56), the equations of vorticity transport for a Stokes flow are derived,

$$\nabla \cdot \omega = 0$$

$$\nabla^2 \omega = \mathbf{0} \tag{57}$$

Introducing now a stream function $\Psi$ such that the velocity components are given by

$$v_r = \frac{1}{r}\frac{\partial \Psi}{\partial \theta} \tag{58}$$

$$v_\theta = -\frac{\partial \Psi}{\partial r} \tag{59}$$

the vorticity can be expressed in terms of the stream function

$$\omega = \nabla \times \mathbf{v} = -\nabla^2 \Psi \mathbf{k} \tag{60}$$

Since the unit vector $\mathbf{k}$ perpendicular to the plane is constant, Eq. (60) can be introduced into Eq. (57) obtaining the biharmonic condition for the stream function,

$$\nabla^4 \Psi = 0, \quad \forall\ (r,\theta) \in \Omega_i \tag{61}$$

The relevance of the asymptotic analysis in fluid mechanics problems involving viscous fluids is clearly evidenced by the fact that there are many viscous flows in which the fluid must negotiate corners (see, e.g., the flow in a driven cavity or sector [111–114] or the flow over rectangular cylinders or cones [115,116]). Contact-line problems [117,118], such as those related to meniscus problems [118], are usually treated as single-fluid systems in which the second fluid is regarded as a passive gas. However, if the flow of both fluids is important, the flow of two immiscible fluids needs be considered [117,119].

This problem can be generalized, in principle, to three or even $n$ immiscible fluids with different viscosities negotiating a corner (see Fig. 13). The boundary conditions at the interfaces impose vanishing velocities in the circumferential direction and the continuity of the velocities in the radial direction, as well as of the tangential and circumferential stresses,

$$v_\theta^i(r,\gamma_{i+1}) = v_\theta^{i+1}(r,\gamma_{i+1}) = 0$$

$$v_r^i(r,\gamma_{i+1}) = v_r^{i+1}(r,\gamma_{i+1})$$

$$\tau_{r\theta}^i(r,\gamma_{i+1}) = \tau_{r\theta}^{i+1}(r,\gamma_{i+1})$$

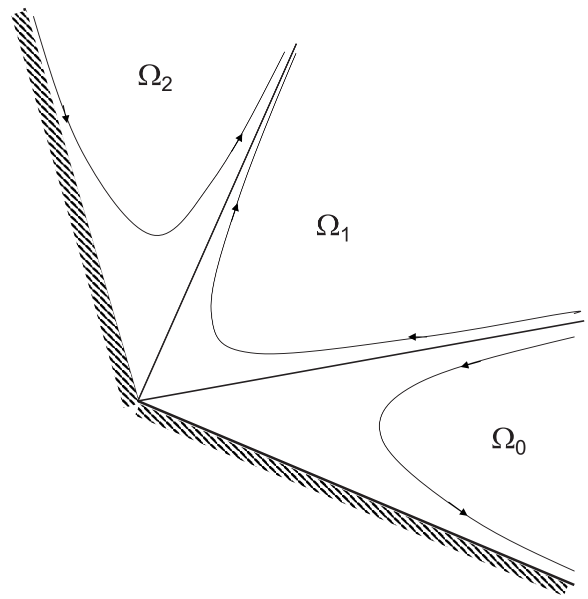


**Fig. 13 A scheme of a corner negotiated by three immiscible viscous fluids**

$$\sigma_\theta^i(r,\gamma_{i+1}) = \sigma_\theta^{i+1}(r,\gamma_{i+1}) \tag{62}$$

These boundary conditions permit us to find out the *local solutions* to the problem and apply in the case of fluids characterized by a small capillary number (see, e.g., Ref. [119] for a comprehensive discussion on this problem). When the capillary number is exactly equal to zero, all the aforementioned local boundary conditions are imposed, with the exception of the circumferential-stress boundary condition, which is identically satisfied. The corresponding solutions are then referred to in the literature as *partial local solutions* since they provide an approximation of the velocity field valid only for a zero capillary number [119].

As far as the boundary conditions at the wedge edges defined by the angle $\bar{\theta}$ are concerned, the boundary may be rigid walls on which the fluid velocity is prescribed, or surfaces on which the stress is prescribed (with the pressure distribution to keep it plane). Three distinct categories of flow can be described by these boundary conditions. In the first category, a nonzero velocity or stress is prescribed on one or both boundaries, and the flow is described by a particular integral of the biharmonic equation satisfying appropriate inhomogeneous boundary conditions. One of these situations was addressed by Taylor [120], who described the case when a viscous fluid is entrapped in a rigid plane, which is scraped along another at a constant velocity and inclination angle.

In the second category, either a velocity or the tangential stress vanishes on each boundary. The flow near the corner is induced by a general motion at a large distance from the corner. In this case, the boundary conditions are homogeneous and the problem seems to have been first considered by Rayleigh [121]. The flows in this category all have a finite velocity at the corner vertex.

In the last category, perturbance is present near the origin, and the velocity in the corresponding flow is infinite at $r=0$ but tends to zero as $r\to\infty$. Such solution may describe the flow at a large distance from the intersection of two planes when some steady disturbance at $r\to 0$ takes place.

Considering the second category in more detail, the following boundary conditions are usually imposed on the wedge edges at $\theta=\bar{\theta}$:

$$\left.\begin{array}{l} v_r(r,\bar{\theta}) = 0 \\ v_\theta(r,\bar{\theta}) = 0 \end{array}\right\} \quad \text{for a rigid surface} \tag{63}$$

$$v_\theta(r,\bar{\theta})=0$$
$$\tau_{r\theta}(r,\bar{\theta})=0 \quad \text{for a free surface} \tag{64}$$
$$\sigma_\theta(r,\bar{\theta})=0$$

In the case of free surfaces, the condition $\sigma_\theta=0$ can be neglected if the fluid has a zero capillary number, i.e., when we are seeking for partial local solutions.

From this preliminary discussion emerges the observation that the flow of a viscous fluid is governed by the biharmonic equation for the stream function, in close analogy with the biharmonic equation for the Airy stress function in plane elasticity. As a consequence, the eigenfunction expansion method can be profitably applied, as firstly pursued by Rayleigh [121] for the study of the flow of a single fluid inside a corner. Considering a stream function given as the product between a radial term and an angular function,

$$\Psi_i(r,\theta)=\sum_j r^{\lambda_j+1} f_{i,j}(\theta,\lambda_j) \tag{65}$$

he derived an expression for $f_{i,j}$ analogous to that reported in Eq. (14). However, at that time, he assumed that the exponent $\lambda_j+1$ must be a multiple of $\pi/\gamma$, where $\gamma$ is the wedge angle. As a consequence of this assumption, he deduced that the expression of $f$ in Eq. (65) is possible only if $\gamma$ is a multiple of $\pi$; i.e., he restricted $\gamma$ to the values $\pi$ and $2\pi$. This assumption was removed by Dean and Montagnon [122] in 1948, who firstly applied the eigenfunction expansion method to the analysis of the steady motion of a viscous liquid around a corner with rigid surfaces. It has to be noticed that this application of the eigenfunction expansion method precedes that by Williams [67] by four years.

According to Eq. (65), the expressions for $v_r$, $v_\theta$, and $\tau_{r\theta}$ are

$$v_r = r^\lambda f' \tag{66}$$

$$v_\theta = (\lambda+1) r^\lambda f \tag{67}$$

$$\tau_{r\theta} = \mu r^{\lambda-1}[-(\lambda^2-1)f + f''] \tag{68}$$

From these expressions, we readily recognize that Eqs. (66) and (67) are analogous to those for the tangential and the circumferential stresses in plane elasticity reported in Eq. (13), i.e.,

$$v_r = -\frac{r}{\lambda}\tau_{\tau\theta}^{\text{el}}$$

$$v_\theta = \frac{r}{\lambda}\sigma_\theta^{\text{el}} \tag{69}$$

where the superscript (el) denotes quantities related to biharmonic problems in classical elasticity. Therefore, it is possible to state that the problem of the flow of a viscous fluid in a single wedge with rigid surfaces has its analogous counterpart in plane elasticity with the stress field of an angular plate with free surfaces.

The eigenequation obtained by Dean and Montagnon [122] in 1948 for the analysis of the steady motion of a viscous liquid around a corner with rigid surfaces is then coincident with the well-known expression derived by Williams in 1952 for the stress singularities in angular plates with free boundaries [67],

$$\frac{\sin(\lambda\gamma)}{\lambda\gamma} = \mp\frac{\sin\lambda}{\lambda} \tag{70}$$

The negative sign in Eq. (70) corresponds to Mode I deformation and symmetric flow, whereas the positive sign is related to Mode II deformation and skew-symmetric flow (see Fig. 14 for a schematic visualization of these flows).

Another interesting point concerns the eigenvalues that can be computed as the roots of Eq. (70). Williams computed the eigenvalues $\lambda$ and plotted their real part as a function of the wedge angle, $\gamma$. For free-free boundary conditions, he found that $\lambda$ is real and in the range $0<\lambda<1$ for obtuse angles, i.e., for plates with reentrant corners. No particular comments were provided by Williams about the transition between real and complex eigenvalues since this case occurs corresponding to $\gamma \cong 146$ deg, i.e., for $\text{Re}\,\lambda>1$ leading to nonsingular stress fields. On the contrary, this transition was carefully analyzed by Dean and Montagnon [122] four years before Williams. This is because in the analogous problem in fluid mechanics, the fluid velocity is always finite and the whole range of variation of $\lambda$ is of interest. Moffatt [123] in 1964 showed that the complex eigenvalues observed for $\gamma<146$ deg imply the existence of a flow near the corner consisting in a sequence of eddies of decreasing size and rapidly increasing intensity (see Fig. 15 for a visualization of such resistive eddies).

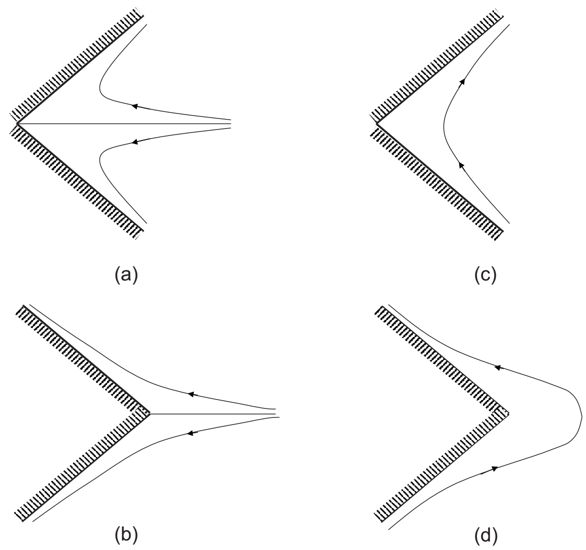


**Fig. 14 Sketch of the stream lines for symmetric (*a*) and (*b*) and skew-symmetric (*c*) and (*d*) flows around a corner**

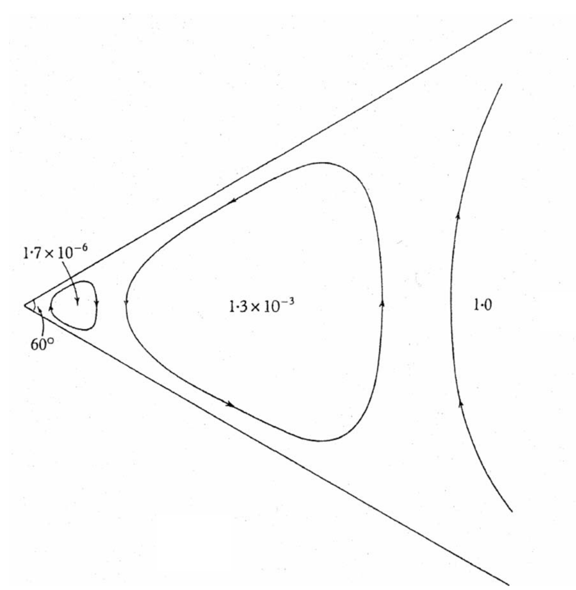


**Fig. 15 Sketch of the stream lines in corner eddies for $\gamma$=60 deg, reprinted from Ref. [123]. The relative dimensions of the eddies are approximately correct and the relative intensities are indicated.**

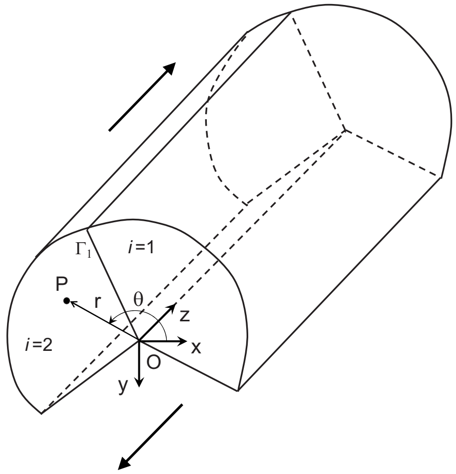


**Fig. 16 Scheme of a composite material subjected to antiplane shear deformation**

## 4 Out-of-Plane Loading

**4.1 Statement of the Harmonic Problem.** Out-of-plane loading due to antiplane shear (Mode III) on composite wedges can lead to stresses that can be unbounded at the junction vertex $O$ (see a scheme of this multimaterial problem depicted in Fig. 16)

When out-of-plane deformations only exist, the following displacements in cylindrical coordinates with the origin placed at the vertex point $O$ can be considered:

$$u_r = 0$$

$$u_\theta = 0$$

$$u_z = u_z(r,\theta) \tag{71}$$

where $u_z$ is the out-of-plane displacement. For such a system of displacements, the strain field becomes

$$\varepsilon_r = \varepsilon_\theta = \varepsilon_z = \gamma_{r\theta} = 0$$

$$\gamma_{rz} = \frac{\partial u_z}{\partial r}$$

$$\gamma_{\theta z} = \frac{1}{r}\frac{\partial u_z}{\partial \theta} \tag{72}$$

and from Hooke's law, the stresses reduce to

$$\sigma_r = \sigma_\theta = \sigma_z = \tau_{r\theta} = 0$$

$$\tau_{rz} = G_i\gamma_{rz} = G_i\frac{\partial u_z}{\partial r}$$

$$\tau_{\theta z} = G_i\gamma_{\theta z} = \frac{G_i}{r}\frac{\partial u_z}{\partial \theta} \tag{73}$$

where $G_i$ is the shear modulus of the $i$th material component. The equilibrium equations in the absence of body forces reduce to

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{1}{r}\tau_{rz} = 0, \quad \forall\ (r,\theta) \in \Omega_i \tag{74}$$

Introducing Eq. (73) into Eq. (74), the harmonic condition upon $u_z$ is derived,

$$\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} = \nabla^2 u_z = 0, \quad \forall\ (r,\theta) \in \Omega_i \tag{75}$$

In contrast to its biharmonic counterpart in elasticity, which has been the subject of a considerable number of investigations, the characterization of the singularities of this harmonic problem in composite regions appears to have received a minor attention. Antiplane shear problems for isotropic and anisotropic bimaterial wedges were addressed by Ma and Hour [124,125] in the late 1980s using the Mellin transform technique. The antiplane shear deformation problem of isotropic as well as anisotropic wedges was solved under different boundary conditions by Karganovin et al. [126] and Shahani [127], respectively. They found, using the Mellin transform technique, analytical solutions for the displacement and stress fields and derived the order of the stress singularity at the wedge apex. Shahani and Adibnazari [128] considered the antiplane shear deformation problem of two-material junctions with an interfacial crack.

Extensions to the computation of the eigenvalues and of the eigenfunctions for bonded and debonded trimaterial junctions were provided by Pageau et al. [73] in 1995 using the eigenfunction expansion method. More recent developments on this subject have concerned the use of cohesive laws instead of the classical boundary conditions for reentrant corners in antiplane shear in order to free them from stress singularities [129].

**4.2 Boundary Conditions.** Along the $i$th perfectly bonded interface defined by the angle $\gamma_{i+1}$, continuity conditions for stress and displacement have to be imposed,

$$u_z^i(r,\gamma_{i+1}) = u_z^{i+1}(r,\gamma_{i+1})$$

$$\tau_{\theta z}^i(r,\gamma_{i+1}) = \tau_{\theta z}^{i+1}(r,\gamma_{i+1}) \tag{76}$$

Moreover, in the case of either a reentrant corner or a crack at the angle $\bar{\theta}$, the following classical stress or displacement boundary conditions can be imposed along the edges:

$$\tau_{\theta z}^i(r,\bar{\theta}) = 0 \quad \text{for an unrestrained stress-free edge}$$

$$u_z^i(r,\bar{\theta}) = 0 \quad \text{for a fully restrained edge} \tag{77}$$

When the problem is characterized by a geometric symmetry, it is worth noting that the problem can be reduced to the solution of two separate subproblems by considering the following boundary condition along the symmetry line ($\bar{\theta}=0,\pi$):

$$\tau_{\theta z}^i(r,\bar{\theta}) = 0 \quad \text{for a symmetric deformation}$$

$$u_z^i(r,\bar{\theta}) = 0 \quad \text{for a skew-symmetric deformation} \tag{78}$$

Cohesive-law boundary conditions have also been considered by Sinclair [129] and were imposed on symmetry lines and on bimaterial interfaces instead of their classical counterparts. These laws admit to relative, albeit small, deflections between the two connected parts of the wedge, something not permitted with traditional boundary conditions. To this aim, a constant stiffness $K$ for a simplified linear version of a cohesive stress versus separation law is introduced, and the following boundary condition can be specified along the $i$th interface:

$$\tau_{\theta z}(r,\gamma_{i+1}) = K[u_z^{i+1}(r,\gamma_{i+1}) - u_z^i(r,\gamma_{i+1})] \tag{79}$$

For the special case of stress-free single-material wedges of angle $\gamma$ in antiplane shear, Sinclair [129] demonstrated that the use of traditional boundary conditions yields the following asymptotic behavior ($r \rightarrow 0$) of the tangential stresses:

$$\tau \propto \begin{cases} r^{(2\pi/\gamma-1)} & \text{for a symmetric deformation} \\ r^{(\pi/\gamma-1)} & \text{for skew-symmetric deformation} \end{cases} \tag{80}$$

Hence, for reentrant corners, i.e., for $\pi/2 < \gamma < \pi$, the skew-symmetric stress field is singular. On the contrary, the adoption of the cohesive separation law along the symmetry line (wedge bisector) permits us to avoid the occurrence of these stress singularities.

### 4.3 Mathematical Methods

*4.3.1 Eigenfunction Expansion Method*. In the framework of the eigenfunction expansion method, the following separable form for the longitudinal displacement $u_z^i$ can be adopted for the $i$th material region:

$$u_z^i(r,\theta) = \sum_j r^{\lambda_j} f_{i,j}(\theta,\lambda_j) \tag{81}$$

where $\lambda_j$ and $f_{i,j}$ are referred to as eigenvalues and eigenfunctions, respectively, as already introduced for the analysis of biharmonic problems. Also in this case, the summation with respect to the subscript $j$ is introduced in Eq. (81) since it is possible to have more than one eigenvalue for each problem, and the superposition principle can be applied.

Introducing Eq. (81) into Eq. (73), tangential stresses can be expressed in terms of the eigenfunction and its prime derivative,

$$\tau_{rz}^j = G_i \lambda r^{\lambda-1} f_{i,j}$$

$$\tau_{\theta z}^j = G_i r^{\lambda-1} f'_{i,j} \tag{82}$$

Furthermore, the harmonic condition requires $f_{i,j}$ to be of the following form:

$$f_{i,j}(\theta,\lambda_j) = A_{i,j}\sin(\lambda_j\theta) + B_{i,j}\cos(\lambda_j\theta) \tag{83}$$

where $A_{i,j}$ and $B_{i,j}$ are undetermined constants. To simplify the mathematical notation, the subscript $j$ in Eq. (83) is dropped in the sequel.

Then, introducing Eq. (83) into Eq. (82), the longitudinal displacement and the tangential stresses can be explicitly written in terms of the unknowns $A_{i,j}$, $B_{i,j}$, and $\lambda$,

$$u_z^i = r^\lambda [A_{i,j}\sin(\lambda\theta) + B_{i,j}\cos(\lambda\theta)]$$

$$\tau_{rz}^j = G_i\lambda r^{\lambda-1}[A_{i,j}\sin(\lambda\theta) + B_{i,j}\cos(\lambda\theta)]$$

$$\tau_{\theta z}^j = G_i\lambda r^{\lambda-1}[A_{i,j}\cos(\lambda\theta) - B_{i,j}\sin(\lambda\theta)] \tag{84}$$

Considering the problem consisting in a multimaterial junction with $n$ material regions, Eq. (84) has to be introduced into the interface matching conditions (Eq. (76)). In this way, a set of $2n$ equations in the $2n+1$ unknowns $A_{i,j}$, $B_{i,j}$, and $\lambda_j$ can be symbolically written as:

$$\mathbf{\Lambda v} = \mathbf{0} \tag{85}$$

where $\mathbf{\Lambda}$ denotes the coefficient matrix that depends on the eigenvalue, and $\mathbf{v}$ represents the vector that collects the unknowns $A_{i,j}$ and $B_{i,j}$.

The coefficient matrix in Eq. (85) is characterized by a sparse structure, as for the corresponding biharmonic problem,

$$\mathbf{\Lambda} = \begin{bmatrix} \mathbf{M}_{\gamma_1}^0 & -\mathbf{M}_{\gamma_1}^1 & & & & & & \\ & \mathbf{M}_{\gamma_2}^1 & -\mathbf{M}_{\gamma_2}^2 & & & & & \\ & & \cdots & \cdots & & & & \\ & & & \mathbf{M}_{\gamma_i}^{i-1} & -\mathbf{M}_{\gamma_i}^i & & & \\ & & & & \cdots & \cdots & & \\ & & & & & \mathbf{M}_{\gamma_{n-1}}^{n-2} & -\mathbf{M}_{\gamma_{n-1}}^{n-1} \\ -\mathbf{M}_{\gamma_0=0}^0 & & & & & & \mathbf{M}_{\gamma_n=2\pi}^{n-1} \end{bmatrix} \tag{86}$$

The elementary matrix $\mathbf{M}_\theta^i$ is given by

$$\mathbf{M}_\theta^i = \begin{bmatrix} \sin(\lambda\theta) & \cos(\lambda\theta) \\ G_i\cos(\lambda\theta) & -G_i\sin(\lambda\theta) \end{bmatrix} \tag{87}$$

and the components of the vector $\mathbf{v}$ are

$$\mathbf{v} = \{\mathbf{v}^0, \mathbf{v}^1, \mathbf{v}^2, \ldots, \mathbf{v}^i, \ldots, \mathbf{v}^{n-2}, \mathbf{v}^{n-1}\}^T \tag{88}$$

where

$$\mathbf{v}^i = \{A_{i,j}, B_{i,j}\}^T \tag{89}$$

A nontrivial solution of the equation system (Eq. (85)) exists if and only if the determinant of the coefficient matrix vanishes. This condition yields an eigenequation that has to be solved for the eigenvalues. For the present purposes, we are concerned only with those values of $\lambda_j$ that may lead to singularities in the stress field. This fact, together with the condition of continuity of the displacement field at the vertex where regions meet, implies that we are looking for eigenvalues in the range $0 < \text{Re}\,\lambda_j < 1$.

*4.3.2 Mellin Transform Technique*. Ma and Hour [124,125] tackled the harmonic problem represented by Eq. (75) by means of the Mellin transform defined in Eq. (27). The application of this technique to the harmonic equation yields an ordinary differential equation for the out-of-plane displacement $(u_z^i)^*$ in the transformed plane $(s,\theta)$, the general solution of which is

$$(u_z^i)^*(s,\theta) = a_{i,j}(s)\sin(s\theta) + b_{i,j}(s)\cos(s\theta) \tag{90}$$

Taking the Mellin transform of $r$ times the tangential stresses, these quantities can be recast in the transformed domain

$$\mathcal{M}[r\tau_{rz}^j, s] = -G_i s[a_{i,j}\sin(s\theta) + b_{i,j}\cos(s\theta)]$$

$$\mathcal{M}[r\tau_{\theta z}^j, s] = G_i s[a_{i,j}\cos(s\theta) - b_{i,j}\sin(s\theta)] \tag{91}$$

Applying the Mellin transform to the interface boundary conditions (Eq. (76)) and then introducing Eq. (91) into them, a matrix form analogous to Eq. (85) can be written. To be more specific, noting that the complex parameter $s$ is related to $\lambda$ by

$$s = -\lambda \tag{92}$$

we obtain the following expression for the elementary matrix $\mathbf{M}_\theta^i$:

$$\mathbf{M}_\theta^i = \begin{bmatrix} -\sin(\lambda\theta) & \cos(\lambda\theta) \\ -G_i\cos(\lambda\theta) & -G_i\sin(\lambda\theta) \end{bmatrix} \tag{93}$$

In this formulation, the components of the vector $\mathbf{v}$ are

$$\mathbf{v}^i = \{a_{i,j}, b_{i,j}\}^T \tag{94}$$

Hence, the elementary matrix in Eq. (93) coincides with that in Eq. (87) obtained by applying the eigenfunction expansion method if the following change of variables is made:

$$a_{i,j} = -A_{i,j}$$

$$b_{i,j} = B_{i,j} \tag{95}$$

Clearly, the eigenequations derived from the condition of a vanishing determinant of the matrix $\mathbf{\Lambda}$ are not influenced by this change of variables. The eigenvalues computed as the zeros of the eigenequations are then the same in both the formulations.

**4.4 Steady-State Heat Transfer Analogy.** The analogy between steady-state heat transfer and antiplane shear in composite regions was firstly addressed by Sinclair [130] in 1980. In both problems, the field equations for the longitudinal displacement $u_z$ and for the temperature $T$ are harmonic. As a result, the following correspondences between these two problems can be considered:

$$\nabla^2 T = 0 \Leftrightarrow \nabla^2 u_z = 0 \tag{96}$$

$$q_r = -k_i \frac{\partial T}{\partial r} \Leftrightarrow \tau_{rz} = G_i \frac{\partial u_z}{\partial r} \tag{97}$$

$$q_\theta = \frac{k_i}{r}\frac{\partial T}{\partial \theta} \Leftrightarrow \tau_{\theta z} = \frac{G_i}{r}\frac{\partial u_z}{\partial \theta} \tag{98}$$

where $q_r$ and $q_\theta$ are, respectively, the heat flux in the radial and circumferential directions and $k_i$ is the thermal conductivity of the $i$th material region.

Concerning the boundary conditions at the interfaces, the continuity of the longitudinal displacement and of the tangential stress reported in Eq. (76) can be replaced by the continuity conditions of temperature $T$ and heat flux $q_\theta$

$$T^i(r,\gamma_{i+1}) = T^{i+1}(r,\gamma_{i+1})$$

$$q_\theta^i(r,\gamma_{i+1}) = q_\theta^{i+1}(r,\gamma_{i+1}) \tag{99}$$

Moreover, the stress or displacement boundary conditions imposed along the edges at $\theta=\bar{\theta}$ of reentrant corners reported in Eq. (77) can be replaced by insulated boundary or temperature prescribed boundary conditions,

$$q_\theta^i(r,\bar{\theta}) = 0 \quad \text{for insulated boundary}$$

$$T^i(r,\bar{\theta}) = 0 \quad \text{for temperature prescribed} \tag{100}$$

When the problem is characterized by a geometric symmetry, the following boundary conditions analogous to those in Eq. (78) can be imposed along the symmetry line at $\bar{\theta}=0,\pi$ to reduce the problem to two separate subproblems:

$$q_\theta^i(r,\bar{\theta}) = 0 \quad \text{for symmetric problems}$$

$$T^i(r,\bar{\theta}) = 0 \quad \text{for skew-symmetric problems} \tag{101}$$

Cohesive-law boundary conditions in Eq. (79) have their analogous expression in the convective conditions for the steady-state heat conduction problems [131],

$$q_\theta(r,\gamma_{i+1}) = h_c[T^{i+1}(r,\gamma_{i+1}) - T^i(r,\gamma_{i+1})] \tag{102}$$

where $h_c$ denotes the convective heat transfer coefficient.

Therefore, according to this analogy, the singularities characterizing the heat fluxes $q_r$ and $q_\theta$ are the same as those for the stress singularities in $\tau_{rz}$ and $\tau_{r\theta}$ due to antiplane shear.

**4.5 de Saint-Venant Torsion Analogy.** The de Saint-Venant torsion problem of cylindrical bars and the related analogies have a longstanding fascinating history, which is reviewed in Refs. [132–134]. The mathematical formulation of this problem can be traced back to the middle of the 19th century [135,136]. From the studies on the torsional deformation of cylinders with general cross sections, the following fundamental properties can be summarized.

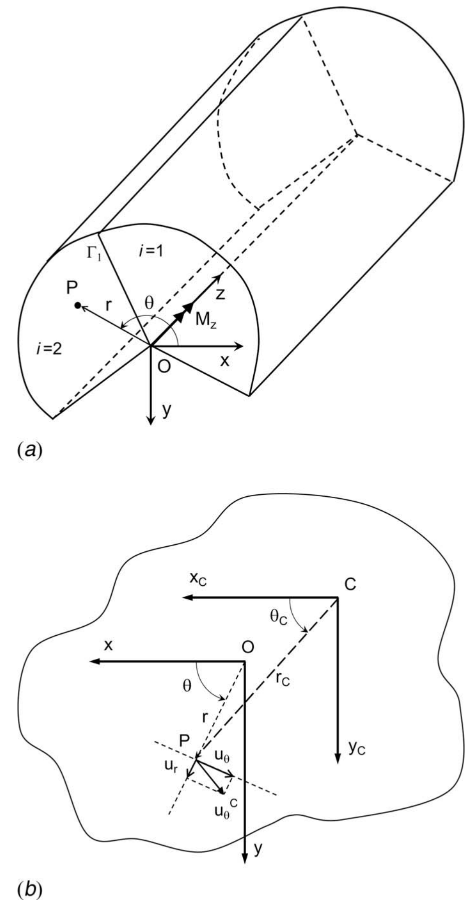


**Fig. 17 Scheme of the St. Venant torsion problem with a composite cylinder (*a*). Relationship between the displacements in polar coordinates computed in the reference systems with origin *O* (singular point) and *C* (center of twist) (*b*).**

1. The projection of each section perpendicular to the longitudinal axis on the $(x,y)$ plane rotates as a rigid body about the center of twist.
2. The amount of projected section rotation varies linearly with the axial coordinate, i.e., $\phi_z=\Theta z$, where $\Theta$ is defined as the angle of rotation between two cross sections having a unitary distance.
3. Plane cross sections do not remain plane after deformation, thus leading to a warping displacement.

The problem of stress singularity at the junction vertex of composite bars seems to have been firstly addressed by Rao in 1971 using the eigenfunction expansion method [64]. Polar coordinates were introduced in that analysis, with the origin of the reference system, $O$, placed at the junction point, as usually performed in the asymptotic analyses (see Fig. 17(*a*)). However, at that stage, no attention was paid to the position of the center of twist, $C$, whose location might not coincide with the above-mentioned origin of the reference system, $O$. If points $O$ and $C$ are more appropriately distinguished in the analysis, then two different reference systems need to be considered: the former is placed at the origin $O$, where stress singularities are expected, whereas the latter is placed at the center of twist, $C$. Then, the displacements of a generic point $P$ in polar coordinates with respect to the center of twist are

$$u_r^C = 0$$

$$u_\theta^C = \Theta z r_C$$

$$u_z^C = u_z^C(r_C, \theta_C) \tag{103}$$

where $r_C$ and $\theta_C$ are the polar coordinates of the generic point $P$ in the reference system centered in $C$ (see Fig. 17(*b*)).

These displacement components can be expressed in the reference system centered in point $O$ as follows:

$$u_r = u_\theta^C \sin(\theta - \theta_C) = \Theta z r_C \sin(\theta - \theta_C)$$

$$u_\theta = u_\theta^C \cos(\theta - \theta_C) = \Theta z r_C \cos(\theta - \theta_C)$$

$$u_z = u_z(r, \theta) \tag{104}$$

For such a system of displacements, the strain field becomes

$$\varepsilon_r = \varepsilon_\theta = \varepsilon_z = \gamma_{r\theta} = 0$$

$$\gamma_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} = \Theta r_C \sin(\theta - \theta_C) + \frac{\partial u_z}{\partial r}$$

$$\gamma_{\theta z} = \frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta} = \Theta r_C \cos(\theta - \theta_C) + \frac{1}{r}\frac{\partial u_z}{\partial \theta} \tag{105}$$

and from Hooke's law, the stresses reduce to

$$\sigma_r = \sigma_\theta = \sigma_z = \tau_{r\theta} = 0$$

$$\tau_{rz} = G_i \gamma_{rz} = G_i \Theta r_C \sin(\theta - \theta_C) + G_i \frac{\partial u_z}{\partial r}$$

$$\tau_{\theta z} = G_i \gamma_{\theta z} = G_i \Theta r_C \cos(\theta - \theta_C) + \frac{G_i}{r}\frac{\partial u_z}{\partial \theta} \tag{106}$$

where $G_i$ is the shear modulus of the $i$th material component.

The only nonvanishing equilibrium equation in the absence of body forces reduces to the well-known harmonic condition upon $u_z$, as previously derived for the antiplane shear deformation,

$$\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} = \nabla^2 u_z = 0, \quad \forall\, (r, \theta) \in \Omega_i \tag{107}$$

At this point, it is remarkable to mention that the above-described solution strategy of the torsion problem is usually referred to as *displacement formulation*, in which the problem is solved for the warp function $u_z$. In this case, the field equation is represented by the Laplace equation (Eq. (107)) with Dini-Neumann traction boundary conditions imposed on the value of the normal derivative of $u_z$ at the points pertaining to the border of the cross section.

Other solution strategies were also pursued in the literature. For instance, de Saint-Venant [136] postulated the existence of a stress function $\phi$ such that the tangential stresses can be found from it by differentiation. However, the introduction of these expressions in the compatibility equations written in terms of stresses, i.e., the so-called Beltrami–Michell equations [137,138], yields a Poisson-type governing equation for $\phi$. In this scheme, the boundary condition simplifies to a simpler Dirichlet boundary condition on the stress function; i.e., $\phi$ must be constant or must vanish at the border of the cross section. According to this solution strategy, the stress function formulation allows us to interpret the torsion problem through mechanical analogies between the stress function $\phi$ and the deformation of a membrane in tension [139], with the stream function of an ideal nonviscous fluid with constant vorticity [140] and with the equimomental lines of the electric field.

For the stress-singularity problem under consideration, the analogy has to be set between the antiplane shear deformation and the displacement formulation for the de Saint-Venant torsion since in both cases the governing variable obeys the Laplace differential equation. Moreover, it has to be pointed out that this analogy is valid for an asymptotic analysis only, i.e., for the study of the singular stress field when $r \to 0$. In fact, a direct comparison between Eqs. (106) and (73) shows that the tangential stresses for the torsion problem are the sum of two components: a regular term depending solely on $\theta$ and a possibly singular term, analogous to that for the antiplane shear deformation. Therefore, it is possible to state that the St. Venant torsion and the antiplane shear deformation problems can be considered analogous in the asymptotic sense, i.e., if and only if the analysis is restricted to the stress field near point $O$ with $r \to 0$.

## 5 Advanced Issues for Nonhomogeneous Materials

Composites frequently involve situations where nonhomogeneous materials are either present naturally or used intentionally to attain a required mechanical performance. Functionally graded materials (FGMs) are an illustrative example of two-phase synthesized materials designed in such a way that the volume fractions of the constituents vary continuously along the thickness direction to give a predetermined composition profile [37,141–143]. The potentials of these new material microstructures characterized by a given grading on the elastic modulus are under current investigation by the scientific community. In 1983, Erdogan [6] stated that "…if the crack is embedded into a nonhomogeneous medium with smoothly varying elastic properties the square root nature of the stress singularity seems to remain unchanged." The square root singularity was mathematically proven in 1987 by Eischen [144] under the assumption that the elastic modulus varies both radially and angularly with respect to the crack tip position at the same time,

$$E(r, \theta) = E_0\left(1 + rE_1(\theta) + \frac{r^2}{2}E_2(\theta) + O(r^3)\right), \quad r \to 0, \tag{108}$$

where $E_0$=const and $E_1(\theta)$ and $E_2(\theta)$ are smooth, bounded functions of $\theta$. According to this expression, cracks perpendicular, parallel, and arbitrarily oriented with respect to the direction of the elastic gradient are situations where the nature of the square root singularity remains unchanged [141,145–147].

In 2005, Carpinteri and Paggi [38] extended the mathematical formulation for the study of stress singularities in junctions between different homogeneous materials to junctions composed of angularly nonhomogeneous materials. It is, in fact, possible to assume that due to bonding, a smooth angular variation of the elastic constants takes place in the regions close to the interfaces (see Fig. 18).

Hence, a general angular grading on Young's modulus was allowed, i.e., $E_i$=$E_i(\theta)$. This possibility was not contemplated in the mathematical analysis proposed by Eischen [144]. In fact, when the elastic modulus in Eq. (108) is independent of $r$, it turns out to be equal to $E_0$, which is a constant. Functions $E_i(\theta)$ are continuous, bounded, positive, and differentiable inside any material region.

When the stress-strain relations and Eqs. (4*a*)–(4*e*) are substituted into the strain compatibility equation (Eq. (6)), the following governing equation for the Airy stress function in plane stress conditions is derived:

$$\nabla^4\Phi_i + \left[\frac{2}{E_i^2}\left(\frac{dE_i}{d\theta}\right)^2 - \frac{1}{E_i}\left(\frac{d^2E_i}{d\theta^2}\right)\right]\left[\frac{1}{r^3}\frac{\partial\Phi_i}{\partial r} + \frac{1}{r^4}\frac{\partial^2\Phi_i}{\partial\theta^2} - \frac{\nu_i}{r^2}\frac{\partial^2\Phi_i}{\partial r^2}\right]$$
$$+ \frac{1}{E_i}\frac{dE_i}{d\theta}\left[\frac{2\nu_i}{r^2}\frac{\partial^3\Phi_i}{\partial r^2\partial\theta} - \frac{2}{r^4}\frac{\partial^3\Phi_i}{\partial\theta^3} - \frac{2}{r^3}\frac{\partial^2\Phi_i}{\partial r\partial\theta}\right]$$
$$+ \frac{1+\nu_i}{E_i}\left[\frac{dE_i}{d\theta}\left(-\frac{2}{r^2}\frac{\partial^3\Phi_i}{\partial r^2\partial\theta} + \frac{2}{r^3}\frac{\partial^2\Phi_i}{\partial r\partial\theta} - \frac{2}{r^4}\frac{\partial\Phi_i}{\partial\theta}\right)\right] = 0 \tag{109}$$

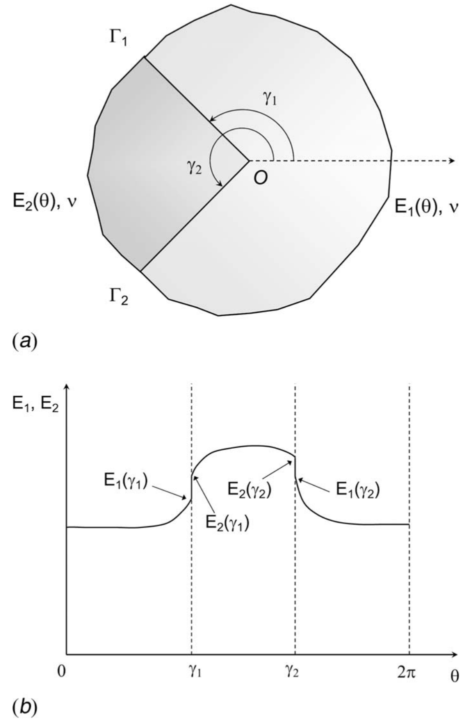


**Fig. 18 Scheme of a FGM bimaterial junction (*a*) and variation of the elastic moduli in the two-material regions (*b*). This variation can be completely general, without any kind of symmetry.**

At this point, the eigenfunction expansion method can be profitably applied [144] by considering the leading term only in the eigenfunction expansion, which contributes to the singular stress components as $r \to 0$,

$$\Phi_i(r,\theta) = r^{\lambda_j+1} f_{i,j}(\theta) \tag{110}$$

Upon substituting Eq. (110) into Eq. (109), the following fourth order, linear, homogeneous ODE with nonconstant coefficients for the unknown function $f_{i,j}(\theta)$ is found:

$$f_{i,j}^{\mathrm{IV}} - 2\frac{E_i'}{E_i} f_{i,j}''' + \left[2(\lambda_j^2+1) + 2\left(\frac{E_i'}{E_i}\right)^2 - \frac{E_i''}{E_i}\right] f_{i,j}'' + \left\{2\frac{E_i'}{E_i}[-\lambda_j^2 + (\nu_i - 1)\lambda_2 - 1]\right\} f_{i,j}' + \left\{(\lambda_j^2-1)^2 + (\lambda_j+1)(\lambda_j\nu_i - 1)\left[-2\left(\frac{E_i'}{E_i}\right)^2 + \frac{E_i''}{E_i}\right]\right\} f_{i,j} = 0 \tag{111}$$

where primes denote derivatives with respect to $\theta$. When $E_i$=const, we obtain

$$f_{i,j}^{\mathrm{IV}} + 2(\lambda_j^2+1) f_{i,j}'' + (\lambda_j^2-1)^2 f_{i,j} = 0 \tag{112}$$

which corresponds to the fourth order, linear, homogeneous ODE with constant coefficients resulting from the biharmonic condition on the Airy stress function in homogeneous materials.

The general solution of Eq. (111) has a more involved character than that of Eq. (112) since the coefficients multiplying the derivatives of $f_{i,j}(\theta)$ depend on the elastic moduli, which are, in turn, functions of the angular coordinate. It has to be noticed that if the elastic moduli have an exponential variation with respect to $\theta$, then we have a fourth order ODE with constant coefficients. In that case, in fact, the ratios $E_i'/E_i$ and $E_i''/E_i$ are constants.

In any case, at least theoretically, once the angular variation of the elastic parameters inside the material regions is specified, the integral of Eq. (111) can be determined within four unknown constants, as for the homogeneous case (see, e.g., Eq. (15)). Then, stress and displacement components can be computed according to Eq. (7). Finally, interface matching conditions (Eq. (9)) for the fully bonded case can be specified. This procedure allows us to formulate an eigenvalue problem, which is completely analogous to that illustrated for homogeneous multimaterial junctions (see Eq. (18)). As a result, eigenvalues and eigenvectors can be computed in order to describe the state of stress at the singular point.

In summary, it has to be remarked that the effect of an angular grading on the elastic moduli of the material regions leads to a different governing equation for the Airy stress function as compared to that for homogeneous materials. As a result, also in the case of a vanishing elastic mismatch at interfaces, i.e., $E_i(\gamma_i) = E_{i+1}(\gamma_i)$ (see Fig. 18), the state of stress inside the material regions depends on the specified grading, and it differs from that in homogeneous materials. Carpinteri and Paggi [38] explored two problems for the special case of an exponential angular grading on Young's modulus: a crack inside an angular FGM material and a trimaterial junction with a single FGM transition wedge. In the former case, they demonstrated that the order of the stress singularity is bounded and it cannot be higher than 0.5. In the latter, it has been shown that the presence of a FGM intermediate material is favorable since it significantly reduces the order of the stress singularity as compared to the same trimaterial junction involving homogeneous different materials. This was particularly evident when an interface crack is introduced and supersingularities can be avoided.

## 6 Conclusions and Future Perspectives

The state of the art of stress-singularity identification in classical elasticity includes a large variety of problems concerning inplane loading of a plate, antiplane shear of a wedge, plate bending, axisymmetric torsion of a cylinder, axisymmetric axial loading at a vertex, and axisymmetric axial loading at a cylindrical boundary. For all of these classes of configurations, the asymptotic expression of the singular stress field can be found in the fundamental review article by Sinclair [9,10].

In this contribution, the specific problem of stress singularities arising at multimaterial interfaces in two-dimensional linear-elastic problems has been addressed. The relevance of this subject and interdisciplinarity has contributed to the development of several research papers. They are mainly concerned with the computation of the quantities associated to stress singularities, such as the eigenvalues, the eigenfunctions, and the stress-intensity factors, for a variety of junction and wedge problems with different boundary conditions, geometrical configurations, and mechanical parameters.

Besides this huge effort in the identification of stress singularities for specific case studies, there has also been a considerable development of mathematical methods for the asymptotic analysis of stress singularities. The eigenfunction expansion method, firstly introduced by Wieghardt [66] in 1907, was then applied by Williams [67] in 1952 to the famous problem of reentrant corners of plates in extension. Thereafter, this method was largely confined to the analysis of problems admitting real eigenvalues. In the early 1970s, the Mellin transform technique and the complex function representation were profitably applied to the problems of bimaterial wedges and junctions. In 1974, Theocaris [23] firstly extended the complex function representation to the problem of multimaterial wedges and proposed a modification of the complex potentials to handle with complex singularities. In the late 1970s, Dempsey and Sinclair [7] reconsidered the eigenfunction expansion method "to remove any suggestion that it has a lacuna with respect to the Mellin transform approach…to derive conditions for a logarithmic stress singularity." However, only in 1996 it was shown by Pageau et al. [103] that "the eigenfunction expansion method can be used to determine expressions for the displacement and stress fields when the order of the stress-singularity is complex, although at first glance the formulation may appear to be limited to the case of real roots." This long debate on the equivalence of the mathematical methods has clearly produced a lack of standardization well evidenced by the large number of case studies duplicated in the literature.

Hence, the main product of this review article was to provide, in a historical retrospective, a unification of the available mathematical techniques for the computation of the order of the stress singularity in two-dimensional interface problems. Presenting the problem in an extremely synthetic matrix form, the equivalence of the eigenfunction expansion method, of the Muskhelishvili complex function representation and of the Mellin transform technique has been mathematically demonstrated. In fact, it has been shown that all these mathematical methods lead to the same expression of the coefficient matrix of the nonlinear eigenvalue problem. As a result, all these techniques can be used for the computation of real and complex eigenvalues.

Concerning the in-plane loading problem in elasticity, the existing analogy with the Stokes flow of dissimilar immiscible fluids has been put into evidence. From the historical point of view, the existence of this analogy implies that the eigenfunction expansion method was independently applied in fluid dynamics even before the fundamental paper by Williams [67]. As regards the out-of-plane loading, the analogies with the steady-state heat transfer across different materials and with the St. Venant torsion of composite bars have also been discussed.

Finally, the recent advanced issues for the stress singularities due to joining of angularly nonhomogeneous elastic wedges have been presented. Clearly, much work remains to be done in this research area in order to fully elucidate the potentials of the use of FGMs for the removal of stress singularities in junction problems.

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