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Letter to the Editor

Comments on "Is the cause of size effect on structural strength fractal or energetic-statistical?" by Bažant & Yavari [Engng Fract Mech 2005;72:1–31]

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Abstract

Aim of this letter to the Editor is at replying to the criticisms raised by Bažant and Yavari [Bažant ZP, Yavari A. Is the cause of size effect on structural strength fractal or energetic – statistical? Engng Fract Mech 2005;72:1–31] against the fractal approach to the size-scale effects on the mechanical properties of materials and the concept of the Multi-Fractal Scaling Law presented by Carpinteri [Carpinteri A. Scaling laws and renormalization groups for strength and toughness of disordered materials. Int J Solids Struct 1994;31:291–302]. These criticisms will be analysed thoroughly, showing how they also contain some mistakes and misunderstandings. The presented elucidations should redirect the discussion to a more correct scientific debate.

Keywords: Size-scale effects; Fractals; Strength; Fracture energy

1. Introduction

In the paper by Bažant and Yavari [1] several criticisms are raised against the concept of the Multi-Fractal Scaling Law presented by Carpinteri [2]; the Authors question its validity and even argue that it lacks sound physical and mathematical basis. Quoting from their paper (p. 13, item 3): "The 'MFSL' was based on a series of hypotheses but does not follow from these hypotheses by a valid mathematical procedure".

The paper by Bažant and Yavari, as well as two previous papers of the first author [3,4], seriously question the scientific work by Carpinteri and co-workers. Research on the "fractal conjecture" by the group from the Politecnico di Torino, carried out since 1992, has resulted in more than 60 peer-reviewed papers published in the most prestigious journals in the fields of fracture mechanics, mechanics of materials and structural engineering. During these years, Carpinteri and co-workers have constantly presented their results in the most

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important conferences throughout the world, accepting the debate and receiving support from several colleagues.

It is worth noting that experimental tests, as evidenced from the first publications early in 1992 [5,6], have often supported the soundness of the fractal approach and of the Multi-Fractal Scaling Law (MFSL). Indeed, Carpinteri and co-workers have always defined clearly the limits of this law, i.e., that when a strong, energy-driven fracture process is activated, as in the presence of deep notches in the structure, an opposite curvature in the bi-log strength vs. size diagram should be considered. This is exactly the case when Bažant's original Size Effect Law (SEL) applies. On the contrary, when the role of microstructural disorder and of self-similar features (i.e., fractality) dominates the damage and fracturing processes, the MFSL permits interpolation of the experimental data better than SEL.

On the other hand, Bažant appears to appreciate this, as his strategy in arguing against the "fractal argument" clearly changed during the years. At the beginning, Bažant simply tried to oppose his 1984 formula (SEL [7]) to the MFSL by Carpinteri. His opposition exploited the same tactics he followed to demonstrate that Weibull-type size effect is not applicable to concrete structures.

At the FraMCoS-2 Conference of Zurich in 1995, a report [8] was distributed to the participants showing how the experimental tests in the literature strongly support the MFSL conjecture when large notches are absent. The reaction of Bažant was a fierce opposition to the existence of a finite asymptotic strength for large structural sizes, which was in contradiction with the SEL. In the following years, however, Bažant realised that laboratory tests, and more generally, structural mechanics, could not be forced to fit SEL. Therefore, he introduced the so-called "Universal SEL" [9,10], where an asymptote can be reached for both small and large sizes, and, more importantly, the same MFSL upwards concavity may be obtained under certain values of the parameters.

After the introduction of the Universal Size Effect Law, the scientific community tried to redirect the discussion into a widely accepted framework, through the RILEM Technical Committee *"Quasi-brittle fracture scaling and size effect"*, chaired by Bažant himself, with the presence of prominent scientists in the field. The work of the Committee was finalised in the publication of its final report [11], wherein the different theories of size effects (including Weibull's) were described and compared.

In this document, when describing the Multifractal Theory, it is reported that Bažant demonstrated that the MFSL for strength can be obtained as a special case of the Universal SEL. The original SEL (1984) is never quoted in the report, and a non-zero finite term is included in the formula which takes into account a finite strength at large sizes. Although some skepticism about the fractal argument was outlined in the same report by Bažant, Gettu, Jirasek, Planas and Xi, the other members of the Committee did not take a position and, in other papers [12,13], expressed their independent point of view, also showing results in favour of the MFSL.

The criticism by Bažant and co-workers rests on very weak bases. As a matter of fact, the mathematical foundations of the multifractal theory may be considered in the framework of Renormalisation Group Theory. The consistency of the 1/2 hypothesis for the exponents of the MFSLs, originally based on simple statistical arguments, has been also proven by relating the fractal exponents for tensile strength, critical strain and fracture energy [14,15]. Eventually, the so-called fractal mechanics was introduced in more recent years [16–19], based on Fractional Calculus.

Other mistakes by Bažant should be noted. For instance, after more than 10 years, Bažant still seems to confuse lacunar fractals (where stress is defined) with invasive fractals (where energy is dissipated). This situation makes serious and correct scientific debate more difficult.

Finally, we recall that important code-of-practice formulae taking into account size effects (see e.g. the FIB formulas [20] suggested for shear strength in reinforced concrete) agree with the MFSL conjecture.

2. Slope of the MFSL asymptote at the smaller scales

Bažant and Yavari [1] aim at showing that the scaling law for strength at the smaller scales, i.e. $\sigma_N \propto D^{-1/2}$, cannot be based on the statistical treatment presented in [2]. They insist on this point throughout the whole paper; quoting again from [1]:

(i) (p. 13, item 4): "the value $-1/2$ is an unproven conjecture which does not follow from the fractal hypothesis";
(ii) (p. 26, item 1): "The exponent $-1/2$ attributed to the small-size asymptotic scaling law is supposed to be solely a consequence of a peculiar situation called 'the extreme disorder'";
(iii) (p. 26, last paragraph): "the property that the left-size asymptote of the MFSL in a bi-logarithmic plot should have the slope $-1/2$ must be considered as unproven by the fractal argument. [...] If only the fractal viewpoint is considered, this property is merely an empirical assumption".

In this section, we will review and reject these criticisms against the fractal interpretation of the size effects, by clarifying some aspects that have been misunderstood and confused by Bažant and Yavari, and showing how their discussion contains flaws and mistakes.

This slope, not only follows from the statistical treatment presented in [2], but is also explained in the framework of the *Fractal Cohesive Model* [14], that has been confirmed by experiments, in our opinion, very convincingly [14,15]. In this framework, indicating with d_σ, d_ε and d_G the non-integer exponents for tensile strength, critical strain and fracture energy, respectively, it has been shown that the following equation can be written

$$d_\sigma + d_\varepsilon + d_G = 1 \tag{1}$$

At the smaller scales, the collapse is governed by the canonical critical strain ε_c and continuum damage mechanics holds. In this case, the damage is diffused (with uniform strain in the bulk) and one obtains $d_\varepsilon = 0$. Thus, the previous relation becomes $d_\sigma + d_G = 1$. On the other hand, the maximum value for d_G is 1/2, since it implies a fractal dimension of the dissipation domain $\Delta_G = 2.5$. This would correspond to a Brownian crack surface due to kinematic reasons of crack opening and closing. As a consequence, $d_\sigma = 1/2$ is the limit value at the smaller scales.

In any case, the criticisms of the statistical treatment presented by Carpinteri [2] are unjustified. On p. 26 of their paper, Bažant and Yavari [1] affirm that "defects of maximum size $a_{\max}$ cannot have the same probability distribution of a as the ensemble of all defects, but could have only one of the three possible extreme value distributions (Fréchet, Weibull or Gumbel), of which only the Weibull distribution would be realistic here because a non-negative threshold on a exists". In other words, they state that $a_{\max}$ cannot have a power-law distribution and, consequently, the assumption $a_{\max}/b = \text{const.}$ (from which the $-1/2$ (LEFM) slope of the size-scale effects follows) should be unjustified.

A first remark concerns the second part of the statement, which is definitely wrong: the limit distribution for a heavy tailed distribution, such as the Pareto, or the Cauchy, is not the Weibull, as erroneously stated by Bažant and Yavari [1], but the Fréchet (this result was already used by Freudenthal [21] almost 40 years ago). Moreover, the existence of "a non-negative threshold on a" (presumably upper) is, in any case, merely speculative and unproven.

A second remark concerns the fact, rigorously provable in the framework of Extreme Value Theory, that $a_{\max}/b = \text{const.}$ on average [22]. In other words, although it is true that "defects of maximum size $a_{\max}$ cannot have the same probability distribution of a as the ensemble of all defects", it can be shown that, starting from a power-law distribution of flaw sizes, the maximum defect size is proportional, on average, to the structural scale.

In addition, this result has been confirmed by Monte Carlo numerical simulations [22]. Therefore, the hypothesis $a_{\max}/b = \text{const.}$ is well justified both theoretically (in two different alternative ways) and numerically.

In conclusion, the fractal-statistical treatment in [2] should be considered valid. The exponent –1/2 attributed to the small-size asymptotic scaling law is definitely not an assumption, or "solely a consequence of a peculiar situation called 'the extreme disorder'". Rather, it is a consequence of LEFM when a fractal distribution (with power-law tail) describes the flaw size distribution inside the material.

3. Further considerations

(1) The paper by Bažant and Yavari is ill-posed also in its title. In fact, fractals can be deterministic or random, i.e. statistically self-similar [2]. The former are only mathematical models, whereas the latter are the shapes usually met in real systems. Thus, fractals and statistics are not in contrast. Furthermore, an ener-

getic-statistical approach, based on a truncated power-law distribution of the defect size inside a material coupled with LEFM considerations, has been recently proposed in the literature by Carpinteri and co-workers [23–27]. Interestingly, the size effects on tensile strength and fracture energy provided by this model substantially coincide with the ones provided by the multifractal approach. Again, fractal geometry, statistical distributions and energy balances do not contradict each other.

(2) P. 13, item 4: Bažant and Yavari have not caught the relation between the mono-fractal and the multifractal laws. The former should be seen as an approximation of the latter, valid only in a certain size-scale range.

(3) P. 13, item 7: The statement that "the fractality needs to be experimentally observed through about six orders of magnitude for the fractal scaling to be considered a very good model" is without any justification. It is understood that the proportion between smallest and largest aggregates should reflect such a ratio, whereas the fractality – or better the renormalisation group – prevailingly springs from the ratios that the considered property presents at the different observation scales.

(4) P. 18, third paragraph: The results obtained by Yavari in several papers dealing with fractal cracks are summarized, including the results about the size effect. Thus, it seems that some researchers can argue about fractals and size effect (Yavari and co-workers) and some others cannot (Carpinteri and co-workers). Even worse, the "alternative approach" by Yavari is not new, at least for self-similar cracks: Eq. (20) coincides with Eq. (15) in Carpinteri and Chiaia [28].

(5) P. 19, Section 4.3, first and second paragraphs: Bažant and Yavari confuse the ligament fractality (our conjecture) with the crack fractality (a conjecture not taking anywhere).

(6) P. 22, last paragraph, and p. 23, first paragraph: Bažant and Yavari state that a null volume implies a null mass. This is not true since, for example, the archetype of fractal solids, the Menger sponge, has a fractal mass density, so that the mass is finite even if the volume is zero.

(7) P. 23, Section 4.4.3, last paragraph: A definition of fractal stress vector is given although, as stated in the same paper, that definition is not rigorous because the vector normal to an invasive fractal surface is not defined. Nevertheless, that "fractal stress" is used throughout [29]. On the other hand, the fractal stress vector defined in [17] has formally the same expression but is defined on a lacunar fractal set, which presents a unique normal.

(8) P. 24, second and third paragraphs: The fractal generalisation of the stress tensor is immediate, if we consider lacunar fractal infinitesimal areas even in the case of the shearing stresses. Obviously, the same generalisation holds also for the strain tensor, included the shearing strains.

(9) P. 24, fourth paragraph: Bažant and Yavari state that in Carpinteri et al. [17] we used fractional derivatives in the definition of the constitutive equations. This is an incorrect statement, since we simply wrote Eq. (12) to show an example of application of fractional derivatives in mechanics. Furthermore, in Carpinteri et al. [17], there is no need of constitutive equation because that paper deals with the Principle of Virtual Work and, as should be well known, its validity is independent of the material behaviour.

(10) P. 26, second paragraph: Bažant and Yavari erroneously affirm that the exponent β "reduces to 1 for the non-fractal case". By carefully reading p. 299 in [2] it is clear that β has nothing to do with fractals. This exponent simply characterizes the power-law behaviour of the maximum defect size a_{max}.

(11) P. 26, item 3: Bažant and Yavari state: "...exponent N, allegedly 'measuring in some way the degree of disorder', is unclear". On the contrary, the exponent N in Eq. (29) in [2] is an evident measure of disorder: as N increases, the tail of the distribution becomes lighter, and in the limit for $N \to 1$ the tail disappears, being $P(a_0) = 1$. The maximum defect size is in this case deterministic and equal to a_0. On the contrary, as N decreases, the tail becomes larger and the probability of finding defects of size exceeding a_0 increases, indicating a more disordered microstructure.

References

[1] Bažant ZP, Yavari A. Is the cause of size effect on structural strength fractal or energetic – statistical? Engng Fract Mech 2005;72:1–31.

[2] Carpinteri A. Scaling laws and renormalization groups for strength and toughness of disordered materials. Int J Solids Struct 1994;31:291–302.

[3] Bažant ZP. Scaling of quasibrittle fracture and the fractal question. ASME J Mater Technol 1995;117:361–7.
[4] Bažant ZP. Scaling of quasibrittle fracture: hypotheses of invasive and lacunar fractality, their critique and Weibull connection. Int J Fract 1997;83:41–65.
[5] Carpinteri A. Fractal nature of material microstructure and size effects on apparent mechanical properties. Mech Mater 1994;18:89–101. Internal Report, Laboratory of Fracture Mechanics, Politecnico di Torino, No. 1/92, 1992.
[6] Carpinteri A, Ferro G. Apparent tensile strength and fictitious fracture energy of concrete: a fractal geometry approach to related size effects. In: Rossmanith HP, editor. Fracture and damage of concrete and rock (Proceedings of the 2nd international conference, Vienna, Austria, 1992). London: E&FN SPON; 1993. p. 86–94.
[7] Bažant ZP. Size effect in blunt fracture: concrete, rock, metal. J Engng Mech (ASCE) 1984;110:518–35.
[8] Carpinteri A, Chiaia B, Ferro G. Multifractal scaling law: an extensive application to nominal strength size effect of concrete structures. Atti del Dipartimento di Ingegneria Strutturale, Politecnico di Torino, No. 51, 1995.
[9] Bažant ZP. Scaling of quasibrittle fracture: asymptotic analysis. Int J Fract 1997;83:19–40.
[10] Bažant ZP. Size effect in tensile and compression fracture of concrete structures: computational modelling and design. In: Mihashi H, Rokugo K, editors. Fracture mechanics of concrete structures. Freiburg, Germany: Aedificatio Publishers; 1998. p. 1905–22.
[11] Bažant ZP et al. Quasibrittle fracture scaling and size effect. Mater Struct (RILEM) 2004;37:547–68.
[12] van Vliet M, van Mier JGM. Size effect of concrete and sandstone. Heron 2000;45:91–108.
[13] van Mier JGM, van Vliet MRA. Influence of microstructure of concrete on size/scale effects in tensile fracture. Engng Fract Mech 2003;70:2281–306.
[14] Carpinteri A, Chiaia B, Cornetti P. A scale-invariant cohesive crack model for quasibrittle materials. Engng Fract Mech 2002;69:207–17.
[15] Carpinteri A, Chiaia B, Cornetti P. On the mechanics of quasi-brittle materials with a fractal microstructure. Engng Fract Mech 2003;70:2321–49.
[16] Carpinteri A, Cornetti P. A fractional calculus approach to the description of stress and strain localization in fractal media. Chaos Solitons Fractals 2002;13:85–94.
[17] Carpinteri A, Chiaia B, Cornetti P. Static-kinematic duality and the principle of virtual work in the mechanics of fractal media. Comput Methods Appl Mech Engng 2001;191:3–19.
[18] Carpinteri A, Chiaia B, Cornetti P. A disordered microstructure material model based on fractal geometry and fractional calculus. Z Angew Math Mech 2004;84:128–35.
[19] Carpinteri A, Chiaia B, Cornetti P. The elastic problem for fractal media: basic theory and finite element formulation. Comput Struct 2004;82:499–508.
[20] Comité Euro-international du Béton. CEB-FIP Model Code 1990. Thomas Telford Ltd.; 1991.
[21] Freudenthal AM. Statistical approach to brittle fracture. In: Liebowitz H, editor. Fracture, vol. 2. Academic Press; 1968. p. 591–619.
[22] Carpinteri A, Puzzi S. Fractals, statistics and size-scale effects on concrete strength. In: FraMCoS-6 international conference, Catania, Italy, June 17–22, 2007, accepted.
[23] Carpinteri A, Cornetti P. Size effects on concrete tensile fracture properties: an interpretation of the fractal approach based on the aggregate grading. J Mech Behav Mater 2002;13:233–46.
[24] Carpinteri A, Cornetti P, Puzzi S. A stereological analysis of aggregate grading and size effect on concrete tensile strength. Int J Fract 2004;128:233–42.
[25] Carpinteri A, Cornetti P, Puzzi S. Size effect upon grained materials tensile strength: the increase of the statistical dispersion at the smaller scales. Theor Appl Fract Mech 2005;44:192–9.
[26] Carpinteri A, Cornetti P, Puzzi S. Scale effects on strength and toughness of grained materials: an extreme value theory approach. Strength Fract Compl 2005;3:175–88.
[27] Carpinteri A, Cornetti P, Puzzi S. Scaling laws and multiscale approach in the mechanics of heterogeneous and disordered materials. Appl Mech Rev 2006;59:283–305.
[28] Carpinteri A, Chiaia B. Crack-resistance behavior as a consequence of self-similar fracture topologies. Int J Fract 1996;76:327–40.
[29] Yavari A. Generalization of Barenblatt's cohesive fracture theory for fractal cracks. Fractals 2002;10:189–98.