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New unified laws in fatigue: From the Wöhler's to the Paris' regime

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Abstract

Generalizations and unification of the celebrated Paris' and Wöhler's laws for fatigue crack propagation are derived by applying the recently developed quantized (or finite) fracture mechanics. In particular, three generalized Paris', Wöhler's or unified laws are proposed and compared, demonstrating their applicability for predicting the life time of structures containing from small (the Wöhler's regime) to large (the Paris' regime) propagating fatigue cracks.

Keywords: Fatigue; Cracks; Paris; Wöhler; Quantized fracture mechanics

1. Introduction

Fatigue life prediction is still an empirical science rather than a theoretical one, despite being a relatively old subject having nearly 150 years of history [1], as described in a number of books or review papers (e.g., [2–6]). In the old days, strength vs. number of cycles to failure (SN) curves were measured and generally maintained their empirical nature even when simplified equations like the Basquin power law [7] emerged, or when simpler rules for the fatigue limit or for describing various other effects (notch geometry, size-scale, roughness) were recognized. This was generally driven by the need to have engineering rules still used today when designing against fatigue with the safe-life approach, i.e., virtually for infinite life. With the advent of fracture mechanics, a more ambitious task was undertaken, i.e., to predict, or at least, to understand the propagation of cracks. It clearly emerged that the propagation ''speed'' was far to be constant in time: generally, it was clear that the crack advance was larger for increasing stress amplitudes, but also for larger cracks, until the pioneering work of Paris et al. [8,9] who suggested to use the Irwin's stress-intensity factor (more precisely, its range), to characterize the rate of crack advance per cycle, since many data collapsed in a single power law. Since then, a lot of work has been done to understand more of Paris' law, its deviations, but we are still far from a complete understanding (see [10]).

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These different approaches in turn imply that there must be deviations from these two limit power laws (Wöhler's [1,7] and Paris' [8,9] regimes); one key responsible is certainly cyclic plastic deformation at the crack tip, which in the Paris' regime satisfies small-scale yielding, i.e., either sufficiently low loads or sufficiently "long" cracks [11–13]. A "long" crack is nearly always related to the fatigue limit and threshold stress-intensity factor range (see the definition of intrinsic crack from El Haddad et al. [14]) whereas, the definition of "short" is more correctly taken with respect to the size of the process zone. Hence, it should depend also on the load level: in the limit of static failure, the equivalent size for the transition from strength- to toughness-controlled failure [15] is various orders of magnitude larger than the El Haddad size. This means that what is "short" in this range is certainly "long" around the fatigue limit region. Thus, whereas Wöhler's law works only for small cracks, Paris' law can be applied only to large cracks and their unification and transition remains unclear.

We shall not try here to compete with the previous fatigue models (see the review by Newman [10] for an extensive historical perspective). However, in the case of fatigue, mechanical models need to make significant assumptions, since information on cyclic elasto-plasticity is needed.

A possible alternative approach, explored in this paper, for a unified treatment of short and long cracks, is an "intermediate asymptotic matching" of the well-known empirical fatigue laws by Wöhler and Paris, obtained by applying quantized, or finite, fracture mechanics concepts [16–20]. "Universal" fatigue laws are thus derived, that could strongly improve our life-time predictions, e.g., for Air Force and Ministry critical components, usually still based on the celebrated Paris' law.

2. New unified laws in fatigue

Let us consider a structure subjected to an applied load σ, pulsating between $\sigma_{\min}$ and $\sigma_{\max}$, where $\Delta\sigma = \sigma_{\max} - \sigma_{\min}$, and containing a crack of length a. The stress-intensity factor at the crack tip be $K(a,\sigma)$; in addition $K(a,\sigma_{\min}) = K_{\min}(a)$, $K(a,\sigma_{\max}) = K_{\max}(a)$ and $\Delta K(a) = K_{\max}(a) - K_{\min}(a)$. For large crack, the fatigue crack growth can be classically deduced according to the Paris' law, i.e., $\frac{da}{dN} = C(\Delta K(a))^m$, where N is the number of cycles and C, m are (nominally) "material" constants. Thus by integration, the fatigue life in terms of limit cycle number N_C^P is easily estimated. On the other hand, for plain specimens (or more realistically for very small cracks) the fatigue life predictions are usually derived according to the Wöhler's law, i.e., $N_C^W = \frac{\overline{C}}{\Delta\sigma^{\overline{m}}}$, where $\overline{C}, \overline{m}$ are again (nominally) "material" constants. Wöhler's law does not take into account the presence of cracks, differently from the Paris' law. Note that this picture is dual to the static case, where maximum stress and stress-intensity factor criteria for structural strength [15] are used respectively for plan and cracked specimens. These two criteria have been recently unified by means of quantized (or finite) fracture mechanics [16,17], i.e., simply relaxing the assumption of a continuum crack advancement in the Griffith's energy balance (see also [18–20]).

Generalized Paris' law. According to quantized fracture mechanics [17], instead of the stress-intensity factor $K(a)$ we have to consider $K^*(a,\Delta a) = \sqrt{\langle K^2(a)\rangle_a^{a+\Delta a}}$, where Δa is the "fracture quantum", a microstructural parameter (the symbol $\langle\cdot\rangle$ denotes the mean value operator). Note that the fracture quantum is similar to the "intrinsic crack length" already proposed by several authors; the novelty is here represented by the new fatigue laws, involving such a parameter, rather than by its introduction. Thus, in the study of fatigue crack growth we have proposed the following generalized Paris' law [19,21,22]:

$$\frac{da}{dN} = C(\Delta K^*(a,\Delta a))^m \tag{1a}$$

from which the total number of cycles $N_C^{P^*}$ at the fatigue collapse, arising when the crack length has reached its critical final value a_C, can be deduced as

$$N_C^{P^*} = \frac{1}{C}\int_a^{a_C} \frac{da}{(\Delta K^*(a,\Delta a))^m} \tag{1b}$$

In the criterion of Eq. (1b) we can fix Δa to recover, in the limit case of $a \to 0$, the Wöhler's prediction, i.e.,

$$\Delta a: \quad N_C^{P^*}(a \to 0) = N_C^W \tag{1c}$$

Thus, Eq. (1b), with the position of Eq. (1c), can be considered the first generalized law. Note that such a law is of very simple application, and would allow one to study not only the final condition but also the evolution of the fatigue crack growth $N^{P^*}(\tilde{a})$, where $a \leqslant \tilde{a} \leqslant a_C$ is the actual crack length.

Generalized Wöhler's law. According to [23] we can generalize the Wöhler's law by replacing the stress σ with the mean value σ^* of the stress field ahead the crack tip (let us say $\sigma_{yy}(x)$ if the crack is placed along x, $x = 0$ defining the crack tip) along the fracture quantum, i.e., $\sigma^*(a, \Delta a) = \langle \sigma_{yy}(x) \rangle_0^{\Delta a}$. Thus

$$N_C^{W^*} = \frac{\overline{C}}{(\Delta\sigma^*(a, \Delta a))^{\overline{m}}} \tag{2a}$$

Here the parameter Δa must be determined in order to recover, in the limit case $a \to a_C$, the asymptotic Paris' prediction, i.e., $N_C^P(a \to a_C)$:

$$\Delta a\colon\ N_C^{W^*}(a \to a_C) = N_C^P(a \to a_C) \tag{2b}$$

Eq. (2a), with the position of Eq. (2b), is the second generalized law. Note that such a criterion is not of very simple analytical application, requiring the complete stress field at the crack tip.

Unified Paris–Wöhler law. In general, the previous two generalized laws are expected to give similar but not identical results. On the other hand, following the procedure outlined in [24] for static failure (a dynamic extension is proposed in [19]), we can require the results to be identical by removing the assumption of a constant crack advancement. In other words, $\Delta a(a)$ is such that

$$\Delta a(a)\colon\ N_C^{P^*} = N_C^{W^*} \tag{3a}$$

Introducing Eqs. (1b) and (2a) into Eq. (3a), by derivation we can write the differential equation providing the function $\Delta a(a)$:

$$C\overline{C}\frac{\mathrm{d}}{\mathrm{d}a}\left[(\Delta\sigma^*(a, \Delta a(a)))^{-\overline{m}}\right] + (\Delta K^*(a, \Delta a(a)))^{-m} = 0 \tag{3b}$$

Thus, by integration, the function $\Delta a(a)$ can be deduced. The integration constant can be obtained imposing the validity of Eq. (3a). Introducing the derived function $\Delta a(a)$ into Eq. (1b) or into Eq. (2a) will give the same prediction $N_C^{PW^*}$. Thus, Eq. (1b) coupled with Eq. (2a), with the position of Eq. (3a), represents the third unified law. Note that such a criterion is not of very simple analytical application, requiring the complete stress field at the crack tip as well as the solution of a differential equation.

Discussion. Numerically all the three different criteria, formerly introduced by Pugno in [19], can be easily applied. However, Eq. (1) are of very simple application, also analytically, requiring the integration of a known function (the stress-intensity factor for the specified geometry and type of loading, usually available from the related Handbooks). All the criteria recover the prediction of the Wöhler's and Paris' laws for small and large crack sizes respectively, thus representing "universal" laws in fatigue.

The Paris's and Wöhler laws, and thus our three generalizations, do not consider explicitly the influence of the $R = \sigma_{min}/\sigma_{max}$ ratio on fatigue crack growth rate $\mathrm{d}a/\mathrm{d}N$. On the other hand, including such an effect is straightforward, by using Elber's [25] crack closure treatment. Elber noticed that at low loads the stiffness was close to that of an uncracked structure; he rationalized that the low incremental compliance, a consequence of the presence of the crack, at low tensile loads was due to the contact between crack surfaces, i.e., to crack closure. He proposed that crack closure occurs as a result of the development of the crack-tip plastic zone, where the yield stress of the material is exceeded. Thus, as the crack grows, a track of plastically deformed zone is developed while the surrounding body remains linear elastic. During unloading, the plastic zone causes the crack surfaces to contact each other before zero load is reached. Elber further postulated that crack closure decreased the fatigue crack growth rate $\mathrm{d}a/\mathrm{d}N$ by reducing the effective stress-intensity range, ΔK, as a function of the stress ratio R. Thus, he introduced a new stress-intensity range, the "effective" one, ΔK_{eff}, to be used in the Paris' equation as $\Delta K_{eff} = K_{max} - K_{op} = U\Delta K$, $U \approx 0.5 + 0.4R$. Thus, $K_{op} = (1/(1 - R) - U)\Delta K$ represents the stress intensity at which the crack opens. U is the empirical relationship between ΔK and ΔK_{eff} derived by Elber, and confirmed by many subsequent researchers, as a function of R. Clearly in our treatment considering $\Delta K^*_{eff} = U\Delta K^*$ instead of ΔK^* allow us to self-consistently consider the effect of the R ratio in an explicit way.

3. On the Paris' "constants"

Several workers have noticed that the C and m Paris' "constants" exhibit a correlation of the form $C = AB^m$ [26–28], considered having no fundamental significance and a result of the logarithmic method conventionally used to plot the data and of the nature of the dimensions of the physical quantities used in the Paris' equation [26]. For example $A = 7.6 \times 10^{-7}$ m/cycle, $B = 1.81 \times 10^{-2}\ (\mathrm{MPa}\sqrt{\mathrm{m}})^{-1}$ for steels and $A = 2.5 \times 10^{-6}$ m/cycle, $B = 4.26 \times 10^{-2}\ (\mathrm{MPa}\sqrt{\mathrm{m}})^{-1}$ for Al alloys have been experimentally observed [27]. We note that interpreting the final fatigue instability as a brittle fracture, thus arising for $K_{\max}(a_C) = K_C$ (K_C is the material fracture toughness) at the fixed crack speed c, would exactly imply $C = AB^m$ with $A = \mathrm{d}a/\mathrm{d}N|_C = c/\dot{N}$, and $B = [(1 - R)K_C]^{-1}$ (the dot over the symbol represents the time derivation). On the other hand, since A is clearly observed $\dot{N}$ independent (otherwise in the Paris' law the time would appear instead of the number of cycles) the final collapse must not be a pure brittle collapse, but rather intermediate between a brittle and a fatigue crack propagation, perhaps a brittle "grain-trapped" crack propagation. This is confirmed by the fact that $B|_{R=0}$ is found to be of the order of K_C^{-1} for many materials and structures: for example, considering the experiments reported in [27] ($R = 0$), we deduce the reasonable toughness $K_C \approx 55.2\ \mathrm{MPa}\sqrt{\mathrm{m}}$ for steels or $K_C \approx 23.5\ \mathrm{MPa}\sqrt{\mathrm{m}}$ for Al alloys.

Furthermore, as discussed in the previous section, to take into account the R ratio effect Elber [25] basically proposed the following substitution in the Paris' law: $C(R) \to C|_{R=0}(1 + R)^m$, similarly to $C(R) \to C|_{R=0}(1 - R)^{-m}$, derived according to our expression for B. Note that these two dependences for $R \to \mp 0$ are asymptotically identical. Obviously, interpreting the deviation of a "universal" law from that of Paris as a variation of its constants C and m would quantify their variability not only as a function of the R ratio but also of the crack length and stress range. For example, applying the generalized Paris' law with $m > 2$ (usual case) to the Griffith's problem, we find the result identical to that predicted by a Paris' law in which the constant C is replaced by $C(a, \Delta\sigma) = C(1 + \Delta a(\Delta\sigma)/2a)^{m/2-1}$ (see Eqs. (5d) and (5e), next Section). Thus, larger values of C are expected for smaller cracks, in qualitative agreement with the fractal prediction [29], for which $C(a) = C(D{=}1)a^{-(D-1)(1+m/2)}/D$, where $1 < D < 2$ is the fractal dimension of the crack. However, quantitative differences are expected: in particular, our universal law automatically recovers the celebrated Paris' law for large cracks, whereas the fractality for such a case would imply $\mathrm{d}a/\mathrm{d}N = 0$; furthermore we can "universally" quantify the dependence of C also on $\Delta\sigma$, as well as on the structural geometry (that does not appear in the previous example, just considering an infinite plate).

4. Example of application: the Griffith's case

As an example of application, let us consider the Griffith's case (infinite elastic plate with a symmetric crack of length $2a$). For this case the stress-intensity factor (mode I) is $K = \sigma\sqrt{\pi a}$ and the full stress field at the crack tip is $\sigma_{yy} = \frac{1}{\sqrt{(1-(a/(a+x))^2)}}$ (where x is the distance from the tip). Accordingly, by integration

$$K^* = \sigma\sqrt{\pi(a + \Delta a/2)} \tag{4a}$$

$$\sigma^* = \sigma\sqrt{1 + 2a/\Delta a} \tag{4b}$$

Generalized Paris' law. By applying Eq. (1b), it follows:

$$N_C^{P^*} = \frac{1}{C\Delta\sigma^m\pi^{m/2}} \frac{\left(a_C + \frac{\Delta a}{2}\right)^{1-m/2} - \left(a + \frac{\Delta a}{2}\right)^{1-m/2}}{1 - m/2} \tag{5a}$$

From Eq. (1c) Δa can be obtained by solving

$$\frac{1}{C\Delta\sigma^m\pi^{m/2}} \frac{\left(a_C + \frac{\Delta a}{2}\right)^{1-m/2} - \left(\frac{\Delta a}{2}\right)^{1-m/2}}{1 - m/2} = \frac{\overline{C}}{\Delta\sigma^{\overline{m}}} \tag{5b}$$

Assuming $a_C \gg \Delta a$ (for the analyzed Griffith's case, $a_C \to \infty$) gives

$$\Delta a = 2\left(a_C^{1-m/2} - \frac{C\overline{C}\pi^{m/2}(1 - m/2)}{\Delta\sigma^{\overline{m}-m}}\right)^{\frac{1}{1-m/2}} \tag{5c}$$

For $m > 2$ (usual case), Eqs. (5a) and (5c) become

$$N_C^{P^*} = \frac{1}{C\Delta\sigma^m \pi^{m/2}} \frac{\left(a + \frac{\Delta a}{2}\right)^{1-m/2}}{m/2 - 1} \tag{5d}$$

$$\Delta a = 2\left(\frac{C\overline{C}\pi^{m/2}(m/2-1)}{\Delta\sigma^{\overline{m}-m}}\right)^{\frac{1}{1-m/2}} \tag{5e}$$

Let us consider $m = 2$, even if materials usually possess larger Paris' exponents; for such a case Eq. (5a) becomes

$$N_C^{P^*} = \frac{1}{C\Delta\sigma^2\pi} \ln\left(\frac{a_C + \frac{\Delta a}{2}}{a + \frac{\Delta a}{2}}\right) \tag{5f}$$

and thus, by applying Eqs. (1c) and (5c) becomes:

$$\Delta a = \frac{2a_C}{e^{\pi C\overline{C}} - 1} \tag{5g}$$

Generalized Wöhler's law. Applying Eq. (2a)

$$N_C^{W^*} = \frac{\overline{C}}{\Delta\sigma^{\overline{m}}(1 + 2a/\Delta a)^{\overline{m}/2}} \tag{6a}$$

At the zeroth-order Eq. (2b) is approximately verified $(0 = 0)$ for each Δa satisfying (note that $N_C^P = N_C^{P^*}(\Delta a \to 0)$):

$$\Delta a \ll 2a_C \tag{6b}$$

Since for the analyzed geometry $a_C \to \infty$, Eq. (6b) is always fulfilled. Thus, Δa remains a free parameter: it can be used to match the first-order condition of Eq. (2b), e.g., related to the slope $\mathrm{d}N/\mathrm{d}a|_{a_C}$ at failure.

Unified Paris–Wöhler law. Applying Eq. (3b) we obtain

$$\begin{aligned} &\frac{\mathrm{d}\Delta a(a)}{\mathrm{d}a} + G2^{1+\frac{\overline{m}-m}{2}} a^{\frac{\overline{m}-m}{2}} \Delta a(a)^{m-1-\overline{m}/2} + G\Delta a(a)^{m/2} a^{-1} - \Delta a(a) a^{-1} = 0, \\ &G = (\overline{C}C\pi^{m/2}\Delta\sigma^{m-\overline{m}} 2^{1-m/2})^{-1} \end{aligned} \tag{7a}$$

In general this differential equation has to be solved numerically. But for the sake of simplicity let us consider $m = \overline{m} = 2$, just for illustrative purpose. In these hypotheses Eq. (7a) becomes

$$\frac{\mathrm{d}\Delta a(a)}{\mathrm{d}a} + \alpha\frac{\Delta a(a)}{a} + \beta = 0, \quad \alpha = \left(\frac{1}{\pi C\overline{C}} - 1\right), \quad \beta = \frac{2}{\pi C\overline{C}} \tag{7b}$$

The previous differential equation presents variable coefficients, and can be solved by applying the method of the "variation of the arbitrary constants". We found the following solution:

$$\Delta a(a) = -2a + \gamma a^{-\alpha} \tag{7c}$$

where γ is the integration constant. Introducing Eq. (7c) into Eq. (3a) gives

$$\gamma = 2a_C^{\alpha+1} \tag{7d}$$

Accordingly,

$$\Delta a(a) = -2a + 2a_C^{\alpha+1} a^{-\alpha} \tag{7e}$$

Introducing Eq. (7e) into Eq. (1b) or into Eq. (2a) gives, as imposed, the same result, namely,

$$N_C^{PW^*} = \frac{\overline{C}}{\Delta\sigma^2}\left(1 - \left(\frac{a}{a_C}\right)^{\alpha+1}\right) \tag{7f}$$

5. Comparison between the different "universal" laws

To make a simple example of comparison, let us consider the previously treated cases in which $m = \overline{m} = 2$ (for Griffith's geometry). Firstly, note that the classical Paris' and Wöhler's laws would yield, respectively:

$$N_{\mathrm{C}}^{\mathrm{W}} = \frac{\overline{C}}{\Delta\sigma^2} \tag{8a}$$

$$N_{\mathrm{C}}^{\mathrm{P}} = \frac{1}{\pi C \Delta\sigma^2} \ln\left(\frac{a_{\mathrm{C}}}{a}\right) \tag{8b}$$

i.e., two completely different results. In particular it is clear that Paris' law yields meaningless results for $a \to 0$, i.e., an infinite life for defect-free plates, whereas for $a \to a_{\mathrm{C}}$, asymptotically:

$$N_{\mathrm{C}}^{\mathrm{P}}(a \to a_{\mathrm{C}}) \approx \frac{1}{\pi C \Delta\sigma^2} \frac{a_{\mathrm{C}} - a}{a} \tag{8c}$$

Now let us consider the first criterion. According to Eqs. (5d) and (5e), asymptotically:

$$N_{\mathrm{C}}^{\mathrm{P}^*}(a \to 0) = N_{\mathrm{C}}^{\mathrm{W}} \tag{9a}$$

$$N_{\mathrm{C}}^{\mathrm{P}^*}(a \to a_{\mathrm{C}}) = \frac{1}{C\Delta\sigma^2\pi} \frac{a_{\mathrm{C}} - a}{a_{\mathrm{C}} + \Delta a/2} \approx N_{\mathrm{C}}^{\mathrm{P}}(a \to a_{\mathrm{C}}) \quad \text{for } \Delta a \ll 2a_{\mathrm{C}} \tag{9b}$$

On the other hand, according to the second criterion, asymptotically

$$N_{\mathrm{C}}^{\mathrm{W}^*}(a \to 0) = N_{\mathrm{C}}^{\mathrm{W}} \tag{10a}$$

$$N_{\mathrm{C}}^{\mathrm{W}^*}(a \to a_{\mathrm{C}}) \approx 0 = N_{\mathrm{C}}^{\mathrm{P}}(a \to a_{\mathrm{C}}) \quad \text{for } \Delta a \ll 2a_{\mathrm{C}} \tag{10b}$$

Finally, considering the third criterion, asymptotically

$$N_{\mathrm{C}}^{\mathrm{PW}^*}(a \to 0) = N_{\mathrm{C}}^{\mathrm{W}} \tag{11a}$$

$$N_{\mathrm{C}}^{\mathrm{PW}^*}(a \to a_{\mathrm{C}}) = \frac{1}{C\Delta\sigma^2\pi} \frac{a_{\mathrm{C}} - a}{a_{\mathrm{C}}} = N_{\mathrm{C}}^{\mathrm{P}}(a \to a_{\mathrm{C}}) \tag{11b}$$

All the three unified laws match asymptotically the Wöhler's (small crack) and Paris' (large crack) regimes. The full comparison is summarized in Fig. 1. This clearly shows the consistency of the proposed "universal" laws, as "intermediate asymptotic matching" of the well-known empirical fatigue limit laws by Wöhler and Paris. Consequently, our laws must automatically match the empirical observations. Note that in the unified Wöhler's law $\Delta a \ll 2a_{\mathrm{C}}$ remains a free best-fit parameter, that can be fixed to recover the slope of the Paris' law for $a \to a_{\mathrm{C}}$; in contrast, in Fig. 1 we have simply assumed the validity of Eq. (5g).

It is clear that the Wöhler's prediction becomes unreasonable for large cracks, whereas the Paris' prediction fails for describing short crack growth, see Eq. (8) and Fig. 1. On the contrary, our universal laws based on quantized or (finite) fracture mechanics [17–19] represent their unification, useful for treating fatigue growth of cracks, from short to long sizes.

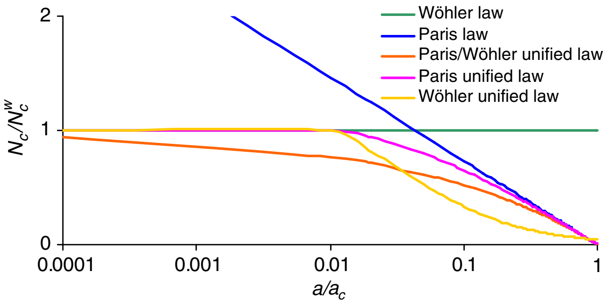

Fig. 1. Comparison between different fatigue laws: classical Paris' and Wöhler's laws and the proposed new universal fatigue laws (Griffith's case, with $C = \overline{C} = \Delta\sigma^2 = 1$; $m = \overline{m} = 2$).

6. Conclusions

We have proposed three different unified fatigue laws, by applying the recently developed quantized (or finite) fracture mechanics [17–19]. Their applicability for predicting the fatigue life of structures containing from short to long cracks is straightforward since they derive from an asymptotic matching between the empirical Wöhler's (small crack) and Paris' (large crack) regimes. Such ''universal'' fatigue laws could thus strongly improve our life-time predictions, e.g., for Air Force and Ministry critical components, usually still based on the celebrated Paris' law.

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