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Fracture mechanics and complexity sciences – Part II. Complex behaviour emerging from simple nonlinear rules: Catastrophes and chaos [1]

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Abstract. The so-called Complexity Sciences are a topic of fast growing interest inside the scientific community. Aim of this paper is to provide an insight into the role of complexity in the field of Materials Science and Fracture Mechanics. The paper is divided into two parts, which deal with the two opposite natural trends of composite systems: order and structure emerging from large, complicated systems and the route towards randomness and chaos arising from simple nonlinear rules. The former trend has been illustrated in the companion paper (Part I); on the other hand, this part will focus on the latter trend. The first example concerns the snap-back instabilities in the structural behaviour of composite structures, which are illustrated in the framework of Catastrophe Theory; the second application deals with the transition towards chaos in the dynamics of cracked beams.

1. The nonlinear cohesive crack model: Snap-back instability as a cusp catastrophe

The first example dates back to the 1980's, when the senior author [1–3] approached the snap-back instability of cracked bodies with a Cohesive Crack model. As will be shown, this instability can be interpreted in the general framework of Catastrophe Theory (Thom [4]).

This first section is thus devoted to a brief review of the ductile-to-brittle transition in the mechanical behaviour of cracked solids, described by means of the Cohesive Crack model, along with some examples for structures in bending and numerical implementations for Mode I cracking. It will be shown how this approach succeeds in capturing the ductile-to-brittle transition by increasing the structural size owing to the difference in the physical dimensions of two materials parameters: the tensile strength σ_u and the fracture energy $\mathcal{G}_F$.

The Cohesive Crack Model was initially proposed by Barenblatt [5,6] and Dugdale [7]. Subsequently, Dugdale's model was reconsidered by Bilby et al. [8], Rice [9] and Willis [10]. Hillerborg et al. [11]

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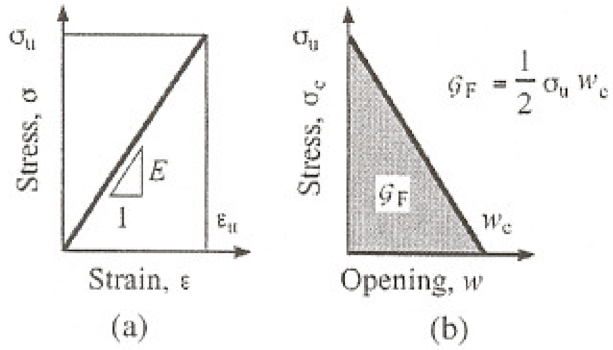


Fig. 1. Constitutive laws of the cohesive crack model: (a) undamaged material; (b) process zone.

proposed the Fictitious Crack Model in order to study crack propagation in concrete. The cohesive crack model is based on the following assumptions [1,11,12]:

1. The cohesive fracture zone (plastic or process zone) begins to develop when the maximum principal stress achieves the ultimate tensile strength σ_u (Fig. 1a).
2. The material in the process zone is partially damaged but still able to transfer stress. Such a stress is dependent on the crack opening displacement w (Fig. 1b). The energy $\mathcal{G}_F$ necessary to produce a unit crack surface is given by the area under the σ–w diagram in Fig. 1b.

The *real crack tip* is defined as the point where the distance between the crack surfaces is equal to the critical value of crack opening displacement w_c and the normal stress vanishes. On the other hand, the *fictitious crack tip* is defined as the point where the normal stress attains the maximum value and the crack opening vanishes (Fig. 1). With some modifications, the cohesive crack model has been applied to model a wide range of materials and fracture mechanisms, most prominently concrete. Regarding this material, there is a very large literature; for a review, the reader is referred to the review papers by Carpinteri and co-workers [13,14].

Before reviewing the numerical results, let us propose a simple analytical interpretation, which easily allows to capture the fracture instability propagation. The linear elastic behaviour of a three point bending, initially uncracked, beam may be represented by the following dimensionless equation:

$$\tilde{P} = \frac{4}{\lambda^3}\tilde{\delta} \tag{1}$$

in which $\tilde{P}$ and $\tilde{\delta}$ are the nondimensional load and midspan deflection, respectively, whilst λ is the beam slenderness (span to depth ratio) [15]. Once the ultimate tensile strength σ_u is achieved at the lower beam edge, a fracturing process in the central cross-section is supposed to start. Such a process admits a limit-situation like that in Fig. 2. The limit stage of the fracturing and deformation process may be considered as that of two rigid parts connected by the hinge A in the upper beam edge. The equilibrium of each part is ensured by the external load, the support reaction and the closing cohesive forces. The latter depend on the distance between the two interacting surfaces: increasing the distance w, the cohesive forces decrease till they vanish for $w \geqslant w_c$. Without entering into details, the load-displacement relation may be obtained as [15]:

$$\tilde{P} = \frac{1}{6}\left(\frac{s_E\lambda^2}{\varepsilon_u\tilde{\delta}}\right)^2, \tag{2}$$

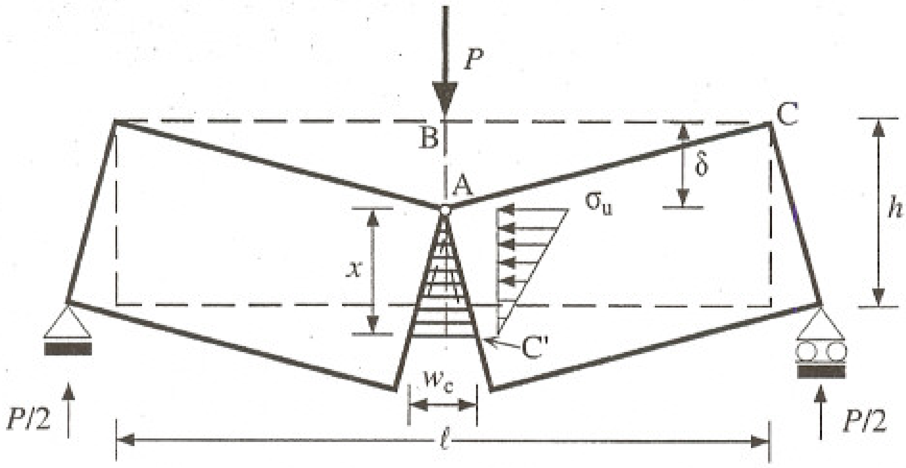


Fig. 2. Limit-situation of complete fracture with cohesive forces.

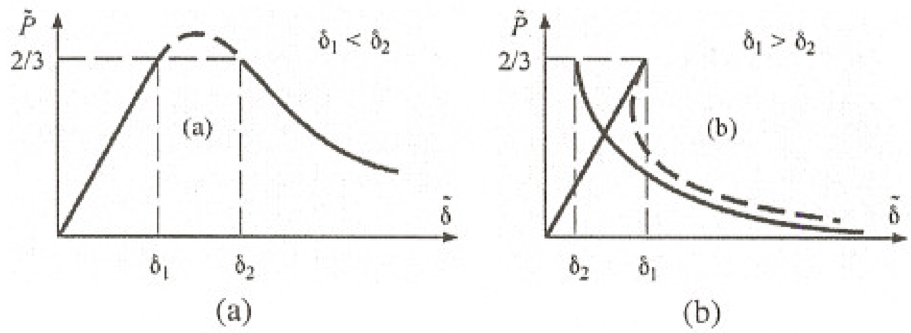


Fig. 3. Load-deflection diagrams: (a) ductile and (b) brittle condition.

where the *Brittleness Number* $s_E = \mathcal{G}_F/\sigma_u h$ is the fundamental quantity ruling the system behaviour [2, 15–17]. Both Eqs (1) and (2) have the same upper validity limit: $\tilde{P} \leqslant 2/3$. Therefore, a stability criterion for elastic-softening beams may be obtained transforming the load bounds into deflection bounds. When the two domains are separated, it is presumable that the two P–δ branches – linear and hyperbolic – are connected by a regular curve (Fig. 3a). On the other hand, when the two domains are partially overlapped, it is well-founded to suppose them as connected by a curve with highly negative or even positive slope (Fig. 3b). Unstable behaviours and catastrophical events (snap-back) may be possible, in the case of the three point bending geometry, for:

$$B = \frac{s_E}{\varepsilon_u \lambda} \leqslant \frac{1}{3}. \tag{3}$$

The system is brittle for low brittleness numbers s_E, high ultimate strains and large slendernesses. It is therefore evident that the relative brittleness for a structure is dependent on loading condition and external constraints, in addition to material properties, size-scale and slenderness. For instance, uniaxial tension is more unstable than three point bending: $B \leqslant 1/2$ (Carpinteri [18]), whilst in bending $B \leqslant 1/3$.

Now, let us quantify the ductile-to-brittle transition by showing synthetically the numerical results for concrete elements in Mode I conditions (Three Point Bending Test – TPBT), based on the cohesive model, obtained using the Finite Element Code FR.ANA. (FRacture ANAlysis, Carpinteri [2,15,18], Carpinteri et al. [12,19,20]).

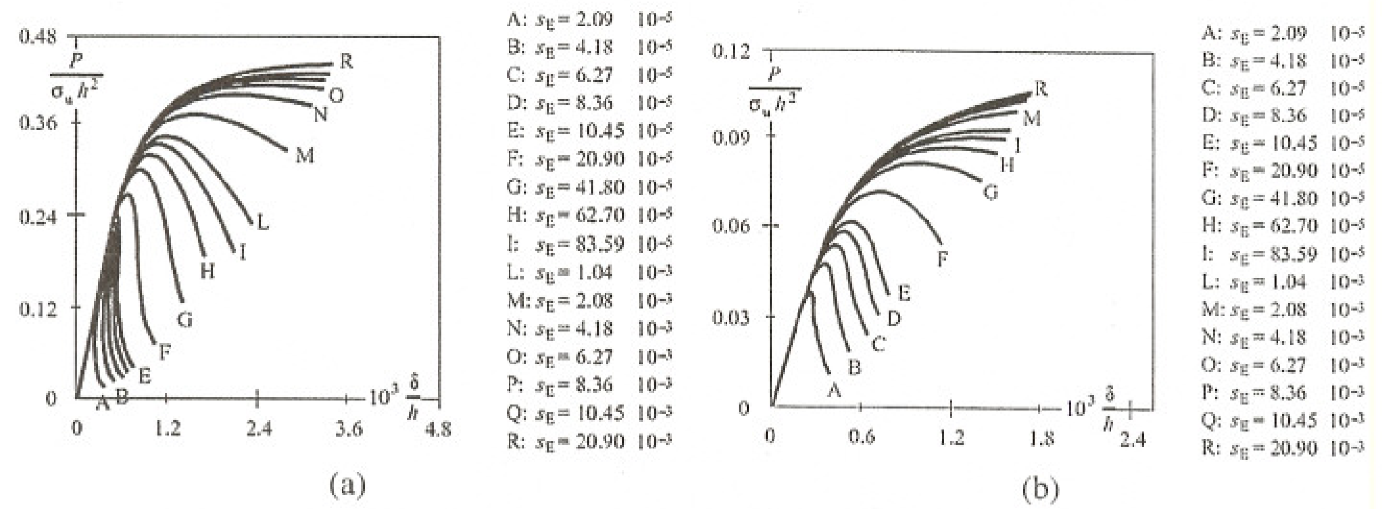


Fig. 4. Dimensionless load vs. deflection diagrams by varying the brittleness number s_E, $a_0/h = 0.0$ (a) and $a_0/h = 0.5$ (b).

Extensive series of analyses were carried out from 1984 to 1989 by A. Carpinteri and co-workers. The experimental results can be found in the RILEM report [21]. The cases described in the reference papers regard three l/h ratios (4, 8 and 16), and four a_0/h ratios (0.00, 0.10, 0.30, 0.50), and a concrete-like material. Figure 4a refers to the case of an initially uncracked beam ($a_0/h = 0.0$), whilst Fig. 4b reports results for the case of an initially cracked beam with $a_0/h = 0.5$.

For each ratio, the response was analyzed for different values of the brittleness number, s_E, ranging from 2×10^{-2} to 2×10^{-5}. As can be seen from the diagrams, the brittleness number s_E has a decisive effect on the structural response of the element: by increasing s_E, the behaviour of the element changes from brittle to ductile, as already stated. Generally speaking, the specimen behaviour is brittle (snap-back) for low s_E numbers, i.e., for low fracture toughnesses $\mathcal{G}_F$, high tensile strengths, σ_u, and/or large sizes, h. In particular, in the case of uncracked beam, for $s_E \leqslant 10.45 \times 10^{-5}$, the P–δ curve presents positive slope in the softening branch and a catastrophical event occurs if the loading process is deflection-controlled. Such indenting branch is not virtual only if the loading process is controlled by a monotonically increasing function of time (Fairhurst et al. [22], Rokugo et al. [23]) like, for example, the displacement discontinuity across the crack (Biolzi et al. [24]).

In the case of the cracked beam, on the contrary, the initial crack makes the specimen behaviour more ductile; for the set of s_E numbers considered in Fig. 4b, the snap-back does not occur. By varying the initial crack depth, it is possible to describe the gradual transition from simple fold catastrophe (softening) to bifurcation or cusp catastrophe (snap-back instability), generating an entire equilibrium surface, or the catastrophe manifold.

Eventually, it is interesting to plot the ratios of the maximum loading capacity P_{Cohes} according to the cohesive crack model (obtained from Figs 4) to the maximum load $P_{\mathrm{L.E.F.M.}}$ of brittle fracture as a function of the inverse of s_E. This ratio may also be regarded as the ratio of the fictitious fracture toughness (given by the maximum load P_{Cohes}) to the true fracture toughness (considered as a material constant). It is evident that for low s_E numbers the results of the cohesive crack model tend to those of linear elastic fracture mechanics (see Fig. 5a), and therefore, the maximum loading capacity can be predicted if the condition $K_{\mathrm{I}} = K_{\mathrm{IC}}$ is applied. It appears that the true fracture toughness K_{IC} of the material can be obtained only with very large specimens.

A similar conclusion may be drawn for the tensile strength, by considering the ratio of the maximum load P_{Cohes} of the cohesive crack model to the maximum load $P_{\mathrm{U.S.}} = 2/3(\sigma_u t h^2/l)$ of ultimate strength.

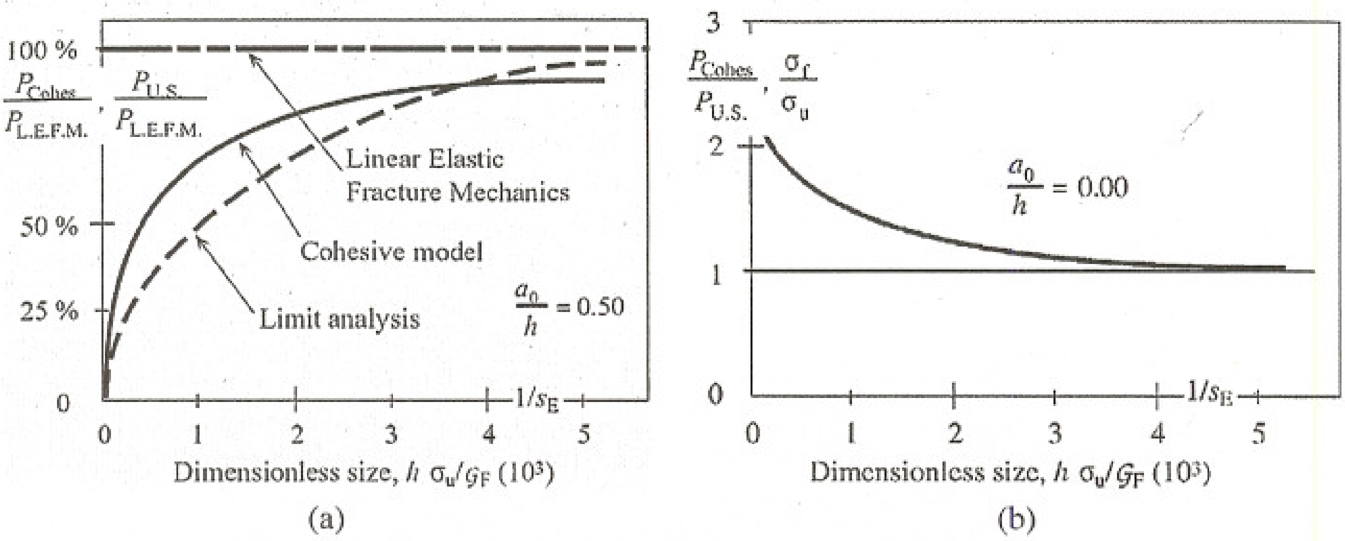


Fig. 5. Increase of fictitious toughness (a) and decrease of fictitious strength (b) with increasing specimen size ($\lambda = 4$, $\varepsilon_u = 0.87 \times 10^{-4}$).

The values of the ratio $P_{\mathrm{Cohes}}/P_{\mathrm{U.S.}}$ may also be regarded as the ratio of the apparent tensile strength σ_f to the true tensile strength (considered as a material constant). Again, it appears that the true tensile strength σ_u of the material can be obtained only with very large specimens (see Fig. 5b).

2. Route towards chaos in the dynamics of cracked beams

The second and last topic is concerned with the dynamical behaviour of cracked beams (Carpinteri and Pugno [25–27]). The damage assessment problem in cracked structures has been extensively studied in the last decade, highlighting that the vibration based inspection is a valid method to detect, localize, and quantify cracks especially in beam structures. Dealing with the presence of a crack in the structure, previous studies have demonstrated that a transverse crack can change its state (from open to closed and vice versa) when the structure, subjected to an external load, vibrates. As a consequence, a nonlinear dynamic behavior is introduced.

In the past many studies have illustrated that a crack in a structure such as a beam, may exhibit nonlinear behavior if it is open during part of the response and closed in the remaining intervals. This phenomenon has been detected during experimental testing performed by Gudmundson [28], in which the influence of a transverse breathing crack upon the natural frequencies of a cantilever beam was investigated. The main result obtained was that the experimentally observed decrease in the natural frequencies of the beam due to the presence of the crack could not be described by a model of crack which is always open. Therefore, it must be concluded that the crack alternately opened and closed giving rise to natural frequencies falling between those corresponding to the always-open and always-closed cases. In fact, if an always-open crack is assumed in the analysis of a beam with a so-called breathing crack, which experimentally both opens and closes during the time interval considered, the reduced decrease in the experimental natural frequencies will lead to an underestimate of the crack depth if determined via a test-model correlation approach.

Several models have been proposed in the past for dealing with cracked vibrating beams [29–33], but, in all these models, the main assumption has been that the crack can be either fully open or fully closed during the vibration. Carpinteri and Pugno [26] recently developed a coupled theoretical and numerical approach to evaluate the nonlinear complex oscillatory behaviour in damaged structures under excitation. In their approach, they have focused their attention on a cantilever beam with several breathing transverse cracks and subjected to harmonic excitation perpendicular to its axis. The method, that is an extension of the super-harmonic analysis carried out by Pugno et al. [34] to subharmonic and zero frequency components, has allowed to capture the complex behavior of the nonlinear system, e.g., the occurrence of period doubling, as experimentally observed by Brandon and Sudraud [35] in cracked beams.

A pioneer work on period doubling was written in 1978, when Mitchell Feigenbaum [36] developed a theory to treat the route from ordered to chaotic states. Even if oscillators showing the period doubling can be of different nature, as in mechanical, electrical, or chemical systems, they all share the characteristic of recursiveness. He provided a relationship in which the details of the recursiveness become irrelevant, through a kind of universal parameter, measuring the ratio of the distances between successive period doublings, the so-called Feigenbaum's delta [37]. His understanding of the phenomenon was later experimentally confirmed [38], so that today we refer to the so-called Feigenbaum's period doubling cascade. However, even if the period doubling has a long history, only recently it has been experimentally observed in the dynamics of cracked structures [35].

The presented method discretizes the differential nonlinear equations governing the oscillations of the continuum structure by the finite element method. Let us consider a multicracked cantilever beam, clamped at one end and subjected to a dynamic distributed force p (with rotating frequency ω). Modeling the breathing cracks as concentrated nonlinear springs, the equation of the motion of each integer beam segment is the classical equation of the beam dynamics. Furthermore, the boundary conditions between two adjacent segments are represented by the continuity of the transversal displacement and of its second and third spatial derivatives (proportional to the bending moment and to the shearing force, respectively), as well as by the compatibility with the crack. This implies that the difference in the rotations between the two adjacent sections must be equal to the rotation of the connecting springs. The problem can be formally written as:

$$\rho A \frac{\partial^2 q(z,t)}{\partial t^2} + EI \frac{\partial^4 q(z,t)}{\partial z^4} = p(z,t) \quad \text{for } z_i < z < z_{i+1}, \tag{4}$$

whereas, for $z = z_i$:

$$q(z_i^-) = q(z_i^+), q''(z_i^-) = q''(z_i^+), q'''(z_i^-) = q'''(z_i^+), q'(z_i^-) - q'(z_i^+) = \frac{EIq''(z_i^{\pm})}{k_i}, \tag{5}$$

where ρ is the density, A the cross-section area, q the transversal displacement, E the Young's modulus, and I the moment of inertia of the beam. k_i is the nonlinear concentrated rotational stiffness (a function of the rotations $q'(z_i^{\pm})$) of the crack placed at z_i (the symbol prime denotes the derivation with respect to z).

Equation (4) can be formally solved by applying the Fourier trigonometric series, searching a solution in the form [26]:

$$q(z,t) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{+\infty} c_k(z)\, \mathrm{e}^{\mathrm{i}k\omega/\Theta t}, \tag{6}$$

where $c_k(z)$ are unknown functions and $\bar{P} = \Theta 2\pi/\omega = \Theta P$ is the period of the response, assumed *a priori* to be different from the period P of the excitation (describing a so-called complex behavior,

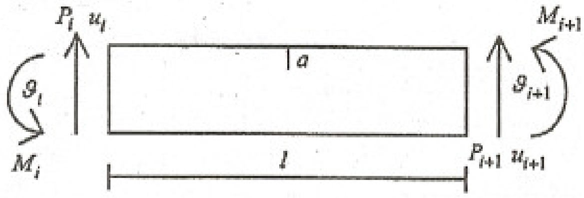


Fig. 6. Cracked element.

thus Θ is a complexity index). On the other hand, if the period of the response tends to infinity, i.e., $\Theta \to \infty$ (nonperiodic response, i.e., chaotic deterministic behaviour), it is well known that Eq. (6) formally becomes a Fourier transform.

By discretizing the structure with the finite element method [26], Eq. (4) can be rewritten as:

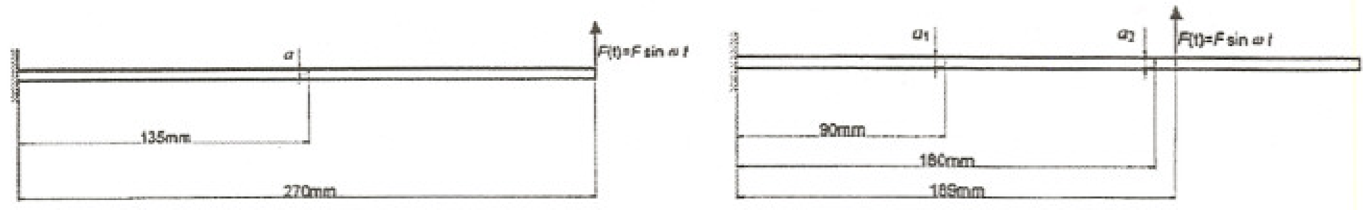
$$[\mathrm{M}]\{\ddot{q}\} + [\mathrm{D}]\{\dot{q}\} + [K]\{q\} + \sum_m [\Delta K^{(m)}] f^{(m)}(\{q\})\{q\} = \{F\}, \tag{7}$$

where [M] is the mass matrix, [D] the damping matrix, $[K] + \sum_m [\Delta K^{(m)}]$ the stiffness matrix of the undamaged beam, and $[\Delta K^{(m)}]$ is half of the variation in stiffness introduced when the m-th crack is fully open. A sketch of the cracked finite element is reported in Fig. 6; details about the calculus of the variation in stiffness due to the presence of the crack can be found in [34]. $\{F\}$ is the vector of the applied forces (with angular frequency ω) and $\{q\}$ is the vector containing the generalized displacements of the nodes (translations and rotations). According to this notation, $f^{(m)}(\{q\})$ is between -1 and $+1$ and models the transition between the conditions of m-th crack fully open and fully closed. The method described assumes, according to [35], that the cracks open and close continuously instead of instantaneously, as suggested by the experiments. Thus, the stiffness varies continuously between the two extremes of undamaged or totally damaged beam (fully open cracks), rather than stepwise. The solution for the elements of the $\{q\}$ vector $\in L^2$ (i.e., $|q_i^2|$ can be integrated according to Lebesgue) can be found by the approximation of Eq. (6), that for the discrete system can be rewritten as:

$$q_i = \sum_{j=0}^{N} \left(A_{ij} \sin j\frac{\omega}{\Theta}t + B_{ij} \cos j\frac{\omega}{\Theta}t \right) \tag{8}$$

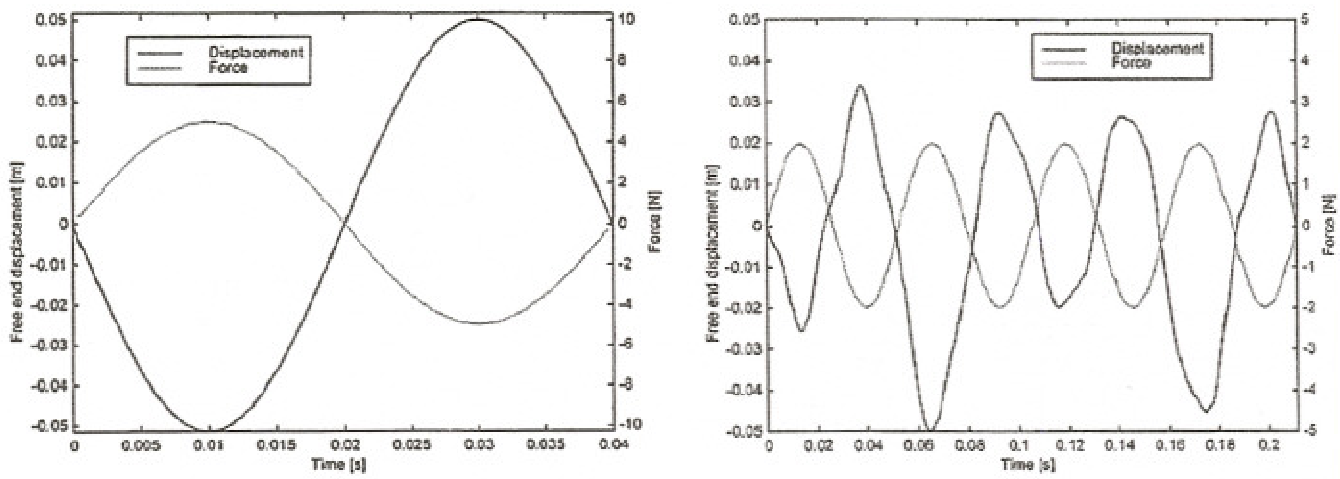
in which the complexity index Θ must be a positive integer, to take into account not only the superharmonics (and offset) but also the subharmonic components of the dynamic response, and theoretically $N = \infty$. The period of the response is not assumed *a priori* equal to the period of the harmonic excitation, as classically supposed (absence of subharmonic components). This is the key to capture the complex behavior of the highly nonlinear structure, e.g., the occurrence of period doubling. Coupling the above formulation with the harmonic balance approach, allows to obtain a nonlinear system of algebraic equations, easy to be solved numerically.

Two different numerical examples will be considered: a weakly nonlinear structure and a strongly nonlinear one. Only in the latter case, the so-called period doubling phenomenon clearly appears. The beam here considered is the same as that described in the mentioned experimental analysis. It is 270 mm long and has a transversal rectangular cross section with base and height, respectively, of 13 and 5 mm. The material is (UHMW)-ethylene, with a Young's modulus of 8.61×10^8 N/m^2 and a density of 935 kg/m^3. The assumed modal damping is 0.002; the beam is discretized with 20 finite elements. Values of the



(a) (b)

Fig. 7. Damaged structures: weakly nonlinear (a) and strongly nonlinear (b).



(a) (b)

Fig. 8. Time history of the free end displacement and of the applied force: weakly (a) and strongly (b) nonlinear structure.

complexity index are $\Theta = 4$ and $N = 16$, respectively. The first natural frequency of the undamaged structure is $f_u = 10.6$ Hz.

For each of the two considered structures (Figs 7a and 7b) it is shown the time history of the applied force and of the free-end displacement (Figs 8a and 8b) and the trajectory in the phase space (Figs 9a and 9b).

In a hypothetical linear structure, the structural response is linear by definition with obviously only one harmonic component at the same frequency of the excitation. In the weakly nonlinear structure of Fig. 7a, the response converges and it appears only weakly nonlinear, as depicted in Fig. 8a. No subharmonic components can be observed. The corresponding phase diagram of the response is shown in Fig. 9a; due to the weak nonlinearity, the trajectory in the phase diagram is close to an ellipse. The diagram is nonsymmetric as the spatial positions of the cracks (placed in the upper part of the beam). The trajectory is an unique closed curve since here the period of the response is equal to the period of the excitation.

In the strongly nonlinear structure of Fig. 7b the nonlinearity increases, as depicted in Fig. 8b. The harmonic components in the structural response are the zero one, the superharmonics as well as the subharmonic ones (see Fig. 10). It should be emphasized that a strong nonlinearity causes the period doubling of the response, i.e., the $\omega/2$ component. The free-end vibrates practically with a period doubled with respect to the excitation. A nonnegligible component at $\omega/4$ is observed too, representing a route to chaos through a period doubling cascade. The corresponding phase diagram of the response is shown in Fig. 9b. The trajectory is again a unique closed curve, since the response is still periodic; it is composed by multiple cycles since here the period of the response is not equal to the period of the excitation. The distortions in the trajectory are consequences of the presence of the super- or subharmonics,

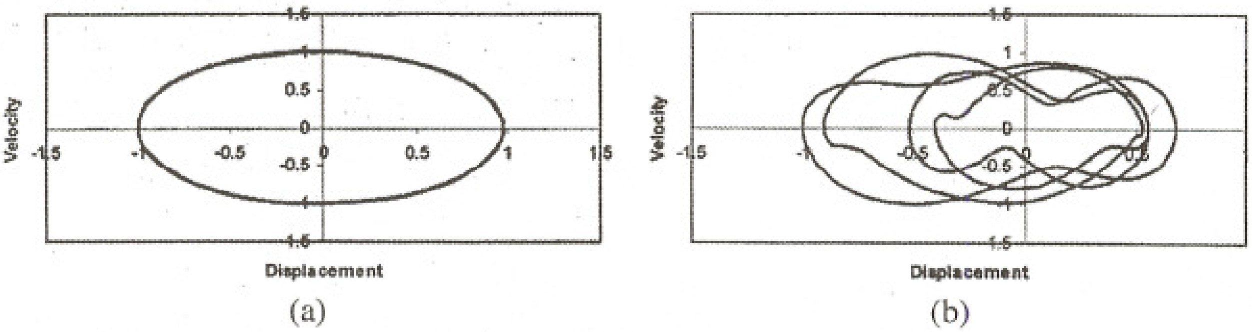


Fig. 9. Dimensionless phase diagram of the response (free-end displacement): weakly (a) and strongly (b) nonlinear structure.

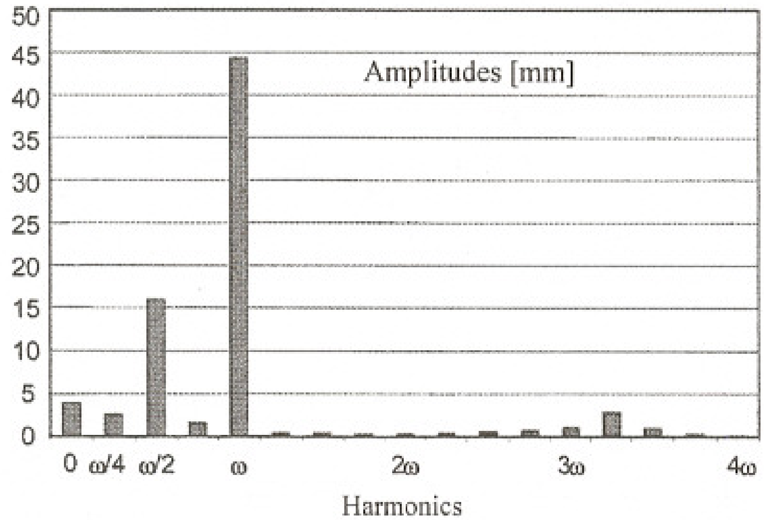


Fig. 10. Zero-, sub-, and super-harmonic components for the free-end displacement of the strongly nonlinear structure.

as well as the multiple cycles emphasize the presence of the subharmonics (four cycles are due to the component $\omega/4$), i.e., the presence of a complexity with associated route to chaos. Also in this case, the diagram is nonsymmetric as the spatial positions of the cracks.

This method is able to catch the transition toward deterministic chaos, like the occurrence of a period doubling, as shown in the numerical examples and experimentally observed in the context of cracked beam by Brandon and Sudraud [35]. In this analysis, of crucial importance appears the complexity index Θ. In fact, as the nonlinearity becomes stronger, offset, and super-harmonic components, as well as subharmonic ones, can be observed in the structural response. As a consequence, in this case, the system dynamics can be caught using a complex index Θ larger than 1. Theoretically, values of Θ tending to infinity (Fourier series become Fourier transforms, with N theoretically tending to infinity too) allow to catch a route to chaos through a period doubling cascade, that here would imply a nonperiodic dynamic response.

3. Conclusions

The so-called "Complexity Sciences" represent a subject of fast-growing interest in the Scientific Community. They have entered also our more circumscribed Communities of Material Science and Material Strength, as the proposed examples may confirm. Under the label of "Complexity Sciences" we usually comprehend a large variety of phenomena, theories, approaches and techniques: nonlinear dynamics, deterministic chaos, nonequilibrium thermodynamics, fractal geometry, intermediate asymptotics, renormalization group theory, catastrophe theory, self-organised criticality, neural networks, cellular automata, fuzzy logic, etc.

Complex systems lie somehow in between order and randomness and exhibit some common characteristics, such as: sensitivity to initial conditions, pattern formation, spontaneous self-organisation, emergence of cooperation and collective properties, hierarchical or multiscale meso-structures, scaling and size effects. We could try to summarize by saying that the nonlinearity in the constitutive laws may produce complex structures and scale-dependent behaviours.

Aim of this paper is that of providing insight into the role of complexity in the fields of Material Strength and Fracture Mechanics. The presented topics are concerned with catastrophes and chaos or, in more general terms, with complex behaviour emerging from simple nonlinear rules; the proposed examples deal with the structural behaviour of composite structures undergoing snap-back instabilities (an example of cusp catastrophe) and the route towards chaos in the dynamics of cracked structures. As shown in these examples, the most interesting behaviours and phenomena can be synthetically interpreted only through the use of new and refined conceptual tools in the framework of "Complexity Sciences".

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