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A fractal approach to indentation size effect

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Abstract

In this paper, we propose an original interpretation of indentation size effect (ISE) in both single crystal and polycrystalline metals, which is based on the experimental evidence of the formation of fractal cellular dislocation patterns during the later stages of plastic deformation, in strain hardening metals, both under tensile loading and in compression. The proposed approach is a generalization of the arguments already proposed by the senior author in order to derive multifractal scaling laws (MFSL), which apply to tensile strength, fracture energy and the critical strain of brittle and quasi-brittle materials.

This approach is thus in the mainstream of the geometrical interpretation of size-scale effects on the strength of solids, which has been counterposed, in recent years, to the mechanical interpretation. The proposed fractal approach aims at strengthening this view, which provides ease of interpretation, and at stimulating discussion on the central key role of dislocation evolution in the plastic deformation of metals, the fractal characteristics of which have not been adequately considered in literature.

The obtained equation for ISE, based on the fractal approach, very closely resembles MFSL for tensile strength in quasi-brittle materials and the scaling equation already proposed by other authors, but it is based on a very different underlying physical model. Some experimental hardness data, obtained from microindentation on copper, have been fitted with MFSL, and show a very good agreement.

Keywords: Hardness; Microindentation; Nanoindentation; Fractals; Size-scale effects

1. The indentation size effect

Indentation tests are perhaps the most common procedures to measure the mechanical properties of materials. In such a test, a hard tip, which could have different shapes, e.g. spherical, conical or cylindrical, is pressed into a sample with a known load. From the standpoint of fracture mechanics, the problem of normal indentation is rather old: in 1953 Muskhelishvili [1] gave the solution to the problem of indentation of the

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Nomenclature

A	cross-section area
A_i	area at scale L_i
A^*_{res}	fractal measure of the Sierpinski carpet
b	structural dimension
D	fractal dimension of the dislocation pattern
d_σ	fractal scaling exponent for strength σ
d_τ	fractal scaling exponent for tangential stress τ
d_H	fractal scaling exponent for hardness H
F	macroscopic force
f	renormalization transformation
f_{t}	asymptotic value of the tensile strength in the limit of infinite structural size
H	hardness
H^*	renormalized fractal hardness
h	indentation depth
k	Tabor's proportionality constant
L^*	scale of observation
l_{ch}	material characteristic length
L_i	scale of observation
N	number of dislocation cells
N_i	number of dislocations at scale L_i
p	scale ratio
x	fractal scaling exponent
$\Delta\tau$	self-similarity dimension
λ	dislocation cell size
σ	tensile strength
σ^*	renormalized fractal tensile strength
τ	tangential stress
τ^*	renormalized fractal tangential stress
τ_i	fictitious microstress at scale L_i

half-plane by a flat-ended rigid punch. His results were later extended by several scientists who treated the indentation problem involving punches with a corner angle that is different from $\pi/2$; in particular, Dundurs and Lee [2] solved the frictionless problem, whilst Gdoutos and Theocaris [3] took friction into account.

From a practical point of view, indentation tests offer several advantages: the size of the specimen can be very small (this is a great advantage in the case of materials which have a fine microstructure, are multi-phase or non-homogeneous) and the procedure is non-destructive. Due to these advantages, indentation is a very broadly used technique. Apart from metals, it is largely applied to ceramics for the determination of fracture toughness, to polymers, sandwich structures and also to materials for bio-medical applications.

Since the 1950s [4], hardness measurements have been recognized to be size dependent; a hardness increase with decreasing indentation depth (or indenter size) is always observed. The ISE has been extensively studied in literature and research in this field has continuously increased over the last decades; this has been partly motivated by the development of advanced nanocomposites and the large-scale application of thin films in electronic components, and partly by the availability of new methods of probing mechanical properties in very small volumes. Several different mechanisms have been suggested to be responsible for ISE. The proposed mechanisms include: the presence of oxides or chemical contamination on the surface, interfacial friction, increased dominance of edge effects with shallow indents, indenter pile-up or sink-in and loading rate [5].

Over the last decade, several authors have proposed that the size dependence of the material mechanical properties (including ISE) results from an increase in strain gradients inherent to small localized zones, which lead to geometrically necessary dislocations that cause additional hardening [6–21]. It is in fact well known that metal deformation is ruled by dislocation formation, dislocation motion and dislocation storage; the latter is responsible for material hardening, according to Taylor's hardening rule. Stored dislocations can be divided into two groups: those generated by trapping each other in a random way are referred to as statistically stored dislocations (SSDs), whilst those which relieve the plastic deformation incompatibilities caused by non-uniform dislocation slip are called geometrically necessary dislocations (GNDs). These GNDs cause additional storage of defects and increase the deformation resistance by acting as obstacles to the SSDs [22]. SSDs are believed to be dependent on the effective plastic strain, while the density of GNDs is directly proportional to the gradient of the effective plastic strain [13,14,23,24]. This dependency on the strain and on the strain gradient is responsible for the size effect: the smaller the scale length, the larger the density of the GNDs relative to the SSDs and, consequently, the larger the plastic strain gradients, compared to the average plastic strains. This approach incorporates the material scale length that is needed to characterize and predict the indentation size effect in the classical plasticity theory.

At present, there is no mutual agreement between researchers regarding strain gradient theories; in a recent review by Voyiadjis and Abu Al-Rub [25] it has been observed that "incorporating the size effect into a phenomenological theory of plastic flow is not necessarily a matter of adding one or more additional parameters to an existing theory. It may require the reformulation of the entire theoretical framework of the gradient-dependent media". Moreover, the real nature of GNDs and their direct experimental quantification is still a matter of very active debate and controversy. A very interesting Viewpoint Set on this latter topic was published in 2003 in the international journal "Scripta Materialia".

A different theory, which gained some success in explaining ISE, is based on the surface to volume ratio (S/V) and it was proposed almost contemporarily by Gerberich et al. [26] and Zhang and Xu [27]. This theory is based on the fact that the work done by an applied indentation load contains both bulk and surface terms. The surface work, which is related to the surface stress and the size and geometry of the indenter tip, prevails if the indentation depth is shallower than a critical depth, while the bulk deformation prevails when the indentation depth is deeper than the same depth.

2. The fractal approach to material strength

The fractal nature of material damage and deformation has been a matter of very intense research during the last two decades. The fractal nature of fracture surfaces in metals was shown more than 20 years ago by Mandelbrot et al. [28]; in their noteworthy paper published in Nature, the authors studied the roughness of crack surfaces in metals fractured either by tensile or impact loading, which were shown to develop fractal characteristics over more than three orders of magnitude. Fractal patterns in metals were also later observed by Kleiser and Bocek [29], who documented the formation of fractal slip bands where the strain concentrates, and, more recently, by several authors [30–33], both in tension and in compression of f.c.c. metals.

Rather surprisingly, this experimental evidence did not inspire a fractal-based explanation of size effects in metal plasticity. Conversely, this happened in the field of quasi-brittle materials, where observations have shown that fracture surfaces display self-affine scaling properties, at least in a certain range of scales, which is in most cases very large and which greatly depends on the material microstructure. This is true for a large variety of quasi-brittle heterogeneous materials such as rock, wood, sea ice, concrete, clay, and ceramics [34,35].

Fractal sets are characterized by non-integer dimensions [36]. For instance, the dimension α of a fractal set in the plane can vary between 0 and 2. Accordingly, increasing the measure resolution, its length tends to zero if its dimension is smaller than 1 or tends to infinity if it is larger. In these cases, the length is a nominal, useless quantity, since it vanishes or diverges as the measure resolution increases. A finite measure can only be achieved using non-integer units, such as meters raised to α. The archetypes of lacunar and invasive fractals in the plane are represented in Fig. 1, where the Cantor dust and the von Koch curve are reported. Obviously, natural fractals do not possess the symmetry of mathematical fractals, but they are characterized by the presence of the same anomalous non-integer dimensions.

Analogously, if the stress and strain localization occurs in a fractal damaged zone, the nominal quantities (ultimate strength, critical strain, fracture energy) should depend on the resolution used to measure the set

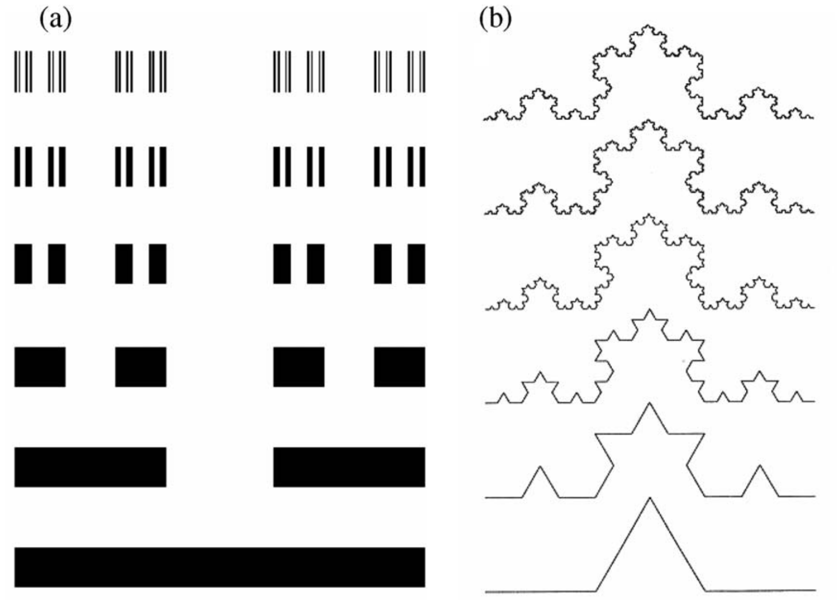

Fig. 1. Archetypes of lacunar and invasive fractals in the plane: Cantor dust (a) and von Koch curve (b).

where stress, strain and energy dissipation take place. In the limit of a very high measure resolution, the stress and the strain should be infinite, while the dissipated energy should be zero. Finite values can only be obtained introducing fractal quantities, i.e., mechanical quantities with fractal (non-integer) physical dimensions: the fractal tensile strength, the fractal critical strain and the fractal fracture energy [37–39]. Considering these quantities intrinsically introduces the fractal dimensions of the sets where stress, strain and energy dissipation localize [37–42].

Before introducing the fractal interpretation of ISE, let us summarize the procedure followed by Carpinteri to derive the MultiFractal Scaling Law for the tensile strength of brittle and quasi-brittle materials. Let us consider a concrete specimen subjected to simple tension, as in Fig. 2(a). A consistent modelling of damage in concrete [36] can be achieved by assuming that the rarefied resisting sections, in correspondence to the critical load, can be represented by stochastic lacunar fractal sets with dimension $2 - d_\sigma$ ($d_\sigma \geqslant 0$). From fractal geometry, it is well known that the area of lacunar sets is scale dependent and tends to zero as the resolution increases: the tensile strength should be infinite, which is meaningless. Finite values can be only obtained with

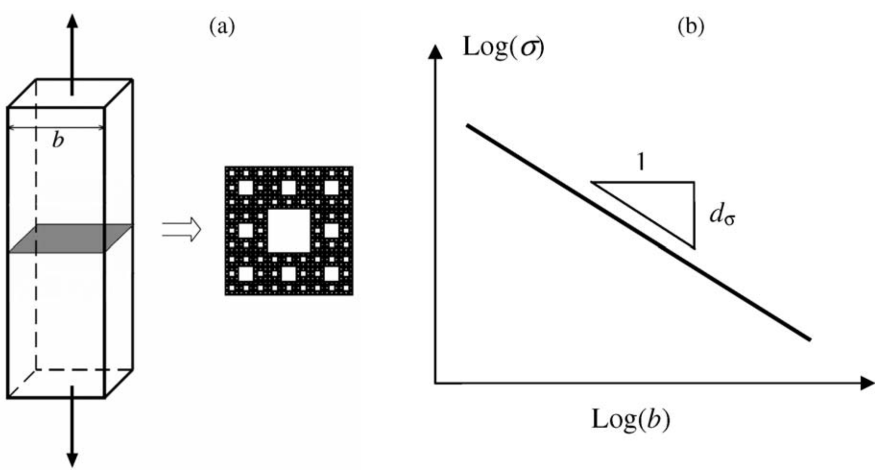

Fig. 2. Lacunar cross-section (a) and scaling law for tensile strength σ (b).

non-integer (fractal) dimensions. For the sake of simplicity, the specimen resistant cross-section could be represented by a Sierpinski carpet built on the square of side b (Fig. 2(a)), whose fractal dimension is 1.893 ($d_\sigma = 0.107$). If an Euclidean domain is assumed, the classical continuum theory states that the maximum load F is given by the product of the strength σ times the nominal area $A = b^2$, whereas, in the fractal model, F equals the product of the (Hausdorff) fractal measure $A^*_{\mathrm{res}} \sim b^{2-d_\sigma}$ of the Sierpinski carpet times the fractal tensile strength σ [37]:

$$F = \sigma A = \sigma^* A^*_{\mathrm{res}} \tag{1}$$

where σ^* presents the anomalous physical dimensions $[F][L]^{2-d_\sigma}$. The fractal tensile strength is the true material constant, i.e., it is scale-invariant. The scaling law for tensile strength is readily obtained from Eq. (1):

$$\sigma = \sigma^* b^{-d_\sigma} \tag{2}$$

i.e. a power-law with negative exponent $-d_\sigma$. Eq. (2) represents the negative size effect on the tensile strength, which has experimentally been revealed by several authors (Fig. 2(b)).

If specimens of different sizes, made of the same material, are tested in uniaxial tension, experiments show that the fractal scaling is strictly valid only in a limited scale range, where the fractal dimensions of the supporting domains can be considered to be constant. As the size increases, the concept of geometrical multifractality, which is closely connected to the characteristics of self-affine fractals [38], in fact implies the progressive vanishing of fractality ($d_\sigma \to 0$) with a corresponding homogenization of the domains. Intuitively, since the microstructure of a disordered material is the same, independently of the macroscopic specimen size, the influence of disorder on the mechanical properties essentially depends on the ratio between a characteristic material length l_{ch} and the external size b of the specimen. Therefore, the effect of microstructural disorder on the mechanical behaviour of materials becomes progressively less important at larger scales. As far as the slope of the scaling law at smaller scales is concerned, Carpinteri [38] observed that the experimental values of d_σ always appear within the interval $[0, 1/2]$. The dimensional decrement d_σ tends to the LEFM limit, 1/2, for only extremely brittle and disordered materials. On the basis of this evidence, the multifractal scaling law (MFSL) for tensile strength has been proposed, which can be written in the following analytical form:

$$\sigma = f_{\mathrm{t}}\left(1 + \frac{l_{\mathrm{ch}}}{b}\right)^{1/2} \tag{3}$$

This scaling law is a two-parameter model, where the asymptotic value of the nominal quantity f_{t}, corresponding to the lowest nominal tensile strength, is only reached in the limit of infinite sizes. The dimensionless term in the square brackets, which is controlled by the characteristic length l_{ch}, represents the variable influence of disorder on the mechanical behaviour.

Similar scaling laws have also been proposed for fracture energy [41,42], and, more recently, for critical strain [39,40]. In this research field, these concepts have been applied not only to tensile tests, but also to explain the R-curve material behaviour and to interpret the results of bending and compression tests. An extended review on the application of the formula of Eq. (3) to different testing geometries has been given by Carpinteri et al. [43]. An example, in the case of the Brazilian splitting cylinder test, with a large scale range of 1:30, is reported in Fig. 3. Very interesting results have also been obtained applying fractal concepts to contact and friction problems [44,45]. Applications of this fractal approach also include fatigue data analysis [46] and seismology [47]. Recently, fractal concepts have also been successfully applied to describe damage evolution in fracture induced by the coalescence of numerous microcracks [48,49], and the fractal dimension of the spatial distribution of microcracks has been proposed as a quantitative indicator of damage localization. A very remarkable application of fractals and of the renormalization group theory concerns the modelling of barrier properties in polymer–clay nanocomposites with layered-silicate fillers [50].

All these findings show the general applicability of fractal, purely geometrical arguments, in order to interpret size-scale effects on the mechanical properties of solids [51]. Obviously, the geometrical approach does not exclude mechanical-based explanations of the same effects, but it is proposed as the most general, simple and potentially universal. Fractals are not in fact considered to be the cause of size-scale effects, but they represent one of the general tools, which enable a simple and direct interpretation of the complex multi-scale phenomena involved in material behaviour. Their frequency in several different natural phenomena shows common

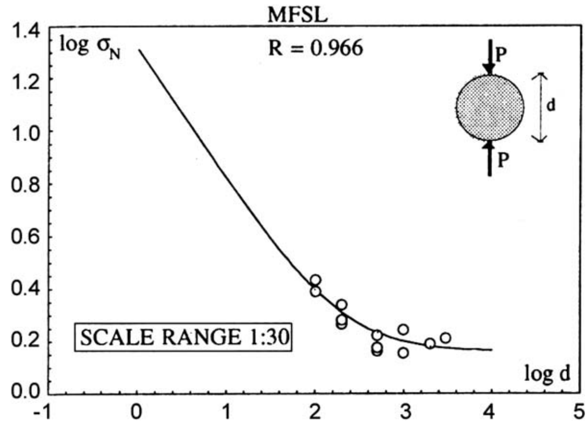

Fig. 3. Multifractal scaling of the apparent strength of concrete; data from Brazilian splitting cylinder tests [68]; from [43].

features in the complexity of real material mechanics. At the same time, the fractal approach requires a breakthrough and the complete re-thinking of concepts, that, in some cases, seem to be well established [36].

Formidable advances have been made in the last few years in the study of the fractal aspects of crack morphology and energy dissipation over fractal domains. The authors wish to emphasize how the hypothesis of a fractal damage domain in materials is not a mathematical abstraction, since fractal patterns have been detected in several experiments; nowadays, the hypothesis of fractality in material damage and failure is well accepted: fractality is a general feature of nature, that researchers are learning to deal with.

3. Fractal explanation of ISE

As far as the plastic deformation of metals is concerned, strain hardening under severe deformation occurs together with the formation of cellular dislocation patterns. When many slip systems are active, these patterns exhibit multiscale behaviour: they are characterized by power-law distributions of cell sizes [52,53,30] and the cell arrangement could be defined as a self-similar 'hole fractal'. In such conditions, cells of all sizes, within a certain range, are present in the microstructure and the concepts of average cell size and of average dislocation densities lose their meaning, as well as all the related modelling attempts.

Fractal microstructures were identified in cold-worked polycrystalline copper [52,53]. More recently, systematic investigations of existing micrographs have shown a more substantial occurrence of fractal dislocation patterns in f.c.c. metals deformed both in tension and in compression [30,54]. Later on, Zaiser et al. [55] reported the results of a fractal dimension analysis performed on dislocation cell patterns developed during severe deformation of polycrystalline NaCl, thus highlighting that such patterns are not confined to f.c.c. metals. A more clear picture of the complex, fractal features of plasticity has been evidenced by means of acoustic emission measurements, which have shown that dislocations dynamics is characterized by intermittency, self-organization of dislocation avalanches into scale-free pattern, power-law distributions of avalanche sizes, fractal distributions of avalanche locations and complex space-time coupling [56,57].

All these findings indicate that dislocation cell structures in metals are often not appropriately characterized by any regular, periodic or almost periodic patterns; therefore, it should be remarked that a microstructure cannot be characterized through an average dislocation density (as assumed, for instance, in models based on the gradient plasticity theory).

In their works, Zaiser and co-workers demonstrate that dislocation cell structures can be envisaged as self-similar hole fractals, which are characterized by power-law distributions of the cell sizes; analytically, the number N of cells of size λ that exceed the size Λ follows a power-law:

$$N(\lambda > \Lambda) \propto \Lambda^{-D} \tag{4}$$

where D is the fractal dimension. The self-similarity of the observed pattern has been verified by using different techniques to determine the Hausdorff (fractal) dimension. It has been shown that the fractal regime of dislocations extends over two orders of magnitude and that the fractal dimension is a function of the applied stress.

Recently, a theoretical evolution model for the development of these fractal patterns has also been provided [58–60]. The dynamic dislocation model is based on the fact that strong long-range interactions arise in dislocation systems, which result in a lack of separability of the length scales. As a consequence, averaged deterministic evolution equations are meaningless and a stochastic approach should be followed. Within this framework, the "driving force" for the evolution of the dislocation patterns is given by the intrinsic fluctuations of the deformation process, which are caused by the collective motion of the dislocations themselves. These stress and strain rates, which are modelled as white noise, stem from the uncorrelated slip events associated with the intermittent passing of mobile dislocations. Therefore, the formation of dislocation patterns is assumed to be induced by sufficiently high levels of noise, in analogy with noise-induced transitions in non-equilibrium phase changes. The transition from homogeneous dislocation structures – in the case of low noise – to scale-invariant fractal structures – in the case of strong noise – is characterized by the emergence of hyperbolic dislocation density spectra, which indicate the spontaneous formation of dislocation-free regions (cell structure). These regions are organized in a self-similar way over a certain range of scales; it has been argued that cell patterning arises as a combination of dynamic recovery processes (easy dislocation recombination and annihilation), and a significant degree of correlated dislocation glide (high stress sensitivity). These stochastic models have been used to predict the occurrence of quasi-periodic and further fractal microstructures. However, according to Kubin and Devincre [61], it is not definitely established that the long-range elastic stresses due to mobile dislocations are the reason for dislocation patterning, nor that the role of the internal stresses from stored dislocations can be neglected. These authors also show the role of junction processes, which make a significant contribution to the flow stress.

This experimental and theoretical evidence of fractality in the plasticity of metals suggests a straight fractal approach to the size-scale effects in indentation. At a given indentation depth, the force P is a (constant) macro-parameter, but can be interpreted through the definition of a fictitious shear microstress τ_n at the scale L_n. By applying the Renormalization Group Theory [62], the force can be expressed for a scale cascade:

$$P = \tau_0 A_0 = \tau_1 N_1 A_1 = \tau_2 N_2 A_2 = \cdots = \tau_n N_n A_n \tag{5}$$

where τ_i is the fictitious shear microstress at the scale L_i, acting on N_i areas A_i. Equating the expressions of P at two subsequent scales, the recursive relation is obtained:

$$\tau_n = \frac{N_{n+1}}{N_n}\frac{A_{n+1}}{A_n}\tau_{n+1} = \frac{N_{n+1}}{N_n}\left(\frac{L_{n+1}}{L_n}\right)^2 \tau_{n+1} \tag{6}$$

which represents the generic transformation of the renormalization procedure. This links the microstress measured at the nth scale to the same stress, measured at the $(n+1)$th scale. In closed form, the application of the Renormalization Group Theory to the stress τ is equivalent to the statement of *self-similarity of the second kind* (or *incomplete similarity*) in the framework of the theory of Intermediate Asymptotics [63]. This means that the fictitious microstress at a given scale is a function of the scale, that is:

$$\tau_n = \tau(L_n) \tag{7}$$

If p denotes the ratio of two subsequent observation scales, Eq. (6) can be rewritten as:

$$\tau(L_n) = \frac{N_{n+1}}{N_n}\frac{A_{n+1}}{A_n}\tau(L_{n+1}) = \frac{N_{n+1}}{N_n}\frac{1}{p^2}\tau\left(\frac{L_n}{p}\right) \quad \text{with } p = \frac{L_n}{L_{n+1}} \tag{8}$$

In several physical phenomena, a scale dependence of the kind of Eq. (7) can be expressed asymptotically by means of a power-law, which represents the *intermediate asymptote* of the physical quantity:

$$\tau(L_n) \propto (L_n)^x \tag{9}$$

Accordingly, Eq. (8) can be expressed in the following form:

$$(L_n)^x = \frac{N_{n+1}}{N_n} p^{-(x+2)} (L_n)^x \tag{10}$$

It is worth noting that this equation is of the form:

$$f(L_n) = L_n \tag{11}$$

which is the equation of a *fixed point* $L_n = L^*$ in the renormalization transformation f. This point is therefore also a *critical point*, at which the correlation length of the phenomenon tends to infinity and the system itself becomes invariant with respect to the transformation. The Renormalization Group Theory in fact states the system invariance with respect to scale transformations in proximity of a critical point. Thanks to this invariance, Eq. (10) maintains its validity for any couple of scales.

From Eq. (10), it is also possible to determine the critical exponent, by observing that the term L_n relative to the scale disappears:

$$x = \log_p\left(\frac{N_{n+1}}{N_n}\right) - 2 = \frac{\ln(N_{n+1}/N_n)}{\ln(p)} - 2 = \Delta\tau - 2 \tag{12}$$

where $\Delta\tau$ is the self-similarity dimension [36], which physically represents the increment in the topological dimension of the dislocation pattern. The critical exponent of the fractal scaling regime is therefore a negative quantity:

$$x = \Delta\tau - 2 = -d_\tau, \quad 0 < d_\tau < 1 \tag{13}$$

and the fractal scaling law can be rewritten as: $\tau_n \propto L_n^{-d_\tau}$. The negative value of the exponent is a direct consequence of the topology of the fractal dislocation pattern and implies a decrease in the stress as the structural size increases. The true scale-independent parameter is, in this case, the renormalized stress τ^*, which can only be obtained by abandoning the classical dimensions of the stress τ ($[F][L]^{-2}$) and introducing non-integer (fractal) dimensions:

$$[\tau^*] = [F][L]^{-(2-d_\tau)} \tag{14}$$

The introduction of fractal stress leads to what Mandelbrot calls a *divergence syndrome* [36]. At a first glance, such a quantity may look strange, but a careful reexamination shows it is acceptable: if a finite non-fractal stress is considered, which acts over a fractal domain (the fractal dislocation arrangement), an infinite force will be obtained and this cannot be accepted. This point is crucial, since it requires a new method of thought. Therefore, the macroscopic (or nominal) stress is size dependent, as experimentally observed:

$$\tau \propto \tau^* L^{-d_\tau} \tag{15}$$

This size-scale effect, which consists of a simple power-law, could be defined as a fractal scaling regime, in analogy to what has been done by Carpinteri [37] for tensile strength and fracture energy.

In order to obtain a scaling law for material hardness, it is sufficient to consider the direct proportionality between τ, σ and H, by assuming validity of the von Mises flow rule and of the well-known Tabor rule (these assumptions are common to several models for ISE in metals):

$$\tau = \sqrt{3}\sigma \tag{16}$$

$$H = k\sigma \tag{17}$$

where k is the Tabor proportionality constant. As a consequence, the same scaling type also applies to hardness H (see Fig. 4):

$$H \propto H^* L^{-d_H} \quad \text{with } d_\tau = d_H \tag{18}$$

Obviously, this scaling regime is valid only in a limited range of scales, where the fractal dimension of the dislocation pattern could be considered constant. If specimens of the same material are tested over a broad range of scales, experiments show that the fractal scaling of H is valid only in a limited scale range, where the fractal dimensions of the supporting domains can be considered to be constant. As the size increases, in fact, the concept of geometrical multifractality implies the progressive vanishing of fractality ($d_H \to 0$), with a corresponding homogenization of the domains, as previously seen in the case of tensile strength. Intuitively, since the dislocation structure of a stressed metal is the same, independently of the macroscopic specimen size, its effect on the mechanical properties essentially depends on the ratio between a characteristic material length and the

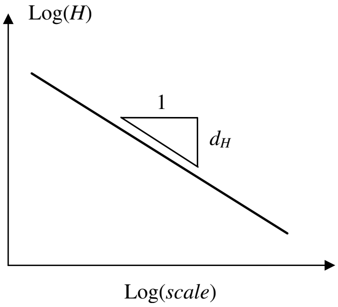

Fig. 4. Fractal scaling of the indentation hardness H.

size of the specimen. Therefore, the effect of the microstructure on the mechanical behaviour of materials becomes progressively less important at the larger scales, whereas it is the fundamental source of the size-scale effects at smaller scales. This transition could be viewed, in the case of ISE, as a homogenization of the dislocation pattern formation at larger scales, which leads to an asymptotic dislocation density in larger structures and to a corresponding asymptotic finite value of hardness. In addition, it should be noted that the occurrence of a cross-over between fractal and Euclidean dimensions has been observed by Gil Sevillano et al. [52] as a function of the strain level. More recently, the presence of an upper cut-off for the fractal regime has been shown to be in relation with the finite size of the stressed volume, and to the presence of grain boundaries in polycrystalline materials [64].

In other words, two asymptotes characterize the dependence of hardness on the observation scale. The first one, valid at smaller scales, where the fractal dimension of the dislocation structure may be considered constant, is given by a power-law dependence of hardness on the observation scale. The second asymptote, referring to larger scales, where fractality vanishes, should be horizontal (the hardness no longer depends on the scale). Any scaling law for indentation hardness should encompass these kinds of asymptotic behaviour.

These arguments can be invoked in order to change the mono-fractal scaling regime of Eqs. (15) and (18) into a multi-fractal scaling regime, as has already been done for the fracture strength of quasi-brittle materials [38]. The result on hardness H is easily expressed as

$$H = H_0\left(1 + \frac{L^*}{L}\right)^{\frac{1}{2}} \tag{19}$$

which could be defined as the Multifractal Scaling Law for hardness, as sketched in Fig. 5. Note the close resemblance to the Multifractal Scaling Law for tensile strength, reported in Eq. (3). It is clear that the structural size tested by the indenter is characterized by the indentation depth h, which could replace the generic structural size L in the above equation.

Since the proposed law is a best-fit law, the fractal model does not provide expressions for H_0 and L^*; these parameters are instead determined from best-fitting procedures. It is only possible to provide an interpretation

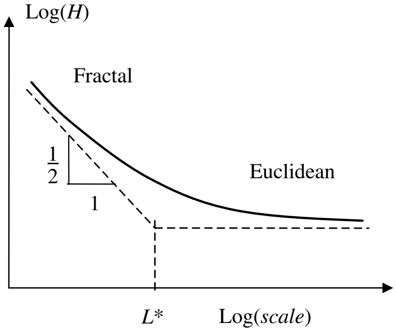

Fig. 5. Indentation hardness H: transition from the fractal to the Euclidean scaling regime.

of their physical meanings: H_0 is the hardness of the material measured at large sizes (large enough to consider the dislocation structure as homogeneous). Thus, a possible expression of H_0 is

$$H_0 = kZ\mu b\sqrt{\rho_0} \tag{20}$$

where k is the Tabor constant, μ is the shear modulus, b is the magnitude of the Burgers vector, ρ_0 is the total dislocation density that causes material hardening, measured at the macroscale, and Z is the Taylor factor, which acts as an isotropic interpretation of the crystalline anisotropy at the continuum level [21]. The values of Z are $\sqrt{3}$ for isotropic solids and 3.08 for FCC polycrystalline metals [63,65]. The key point in Eq. (20) is how to express ρ_0, which may come from dislocation evolution models. In the Hähner and Zaiser model [59], for instance, its value is determined by a stochastic hardening relation with the macroscopical flow stress and a complex interplay among several statistical parameters. Its meaning, however, is very clear: is the mean (homogenous) value of the dislocation density measured at the macroscale, where homogenization of the material deformation occurs.

L^* is the critical material characteristic length, in relation to the microstructure, which individuates the transition between the two regimes, that is, the fractal one and the Euclidean one. Again in this case, a formula cannot be provided by the present model, since L^* is determined from best-fitting procedures. An expression may be obtained from dislocation evolution models. In [58], L^* is a function of the slip-line length and of the fractal dimension of the dislocation pattern itself.

Physically speaking, the value of L^* in Eq. (19) and the experimental upper cutoff for the fractal regime have the same meaning, since for $L < L^*$ the slope of the curve is between 0 and 1/2 (fractal regime), whilst for $L > L^*$ the slope tends to zero (homogeneous regime, where the dislocation pattern ceases to be fractal and could be interpreted by means of a mean, homogenized dislocation density ρ_0).

It is remarkable that Eq. (19) has independently been already proposed to interpret ISE by different authors [6,7,62,66], but on the basis of a very different physical model. The value 1/2 of the fractal exponent came to Carpinteri and coworkers from physical and statistical considerations [37,38]. In the case of dislocation patterns, however, there is no immediate evidence of such an upper bound. A confirmation comes from experiments on ice crystals, where it has experimentally found that the scale-invariant spatial distribution of dislocation avalanches is characterized by a correlation dimension $D = 2.5 \pm 0.1$ [67].

A best fit of experimental data from microindentation experiments on both single crystal and polycrystalline copper is shown in Figs. 6 and 7. The agreement of the MultiFractal Scaling Law for the hardness with the experiments is remarkably good. For polycrystalline Cu we obtained $H_{0,\text{pc}} = 0.834$ GPa and $L^*_{\text{pc}} = 0.464$ μm, whilst for single crystal Cu we obtained $H_{0,\text{sc}} = 0.581$ GPa and $L^*_{\text{sc}} = 1.60$ μm. It is worthwhile noting that L^*_{sc} and L^*_{pc} are both located at about 1 μm, where the upper cutoff for the fractal regime is observed [58]. Furthermore, $L^*_{\text{sc}} > L^*_{\text{pc}}$; this fact strengthens the physical interpretation of L^*, since it was found that the upper cutoff for the fractal regime is smaller in polycrystalline Cu than in polycrystalline Cu.

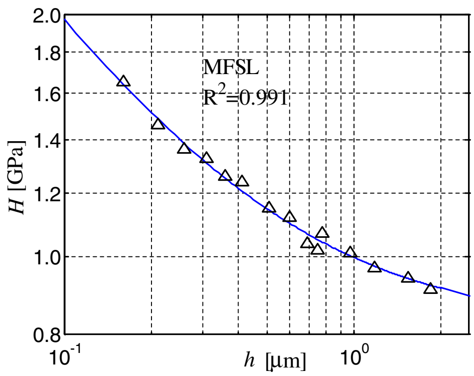

Fig. 6. Indentation hardness H as a function of the indentation depth h: best fit of experimental data from microindentation tests on single crystal copper [7].

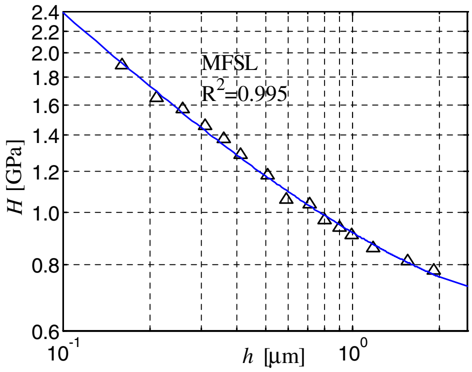

Fig. 7. Indentation hardness H as a function of the indentation depth h: best fit of experimental data from microindentation tests on polycrystalline copper [7].

4. Conclusions

A fractal approach to ISE in metals, which is based on the experimental evidence of fractal patterns in the plastic deformation of metals is presented in this paper. The introduction of a fractal domain, characterized by non-integer physical dimensions, leads to size-scale effects on the mechanical quantities that define the behaviour of the system, the true size-independent quantities being the corresponding fractal mechanical properties, which are characterized by anomalous non-integer dimensions.

The proposed approach has the great advantage of being based on the very general concept of fractality; therefore, it is not limited to ductile metals, but could apply to any type of brittle or quasi-brittle material. In this context, it should be emphasized that the fractal approach explains the size effect associated to macroscopically homogeneous deformation, as in simple tension, therefore without strain gradients, where strain gradient theories do not succeed.

It could be remarked that the fractal approach provides a simple interpretation of ISE. Fractals provide a general tool for dealing with size-scale effects [51]; in this sense, they do not exclude a different mechanical explanation. Nevertheless, their appearance in several natural phenomena subtends some central issues of the mechanics of materials. With regard to ISE in metals, it is clear that fractals only synthetically account for the complex multi-scale phenomena that occur during plastic deformation, which are in relation to the dislocation origin, growth, interaction and evolution. This is the real issue that should be investigated, in order to cast new light on ISE.

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