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Numerical evaluation of generalized stress-intensity factors in multi-layered composites

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Abstract

The problem of the evaluation of the generalized stress-intensity factors for re-entrant corners in multi-layered structural components is addressed. An approximate analytical model based on the theory of multi-layered beams is presented. This approach provides a simple closed-form solution for the direct computation of the Mode I stress-intensity factor for the general problem of a re-entrant corner symmetrically meeting a bi-material interface. For the self-consistency of the theory, re-entrant corners in homogeneous materials and cracks perpendicular to bi-material interfaces can also be gained as limit cases of this formulation. According to this approach, the effects of the elastic mismatch parameters, the value of the notch angle and the thicknesses of the layers on the stress-intensity factor are carefully quantified and the results are compared with FE solutions. FE results are obtained by applying a combination of analytical and numerical techniques based on the knowledge a priori of the asymptotic stress field for re-entrant corners perpendicular to a bi-material interface and on the use of generalized isoparametric singular finite elements at the notch tip. A good agreement between approximate and analytical/numerical predictions is achieved, showing the effectiveness of this approach.
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1. Introduction

Since the pioneering paper by Zak and Williams (1963), cracks in bi-material structural elements have been deeply investigated, both from the theoretical (Cook and Erdogan, 1972) and the experimental (Wang and Chen, 1993) point of view. In spite of this, cracks are only a particular case of the more general problem

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of re-entrant corners. In this context, the power of the stress-singularity at the vertex of re-entrant corners in homogeneous materials has been given by Williams (1952). From a numerical point of view, Walsh (1974) extended conventional finite element procedures to non-zero angle notch problems and the Reciprocal Work Contour Integral was introduced by Carpenter (1984) to compute fracture mechanics parameters for a general corner in a homogeneous structure. Size-scale effects were firstly considered by Leicester (1973) and afterwards experimentally and theoretically investigated by Carpinteri (1987) and by Carpinteri and Pugno (2000).

As far as re-entrant corners in heterogeneous materials are concerned, the evaluation of the power of the stress-singularity for the case of butt tensile joints was extensively investigated in the last decade (Reedy, 1993; Reedy and Guess, 1993). This study was subsequently extended by Qian and Akisanya (1999) to a more general geometrical configuration and a detailed investigation of the corresponding stress-singularity was proposed (see Fig. 1(a)).

In this paper we consider the general geometrical configuration depicted in Fig. 1(b), where the interface lies along the y-axis.

Hence, an approximate analytical solution (Carpinteri and Pugno, 2000) based on the theory of multi-layered beams is herein presented. This model provides a simple closed-form expression for the direct computation of the Mode-I stress-intensity factor for the general problem of a re-entrant corner symmetrically meeting a bi-material interface. For the self-consistency of the theory, re-entrant corners in homogeneous materials and cracks perpendicular to bi-material interfaces can also be gained as limit cases of this formulation. According to this approach, the effects of the elastic mismatch parameters, the value of the notch angle and the thicknesses of the layers on the stress-intensity factor are carefully investigated and quantified. The results are compared with FE solutions that are obtained by using a combination of analytical and numerical techniques based on the knowledge a priori of the asymptotic stress field and on the use of generalized isoparametric singular finite elements at the notch tip. In this case, in fact, the power of the stress-singularity depends both on the notch angle, and on the elastic mismatch parameters. Hence, a preliminary asymptotic analysis of the stress field has to be performed, since the power of the stress-singularity for this problem has not been found in the Literature. Therefore, this result can be considered as a generalization to those obtained by Williams (1952) and by Zak and Williams (1963) concerning, respectively, a re-entrant corner in a homogeneous material and a crack perpendicular to a bi-material interface.

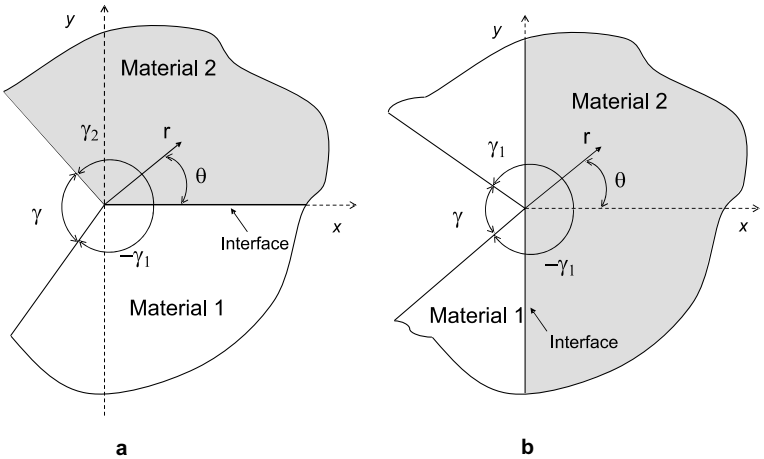


Fig. 1. Schematic of a re-entrant corner in a structure with (a) horizontal interface and (b) with vertical interface.

From the comparison, a good agreement between approximate and analytical/numerical predictions is achieved, demonstrating that the approximate solution can be particularly useful from the engineering point of view. In fact, the values of the stress-intensity factors can be gained by applying a single formula with a significative saving in computational time compared to the FE approach, with the possibility to easily perform parametric analyses. Moreover, it has to be remarked that the model can also be extended to other engineering configurations, such as the problem of a re-entrant corner in a multi-layered plate in tension.

2. Approximate solution

In this section the approach proposed by Carpinteri and Pugno (2000) is briefly summarized.

2.1. Generalized stress-intensity factors

In the study of multi-layered beams under axial load and bending moment (Fig. 2), by assuming the conservation of the plane sections (Carpinteri, 1997), the stress is given by

$$\sigma(y) = \frac{E(y)}{E_r}\left(\frac{N}{A^*} + \frac{M}{I^*}y\right), \tag{1}$$

where E_r is an arbitrary reference value of the Young's modulus; N and M are, respectively, the axial load and the bending moment at the specified section, and

$$A^* = \int_A \frac{E(y)}{E_r}\mathrm{d}A, \tag{2}$$

is the weighted area, as well as

$$I^* = \int_A \frac{E(y)}{E_r}y^2\mathrm{d}A, \tag{3}$$

represents the weighted moment of inertia. The origin of the reference system, the elastic centroid, is defined from the following relationship:

$$S^* = \int_A \frac{E(y)}{E_r}y\,\mathrm{d}A = 0. \tag{4}$$

Considering a linear elastic homogeneous beam with an edge crack subjected to three-point bending (Fig. 2, with $\gamma = 0$ and $E(y) = \text{const.}$), the symmetric stress field around the crack tip can be written as follows:

$$\sigma_{ij} = K_I r^{-1/2} S_{ij}(\theta), \tag{5}$$

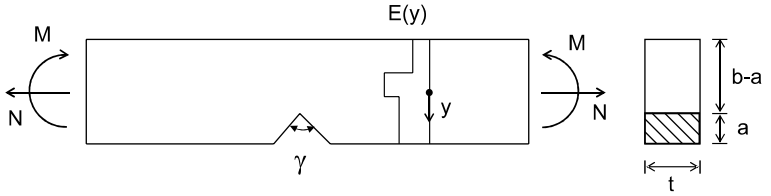


Fig. 2. Multi-layered beam under normal load and bending moment.

where K_I is the stress-intensity factor for the Mode I loading condition, r and θ are the polar coordinates represented in Fig. 1(b) and S_{ij} is a function describing the angular profile of the stress field. Furthermore, the analytical solution for the stress-intensity factor is given by

$$K_I = \frac{Pl}{tb^{3/2}} f(a/b), \tag{6}$$

where t, b and l denote, respectively, thickness, height and span of the beam, whereas f is a shape function depending on the geometry of the structure and on the ratio a/b, and can be expressed as follows ($a/b < 0.6$) (Carpinteri and Pugno, 2000):

$$f(a/b) = 2.9(a/b)^{1/2} - 4.6(a/b)^{3/2} + 21.8(a/b)^{5/2} - 37.6(a/b)^{7/2} + 38.7(a/b)^{9/2}. \tag{7}$$

According to Eischen (1987), the asymptotic stress field at the crack tip, $\tilde{\sigma}_{ij}$, for a structure made of a functionally graded material has the same singularity order as that for a homogeneous material:

$$\tilde{\sigma}_{ij} = \widetilde{K}_I r^{-1/2} S_{ij}(\theta), \tag{8}$$

where the superscript ($\sim$) denotes a heterogeneous structure. Eqs. (5) and (8) can be generalized to the case of a re-entrant corner with angle γ (see Fig. 2). When a re-entrant corner in a homogeneous element is considered and both the notch surfaces are stress-free, the symmetric stress field at the notch tip for the homogeneous structure is given by Carpinteri (1987)

$$\sigma^*_{ij} = K^*_I r^{\lambda-1} S^{(\gamma)}_{ij}(\theta), \tag{9}$$

where the power $\lambda - 1$ of the stress-singularity is provided by the following eigenequation:

$$\lambda \sin(2\pi - \gamma) = \sin[\lambda(2\pi - \gamma)] \tag{10}$$

and ranges between $-1/2$ (when $\gamma = 0$) and zero (when $\gamma = \pi$), see Table 1. In the sequel, superscript ($*$) denotes a structure with a re-entrant corner. On the other hand, when a re-entrant corner in a heterogeneous element is addressed, an analogous expression can be written

$$\tilde{\sigma}^*_{ij} = \widetilde{K}^*_I r^{\lambda-1} S^{(\gamma)}_{ij}(\theta). \tag{11}$$

Furthermore, if Buckingham's theorem for physical similitude and scale modeling is applied and stress and linear size are assumed as fundamental quantities, it is possible to write an equation analogous to Eq. (6) which holds in the case of a re-entrant corner in a heterogeneous structure

$$\widetilde{K}^*_I(\gamma) = \frac{\tilde{P}^* l}{tb^{1+\lambda}} \tilde{f}^*(\gamma, E(y), a/b); \quad [\widetilde{K}^*_I] = [F][L]^{-(1+\lambda)}. \tag{12}$$

When the heterogeneous structure becomes homogeneous (symbols without superscript ($\sim$), $E(y) = \text{const}$) and the notch angle γ vanishes, Eq. (12) must coincide with Eq. (6). On the contrary, when $\gamma = \pi$ the stress-singularity disappears and the generalized stress-intensity factor $\widetilde{K}^*_I$ assumes the physical dimensions of stress and becomes proportional to the nominal stress

$$\widetilde{K}^*_I(\gamma = \pi) = \tilde{\sigma}_u = \frac{\tilde{P}^\pi l}{tb^2} \tilde{g}(E(y), a/b). \tag{13}$$

Function $\tilde{g}(E(y), a/b)$ describes the reduction of the resisting cross-section and can be evaluated by integration over the ligament area according to Eq. (4) in case of a multi-layered beam

$$\tilde{g}(E(y), a/b) = \frac{E_r t b^2 \tilde{y}/4}{\int_{A_{lig}} E(y) y^2 \, dA}, \tag{14}$$

where $\tilde{y}$ denotes the coordinate of the crack tip with respect to the reference system given by Eq. (4). For a homogeneous structure it becomes

$$g(a/b) = \tilde{g}(E(y) = \text{const}, a/b) = \frac{3/2}{(1 - a/b)^2}. \tag{15}$$

On the other hand, the remaining limit case of Eq. (12) concerns a crack in a multi-layered structure. In this situation, the expression of $\tilde{f}(E(y), a/b)$ has to be determined, as well as the general form of $\tilde{f}^*(\gamma, E(y), a/b)$.

2.2. Approximate derivation of the generalized shape function

The analytical expression of the generalized shape function can be obtained by taking into account both the relationship between the critical stress-intensity factor for a re-entrant corner and that for a crack (Seweryn, 1994) and the relationship between the generalized critical stress-intensity factor and the brittleness number $\tilde{s}^*(\gamma)$, by generalizing some of the previous results by Carpinteri and Pugno (2000) strictly derived for homogeneous materials only

$$\widetilde{K}^*_{\mathrm{IC}}(\gamma) = \lambda \tilde{\sigma}_{\mathrm{u}} \left(\frac{2\widetilde{K}_{\mathrm{IC}}}{\tilde{\sigma}_{\mathrm{u}}} \right)^{2(1-\lambda)}, \tag{16a}$$

$$\tilde{s}^*(\gamma) = \frac{\widetilde{K}^*_{\mathrm{IC}}(\gamma)}{\tilde{\sigma}_{\mathrm{u}} b^{(1-\lambda)}}, \tag{16b}$$

where $\tilde{\sigma}_{\mathrm{u}}$ denotes the ultimate strength of the structure. From Eqs. (16) the following expression is derived:

$$\tilde{s}^*(\gamma) = \lambda (2\tilde{s})^{2(1-\lambda)}. \tag{17}$$

In the opposite cases of a crack and a corner angle close to π (a flat angle) we obtain

$$\tilde{s}^*(\gamma = 0) = \tilde{s} = \frac{\widetilde{K}_{\mathrm{IC}}}{\tilde{\sigma}_{\mathrm{u}} \sqrt{b}}, \tag{18a}$$

$$\tilde{s}^*(\gamma = \pi) = 1. \tag{18b}$$

The last trivial equation means that, for an uncracked structure, the generalized brittle and ductile collapses coincide.

Considering a three-point bending specimen and substituting Eq. (16b) into Eq. (12) in the critical condition ($\widetilde{K}^*_I = \widetilde{K}^*_{\mathrm{IC}}$, $\tilde{P}^* = \tilde{P}^*_{\mathrm{C}}$), we obtain the non-dimensional failure load as a function of the generalized brittleness number and of the shape function

$$\frac{\tilde{P}^*_{\mathrm{C}} l}{t b^2 \tilde{\sigma}_{\mathrm{u}}} = \frac{\tilde{s}^*(\gamma)}{\tilde{f}^*(\gamma, E(y), a/b)}. \tag{19}$$

This kind of collapse is always intermediate between brittle ($\widetilde{K}_{\mathrm{I}} = \widetilde{K}_{\mathrm{IC}}$) and ductile ($\tilde{\sigma}^* = \tilde{\sigma}_u$) collapses. Eq. (19) can be evaluated for a crack

$$\frac{\tilde{P}^*_{\mathrm{C}} l}{t b^2 \tilde{\sigma}_{\mathrm{u}}} = \frac{\tilde{s}}{\tilde{f}(E(y), a/b)}, \tag{20}$$

and for a flat angle

$$\frac{\tilde{P}^{\pi}_{\mathrm{C}} l}{t b^2 \tilde{\sigma}_{\mathrm{u}}} = \frac{1}{\tilde{g}(E(y), a/b)}. \tag{21}$$

In case of a structural element with a crack of a given relative depth, the transition between brittle and ductile collapse (Carpinteri, 1987; Carpinteri and Pugno, 2000) arises when the failure loads given by Eqs. (20) and (21) are equal. In this case we obtain

$$\tilde{s} = \frac{\tilde{f}(E(y), a/b)}{\tilde{g}(E(y), a/b)}. \tag{22}$$

If the notch angle is different from zero, the crack becomes a re-entrant corner and the transition arises when the failure loads (19) and (20) are equal for

$$\tilde{s}^*(\gamma) = \frac{\tilde{f}^*(\gamma, E(y), a/b)}{\tilde{g}(E(y), a/b)}. \tag{23}$$

Substituting Eqs. (22) and (23) into Eq. (17), we obtain the generalized shape function for a re-entrant corner in a bi-material structural component

$$\tilde{f}^*(\gamma, E(y), a/b) = \lambda \tilde{g}(E(y), a/b) \left(2 \frac{\tilde{f}(E(y), a/b)}{\tilde{g}(E(y), a/b)} \right)^{2(1-\lambda)}. \tag{24}$$

It has to be remarked that exactly the same result can be gained for the problem of plates in tension.

Substituting the generalized stress-intensity factor (16a) and the shape function (24) into Eq. (12), we obtain the failure load for a multi-layered structural component with a re-entrant corner in a very synthetic form

$$\frac{\tilde{P}_C^*}{\tilde{P}_C^{\pi}} = \left(\frac{P_C^*}{P_C^{\pi}} \right)^{2(1-\lambda)}. \tag{25}$$

At this point, we can observe that the ratio $\tilde{P}_C^*/P_C^*$ is, as a first approximation, independent of the corner angle. Consequently, we have (see Eq. (12))

$$\frac{\tilde{f}^*(\gamma, E(y), a/b)}{f^*(\gamma, a/b)} = \frac{\tilde{g}(E(y), a/b)}{g(a/b)} \left(\frac{\sigma_u \widetilde{K}_{IC}}{\tilde{\sigma}_u \widetilde{K}_{IC}} \right)^{2(1-\lambda)}. \tag{26}$$

This equation is an identity for the limit case of a flat angle. On the other hand, considering the limit case of a crack, we derive the expression of $\tilde{f}(E(y), a/b)$

$$\tilde{f}(E(y), a/b) = \frac{\sigma_u \widetilde{K}_{IC}}{\tilde{\sigma}_u K_{IC}} \frac{\tilde{g}(E(y), a/b)}{g(a/b)} f(a/b), \tag{27}$$

Table 1
Parameters λ and ω (defined in Section 3.2) for different values of γ in case of a re-entrant corner

γ	λ	ω
0°	0.500	0.250
15°	0.500	0.250
30°	0.501	0.251
45°	0.505	0.253
60°	0.512	0.258
75°	0.524	0.266
90°	0.544	0.279

that can be further simplified by assuming $\frac{\sigma_u \widetilde{K}_{IC}}{\bar{\sigma}_u K_{IC}} \simeq 1$. Eventually, by introducing Eq. (27) into Eq. (24), the shape function for a re-entrant corner in a multi-layered structural component is fully determined.

3. Analytical and numerical solution

From a numerical point of view, since the early finite element studies, it was recognized that, unless a specific treatment could be used, it would be necessary to have a very refined mesh at the crack tip to approximate the stress-singularity with normal elements. Three most important approaches that have been put forward in the last decades in order to improve the accuracy and the rate of convergence to the solution: (1) hp-finite elements (Schwab, 1998); (2) enriched elements (Benzley, 1974) and (3) singular elements (Barsoum, 1976).

As far as singular elements are concerned, Barsoum (1976) demonstrated that the inverse square root singularity characteristic of linear elastic fracture mechanics can be obtained both in the 2D-8 nodes (Q8) and in 2D-6 nodes (T6) isoparametric elements when the mid-side nodes near the crack tip are placed at the quarter point. Furthermore, the concept of the quarter-point singular element has been generalized by Staab (1983) in order to capture singularities of order different from $-1/2$. In fact, by varying the placement of the mid-side node between the quarter-point and the mid-point position, it is possible to model a singularity of a generic order. For instance, Lim and Lee (1995) applied this technique for modeling the stress-singularity of a crack perpendicular to a bi-material interface and proposed a powerful method for computing the stress-intensity factor.

In this section we apply the technique proposed by Lim and Lee to the more general problem of re-entrant corners in multi-layered elements by using the public domain FE code FRANC2D (FRacture ANalysis Code 2D) (Wawrzynek and Ingraffea, 1991; Wawrzynek, 1991). To this aim, the preliminary computation of the order of the stress-singularity is required as an input for the numerical method. Hence, since this information is not yet available in the Literature, we perform an asymptotic analysis of the stress field according to the Muskhelishvili complex function representation. This analysis cannot be omitted, since this is an extension of the previous studies by Williams (1952) and by Zak and Williams (1963) concerning, respectively, the limit cases of a re-entrant corner in a homogeneous material and a crack perpendicular to a bi-material interface.

3.1. Asymptotic analysis of the singular stress field at the vertex of re-entrant corners symmetrically meeting a bi-material interface

In this analysis we study the geometry depicted in Fig. 1(b) which contains two stress free surfaces symmetrically oriented at $\theta = \gamma_1$ with respect to the x-axis. The case of plane isotropic elasticity in the absence of body forces is addressed. In this case, according to Muskhelishvili (1953), the stresses and displacements in the material m ($m = 1, 2$) around the interface corner can be expressed in terms of the complex potentials ψ_m and χ_m

$$\sigma_r^m + i\tau_{r\theta}^m = \psi'_m(z) + \overline{\psi'_m(z)} - z\overline{\psi''_m(z)} - \bar{z}z^{-1}\overline{\chi'_m(z)}, \tag{28a}$$

$$\sigma_\theta^m - i\tau_{r\theta}^m = \psi'_m(z) + \overline{\psi'_m(z)} + z\overline{\psi''_m(z)} + \bar{z}z^{-1}\overline{\chi'_m(z)}, \tag{28b}$$

$$u_r^m + iu_\theta^m = \frac{1}{2\mu_m} e^{-i\theta}[\kappa_1 \psi_m(z) - z\overline{\psi'_m(z)} - \overline{\chi_m(z)}], \tag{28c}$$

where $z = re^{i\theta}$ is a complex variable, μ is the shear modulus, v is the Poisson's ratio, $\kappa = 3 - 4v$ for plane strain and $\kappa = 3 - 4v/(1 + v)$ for plane stress. The symbols ($'$) and ($-$) denote, respectively, a derivative with respect to z and the complex conjugate of the variable. Following England (1971), the complex potentials are assumed to have the following form as $z \to 0$:

$$\psi_1 = Az^{\lambda}; \quad \chi_1 = Bz^{\lambda}, \quad \text{for Material 1,} \tag{29a}$$

$$\psi_2 = Cz^{\lambda}; \quad \chi_2 = Dz^{\lambda}, \quad \text{for Material 2,} \tag{29b}$$

where $A = A_1 + iA_2$, $B = B_1 + iB_2$, $C = C_1 + iC_2$ and $D = D_1 + iD_2$ are complex constants and λ defines the power of the stress-singularity. By substituting Eqs. (29) into Eqs. (28), stress and displacement fields in the two materials are derived

$$\begin{aligned}\sigma_r^m = {} & \lambda r^{\lambda-1}\{X_1^m(3-\lambda)\cos[(\lambda-1)\theta] - X_2^m(3-\lambda)\sin[(\lambda-1)\theta] \\ & - Y_1^m\cos[(\lambda+1)\theta] + Y_2^m\sin[(\lambda+1)\theta]\},\end{aligned} \tag{30a}$$

$$\begin{aligned}\sigma_\theta^m = {} & \lambda r^{\lambda-1}\{X_1^m(\lambda+1)\cos[(\lambda-1)\theta] - X_2^m(\lambda+1)\sin[(\lambda-1)\theta] \\ & + Y_1^m\cos[(\lambda+1)\theta] - Y_2^m\sin[(\lambda+1)\theta]\},\end{aligned} \tag{30b}$$

$$\begin{aligned}\tau_{r\theta}^m = {} & \lambda r^{\lambda-1}\{X_1^m(\lambda-1)\sin[(\lambda-1)\theta] + X_2^m(\lambda-1)\cos[(\lambda-1)\theta] \\ & + Y_1^m\sin[(\lambda+1)\theta] + Y_2^m\cos[(\lambda+1)\theta]\},\end{aligned} \tag{30c}$$

$$\begin{aligned}u_r^m = {} & \frac{1}{2\mu_m} r^{\lambda}\{X_1^m(\kappa_m-\lambda)\cos[(\lambda-1)\theta] - X_2^m(\kappa_m-\lambda)\sin[(\lambda-1)\theta] \\ & - Y_1^m\cos[(\lambda+1)\theta] + Y_2^m\sin[(\lambda+1)\theta]\},\end{aligned} \tag{30d}$$

$$\begin{aligned}u_\theta^m = {} & \frac{1}{2\mu_m} r^{\lambda}\{X_1^m(\kappa_m+\lambda)\sin[(\lambda-1)\theta] + X_2^m(\kappa_m+\lambda)\cos[(\lambda-1)\theta] \\ & + Y_1^m\sin[(\lambda+1)\theta] + Y_2^m\cos[(\lambda+1)\theta]\},\end{aligned} \tag{30e}$$

where X^m and Y^m are either A and B when $m = 1$, or C and D when $m = 2$.

By considering the symmetry of the problem, we can write the following boundary conditions that represent the symmetry conditions along $\theta = 0$, the matching conditions along the vertical interface ($\theta = \pi/2$) and the free edge boundary condition along $\theta = \gamma_1$

$$u_\theta^2 = 0 \qquad \theta = 0, \qquad 0 < r < \infty, \tag{31a}$$

$$\tau_{r\theta}^2 = 0 \qquad \theta = 0, \qquad 0 < r < \infty, \tag{31b}$$

$$\sigma_\theta^1 - i\tau_{r\theta}^1 = \sigma_\theta^2 - i\tau_{r\theta}^2 \quad \theta = \pi/2, \quad 0 < r < \infty, \tag{31c}$$

$$u_r^1 + iu_\theta^1 = u_r^2 + iu_\theta^2 \quad \theta = \pi/2, \quad 0 < r < \infty, \tag{31d}$$

$$\sigma_\theta^1 - i\tau_{r\theta}^1 = 0 \qquad \theta = \gamma_1, \qquad 0 < r < \infty. \tag{31e}$$

Substituting Eqs. (30) into Eqs. (31), a homogeneous system of eight linear equations in the eight unknown coefficients A_i, B_i, C_i, D_i ($i = 1, 2$) is derived. A non-trivial solution to the system exists only if the determinant of the coefficient matrix vanishes. This occurs when the eigenvalue, λ, satisfies the following characteristic equation:

$$\begin{aligned}F = {} & \left[1 - \beta + 2\alpha(1 + \alpha - \alpha\lambda^2 + \beta(\lambda^2 - 1))\right]\sin\pi\lambda + \alpha(\beta - \alpha)\lambda^2\sin(\pi\lambda - 2\gamma_1) - (\alpha + \alpha^2 - \alpha\beta) \\ & \times \sin[2\lambda(\pi - \gamma_1)] + \beta\lambda\sin(2\gamma_1) - (1 + \alpha)(\alpha - \beta)\sin 2\lambda\gamma_1 + (\alpha\beta\lambda^2 - \alpha^2\lambda^2)\sin(\pi\lambda + 2\gamma_1) \\ = {} & 0.\end{aligned} \tag{32}$$

This equation depends both on the power of the singularity, λ, and on the composite parameters, α and β, as defined by Zak and Williams (1963)

$$\alpha = \frac{\mu_1/\mu_2 - 1}{1 + \kappa_1}, \tag{33a}$$

$$\beta = \frac{\mu_1}{\mu_2}\frac{1 + \kappa_2}{1 + \kappa_1}. \tag{33b}$$

The eigenequation (32) can be considered as a generalization of the eigenequations given by Williams (1952) and by Zak and Williams (1963) concerning re-entrant corners in homogeneous structures and a crack perpendicular to a bi-material interface, respectively. In fact, we can observe that this expression contemplates both such problems as limit cases. If we consider a homogeneous structure with a re-entrant corner, we can substitute $\alpha = 0$ and $\beta = 1$ into Eq. (32), obtaining

$$F = \lambda \sin 2\gamma_1 + \sin 2\lambda\gamma_1 = 0, \tag{34}$$

which represents the eigenequation given by Williams (1952). On the other hand, if we study a crack perpendicular to a bi-material interface, we can substitute $\gamma_1 = \pi$ into Eq. (32), obtaining

$$F = \sin \lambda\pi[\lambda^2(-4\alpha^2 + 4\alpha\beta) + 2\alpha^2 - 2\alpha\beta + 2\alpha - \beta + 1 - (2\alpha^2 - 2\alpha\beta + 2\alpha - 2\beta)\cos \lambda\pi] = 0, \tag{35}$$

which represents the eigenequation given by Zak and Williams (1963). Eventually, in the particular case of a crack in a homogeneous material ($\gamma_1 = \pi$, $\alpha = 0$ and $\beta = 1$), the above expression becomes

$$F = \sin 2\lambda\pi = 0, \tag{36}$$

giving $\lambda = 1/2$ and the well-known power of the stress-singularity. Analogously, in the case of a flat angle, $\gamma_1 = \pi$, we obtain $\lambda = 1$ and the singularity disappears.

Once the value of λ has been computed from Eq. (32), seven of the height unknown constant terms can be expressed in terms of the height. The last unknown constant can be normalized such that $\sigma_\theta(\theta = 0) = K_{\mathrm{I}}^* r^{\lambda-1}$, $\tau_{r\theta}(\theta = 0) = K_{\mathrm{II}}^* r^{\lambda-1}$. This allows to write the singular stresses and the displacements as

$$\sigma_{ij} = \tilde{\sigma}_{ij} = \widetilde{K}_N^* r^{\lambda-1} S_{ij}^N(\theta), \tag{37a}$$

$$u_i = \tilde{u}_i = \widetilde{K}_N^* r^{\lambda} T_i^N(\theta), \tag{37b}$$

where N is either 1 (Mode 1) or 2 (Mode 2). If a homogeneous structure with a re-entrant corner is considered, the solution of the above problem reduces to that given by Dunn et al. (1997). On the contrary, as far as a crack perpendicular to a bi-material interface is concerned, the explicit expressions can be found in Cook and Erdogan (1972). In both cases the asymptotic expressions of the stress field have been omitted, since they can be found in the above cited papers.

3.2. Generalized singular elements and computation of the Mode I stress-intensity factor

According to the method proposed by Lim and Lee (1995), the position of the intermediate nodes in a typical eight-node isoparametric element are moved from the normal mid-point position (Fig. 3). Placing the origin at node 1 and denoting the length of the side 1–3 with l, then $x_1 = 0$, $x_2 = \omega l$ and $x_3 = l$. In this configuration the x-coordinate is given by

$$x/l = \omega + 0.5\xi + 0.5(1 - 2\omega)\xi^2 \tag{38}$$

and the parameter ω represents the position of the mid-side node. The displacement variation along the horizontal side is in fact given by

$$u = [1 - 3(x/l)^\phi + 2(x/l)^{2\phi}]u_1 + [4(x/l)^\phi - 4(x/l)^{2\phi}]u_2 + [-(x/l)^\phi + 2(x/l)^{2\phi}]u_3, \tag{39}$$

whereas the strain in the x-direction is

$$\varepsilon_x \sim x^{\phi-1}, \tag{40}$$

with a singularity of order $\phi - 1$.

The parameter ω controls the power of the singularity. For cracks in homogeneous materials we set $\omega = 1/4$ and we obtain the typical quarter point element. As far as a crack perpendicular to and terminating at a bi-material interface is concerned, the parameter ϕ is equal to λ, according to Lim and Lee (1995). This

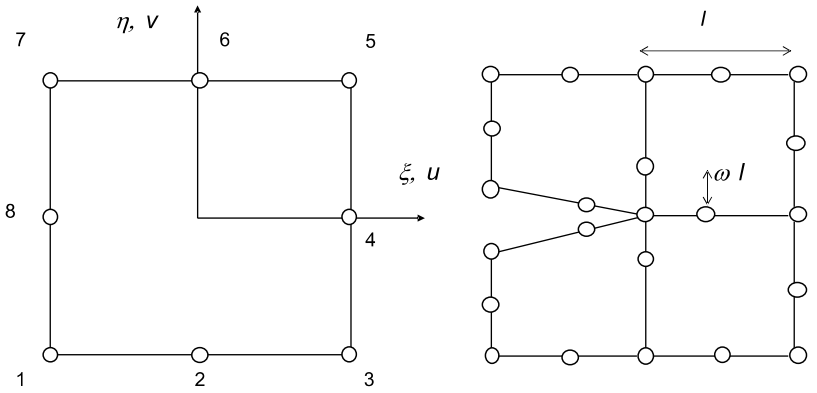


Fig. 3. Q8 isoparametric finite element and Q8 elements near a notch tip.

Table 2
Parameters λ and ω for different values of E_1/E_2 in the case of a crack perpendicular to a bi-material interface

E_1/E_2	λ	ω
0.1	0.709	0.374
0.2	0.661	0.348
0.4	0.597	0.311
0.8	0.524	0.266
1.0	0.500	0.250
1.2	0.475	0.233
2.5	0.399	0.177
5.0	0.322	0.116
10.0	0.250	0.054

result can be extended to the case of a re-entrant corner, where λ is given by the previous asymptotic analysis. The position of the mid-side node, ω, can be directly computed by a best-fit procedure, as demonstrated by Lim and Lee (1995). Parameter ω has been computed for different values of the angle of the re-entrant corner in a homogeneous material (see Table 1) and for different elastic mismatches and for plane stress conditions in the case of a crack perpendicular to and terminating at a bi-material interface (see Table 2).

Finally, the Mode I stress-intensity factor can be computed using a displacement correlation technique (the reader is referred to Lim and Lee (1995) for more details). This method for computing the stress-intensity factor has the advantage to be based on the calculated nodal displacements which, in a finite element calculation based on a displacement formulation, are more accurate than stresses.

4. Comparison between approximate and analytical solutions

This section focuses on the evaluation of the shape function for some structural problems with cracks and re-entrant corners. Considering a three-point bending bi-material beam with a crack, the effect of the elastic mismatch on the Mode I stress-intensity factor is investigated (Fig. 4). Hence, the ratio between the thicknesses of the two layers, h_1/h_2, has been kept constant in the simulations, whereas different values of the ratio between the elastic moduli, E_1/E_2, have been taken into account (see also Table 2). In each sim-

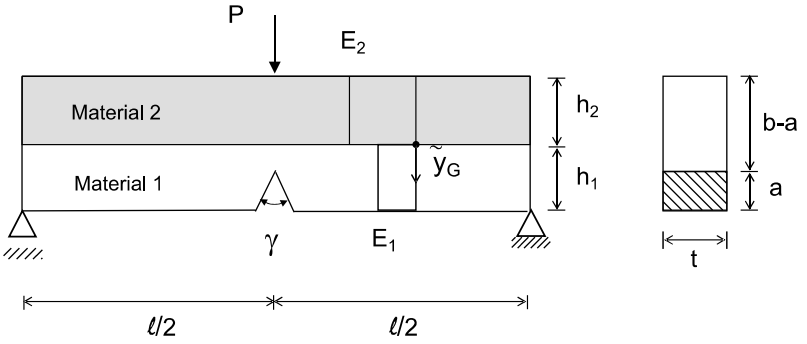
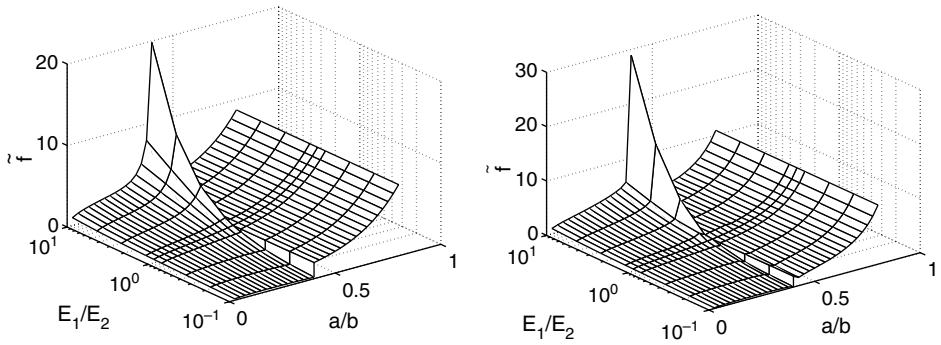


Fig. 4. Notched bi-layered three-point bending beam.



(a) (b)

Fig. 5. Approximate vs. analytical results: generalized shape function $\tilde{f}$ vs. relative crack depth a/b and Young's ratio E_1/E_2 for a bi-material three-point bending beam with a crack ($h_1/h_2 = 2/3$). (a) Approximate and (b) analytical.

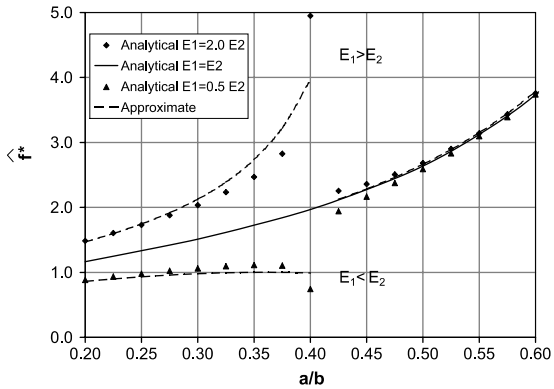


Fig. 6. Approximate vs. analytical results: generalized shape function $\tilde{f}$ vs. relative crack depth a/b for a bi-material three-point bending beam with a crack ($h_1/h_2 = 2/3$) and different modular ratios.

ulation, the non-dimensional crack depth, a/b, was varied from 0.0 to 0.8. The crack meets the horizontal interface at $a/b = 0.4$.

The theoretical shape function predicted according to Eq. (24) is depicted in terms of a/b and E_1/E_2 in Fig. 5(a). The corresponding result from asymptotic analysis and finite element computations is presented in Fig. 5(b). A good agreement between the two approaches is achieved. A detailed comparison between analytical and approximate solutions is presented in Fig. 6, where the cases corresponding to $E_1/E_2 = 0.5$ and $E_1/E_2 = 2.0$ are compared. The shape function corresponding to the limit case of $E_1/E_2 = 1$ is the well-known shape function f for a homogeneous structure given by Eq. (7). When there is an elastic mismatch, generalized shape functions present discontinuities in correspondence of the bi-material interface. Generalized shape functions for configurations having modular ratios E_1/E_2 less than unity slightly decrease when the crack tip approaches the bi-material interface. This implies that cracks are difficult to propagate across the bi-material interface when the substrate is stiffer than the external layer.

On the contrary, when E_1/E_2 is greater than unity, the generalized shape functions clearly increase with the crack length, because of the elastic mismatch. Unstable crack propagations are likely to occur in such cases. Furthermore, the higher is the ratio between the elastic moduli, the higher is the value of the shape function.

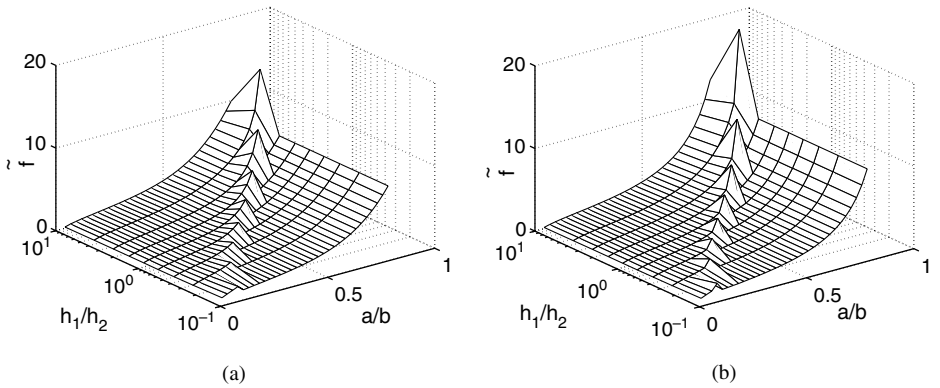


Fig. 7. Approximate vs. analytical results: generalized shape function $\tilde{f}$ vs. relative crack depth a/b and thicknesses ratio h_1/h_2 for a bi-material three-point bending beam with a crack ($E_1/E_2 = 1.8$). (a) Approximate and (b) analytical.

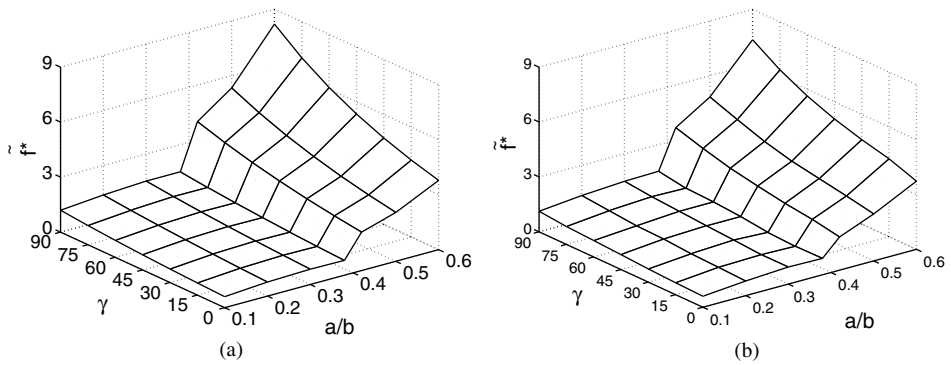


Fig. 8. Approximate vs. analytical results: generalized shape function $\tilde{f}^*$ vs. relative crack depth a/b and notch angle γ for a bi-material three-point bending beam with a re-entrant corner ($E_1/E_2 = 0.4$, $h_1/h_2 = 2/3$). (a) Approximate and (b) analytical.

When the crack tip overcomes the interface, the approximate model predicts a shape function equal to the classical one for homogeneous structures, independently of the ratio E_1/E_2. This result can be extended to multi-layered structures in a straightforward manner: as a first approximation, from a fracture point of view, the behavior of the tip in the last layer is not influenced by the remaining ones.

The effect of the thicknesses of the two layers on the shape function is investigated in Fig. 7. In these simulations, the ratio between the elastic moduli is kept constant ($E_1/E_2 = 1.8$), whereas the ratio h_1/h_2 is varied from 0.1 to 10.0. Discontinuities in the shape functions can be seen whenever the crack tip reaches the bi-material interface at $a/b = h_1/(h_1 + h_2)$. Also in this case, both approximate and analytical predictions agree satisfactorily and we can observe that the higher is the ratio h_1/h_2, the higher is the value of the shape function.

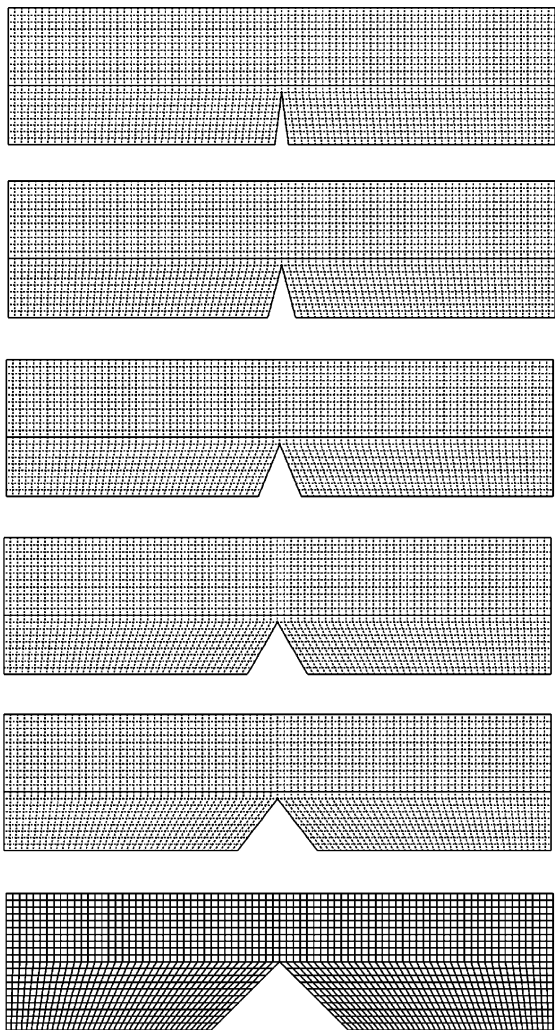


Fig. 9. Some original meshes for the finite element simulations concerning three-point bending beams with notch angles equal to 15°, 30°, 45°, 60°, 75° and 90°.

Eventually, the effect of the notch angle on the shape function is considered in Fig. 8. In this case the ratio between the elastic moduli is kept constant ($E_1/E_2 = 0.4$), as well as the position of the bi-material interface ($h_1/h_2 = 2/3$). Notch angles equal to 0°, 15°, 30°, 45°, 60°, 75° and 90°, with non-dimensional depths a/b from 0.0 to 0.6, are analyzed. Some original meshes for the finite element computations are depicted in Fig. 9. Both approximate and analytical approaches predict discontinuities of the shape functions at the bi-material interface, as in the case of a crack. In this case, the higher is the value of the notch angle, the higher is the value of the shape function.

5. Conclusions

Both approximate and analytical approaches herein presented permit to evaluate the shape function for the computation of the stress-intensity factor in multi-layered structural components with cracks or re-entrant corners. The asymptotic analysis of the stress-field near a re-entrant corner symmetrically meeting a bi-material interface shows that the power of the singularity depends both on the elastic mismatch, and on the notch angle. Limit cases of this geometrical configuration consist in the well-known problems of a crack perpendicular to a bi-material interface and of a sharp re-entrant corner in a homogeneous structural component. The asymptotic analysis provides the basis for a proper numerical modeling of the singular stress field in the framework of generalized singular finite elements.

However, this combination of analytical/numerical techniques is rather complicate and time consuming, particularly if we are interested in performing parametric studies. On the contrary, the approximate model is particularly sound from the engineering point of view, since a single closed-form expression for the computation of the generalized shape function is provided. Approximate and analytical formulations agree satisfactorily, showing the effectiveness of this approach (Carpinteri and Pugno, 2000) which is worth using in parametric analyses for a preliminary design of new components. In fact, it has to be remarked that analogous expressions can be obtained by considering multi-layered plates in tension or even other scenarios involving multi-layered composite geometries.

For the particular benchmark case of a bi-material beam under three-point bending, the effects of elastic mismatch, notch angle and layers thicknesses on the Mode I stress-intensity factor have been quantified and reported in some useful diagrams. The predicted results are particularly important from an engineering point of view, due to the fact that such situations are occurring very frequently in composite structural elements and a proper design should pay attention to them.

Acknowledgement

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