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INFLUENCE OF THE INTERMEDIATE MATERIAL ON THE SINGULAR STRESS FIELD IN TRIMATERIAL JUNCTIONS

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By the method of eigenfunctions, we analyze the stresses formed at the vertex of a multicomponent wedge formed by homogeneous elastic wedges under the conditions of plane stress state or plane deformation. The exponent of the singularity of stresses in the case of opening and shift of the wedges is numerically determined. The singular stresses formed near the vertex are investigated both for softer and stiffer wedges.

Introduction

According to linear elastic fracture mechanics, stress singularities occur in multimaterial junctions. Problems of this sort were extensively studied for composite plates [1–5]. More recently, the experimental and theoretical studies have emphasized me role of interfaces and grain boundaries in understanding the mechanical properties of polycrystalline materials. In 1996, stress singularities at triple junctions in single-phase polycrystals due to freely sliding grain boundaries were investigated by Picu and Gupta [6]. This study was motivated by the need of modeling the nucleation of cracks and cavities in polycrystalline materials. They found that the exponent of the singular stress field depends only on the grain-boundary positions, i.e., on the amplitudes of the wedge angles formed by the grains at the singular point. Furthermore, supersingularities, i.e., singularities more severe than those due to a crack inside a homogeneous material, were found for some particular junction geometries, indicating critical configurations which can give rise to crack nucleation.

In 2001, a theoretical study based on the molecular-dynamics atomistic simulations of the structure and energy conditions for a multiple-twin triple junction in silicon was proposed [7]. By analyzing the calculated excess line and volume energies, as well as the atomic level stress, it was demonstrated that a triple junction can be considered as a true linear defect. It was noticed that these critical points can be a source of residual stresses concentrated at the vertex of the junction.

In the present work, the general problem of multimaterial junctions is formulated within the framework of the eigenfunction expansion method [1, 2]. Then the order of the stress singularity arising from trimaterial junctions is carefully analyzed. More specifically, we focus our attention on material junctions between two different polycrystals embedded into a matrix, which represents an intermediate material. Engineering examples may concern material junctions in three-phase polycrystals with an interphase inclusion. On the basis of the proposed method, the order of the stress singularity is determined for different material combinations. According to the criterion of minimum singularity, as also proposed in [8], it is possible to determine the optimum configurations. The influence of initial defects modeled as cracks inside the softer and stiffer materials is also analyzed. Finally, the issue of size-scale effects on the material strength is addressed.

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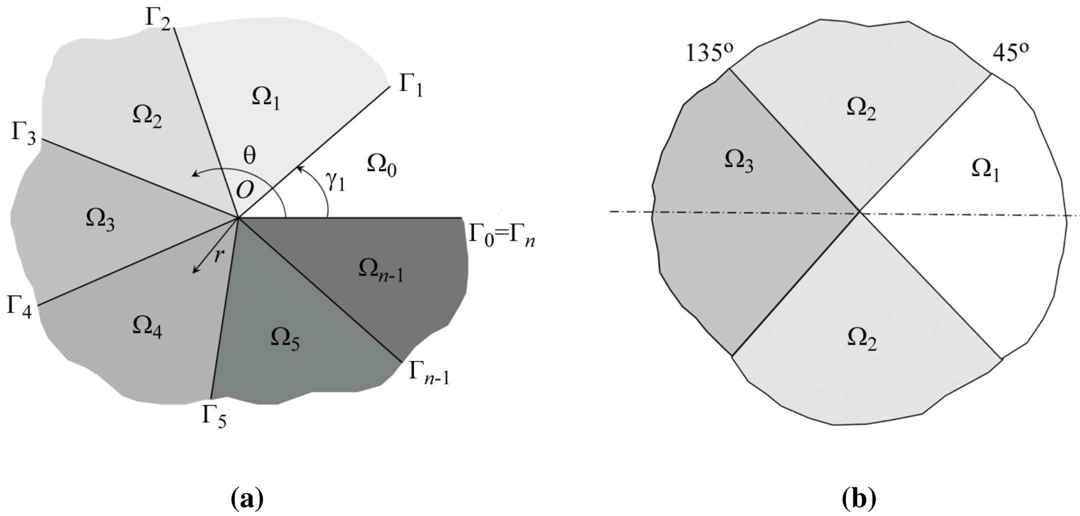

Fig. 1. Schematic diagrams of a general multimaterial problem (a) and the configuration of a trimaterial junction (b) studied in the present work.

Mathematical Formulation

The geometry of a plane problem of n dissimilar wedges with arbitrary angles perfectly bonded along their interfaces intersecting at the same vertex O is depicted in Fig. 1a. Each material region is denoted by Ω_i, with $i = 0, \dots, n-1$, and bounded by the interfaces Γ_i and Γ_{i+1}. The first and the last interfaces corresponding, respectively, to $\theta = 0$ and $\theta = 2\pi$ coincide and are denoted by Γ_0. It is possible to assume that the following separable form holds for the biharmonic stress function Φ_i in the ith subregion [1]:

$$\Phi_i(r, \theta) = \sum_j r^{\lambda_j + 1} f_{i,j}(\theta, \lambda_j), \tag{1}$$

where λ_i and $f_{i,j}$ are called eigenvalues and eigenfunctions, respectively. Polar coordinates (r, θ) are introduced with origin at the vertex of the junction (Fig. 1a).

The summation with respect to j is introduced in Eq. (1), since it is possible to have more than one eigenvalue for each problem. The biharmonic condition requires f_i to be of the form:

$$f_i(\theta, \lambda) = A_i \sin[(\lambda+1)\theta] + B_i \cos[(\lambda+1)\theta] + C_i \sin[(\lambda-1)\theta] + D_i \cos[(\lambda-1)\theta], \tag{2}$$

where A_i, B_i, C_i, and D_i are undetermined constants. For each eigenvalue λ, the corresponding stress and displacement fields are given by the formulas:

$$\sigma_r^i = r^{\lambda-1}[f_i'' + (\lambda+1) f_i], \qquad \sigma_\theta^i = r^{\lambda-1}[\lambda(\lambda+1) f_i], \qquad \tau_{r\theta}^i = r^{\lambda-1}[-\lambda f_i'],$$

$$u_r^i = \frac{r^\lambda}{2G_i}\left\{-(\lambda+1) f_i + \frac{1}{\lambda\kappa_i}[f_i'' + (\lambda+1)^2 f_i]\right\}, \tag{3}$$

$$u_\theta^i = \frac{r^\lambda}{2G_i}\left\{-f_i' - \frac{1}{\lambda(\lambda-1)\kappa_i}\left[f_i''' + (\lambda+1)^2 f_i'\right]\right\},$$

where the primes denote the operation of differentiation with respect to θ, G_i is the shear modulus, and κ_i, in terms of Poisson's ratio, is equal either to $(1+\nu_i)$ for plane stress or to $1/(1-\nu_i)$ for plane strain.

The following boundary conditions on the interface Γ_i can be considered:

$$\begin{aligned} \sigma_\theta^i(r,\gamma_{i+1}) &= \sigma_\theta^{i+1}(r,\gamma_{i+1}), \qquad \tau_{r\theta}^i(r,\gamma_{i+1}) = \tau_{r\theta}^{i+1}(r,\gamma_{i+1}), \\ u_r^i(r,\gamma_{i+1}) &= u_r^{i+1}(r,\gamma_{i+1}), \qquad u_\theta^i(r,\gamma_{i+1}) = u_\theta^{i+1}(r,\gamma_{i+1}) \end{aligned} \tag{4}$$

by taking into account the fact that the interface Γ_0 must be defined by $\gamma = 0$ for region 1 and $\gamma_{n+1} = 2\pi$ for region n. This method yields a set of $4n$ equations in $4n+1$ unknowns A_i, B_i, C_i, D_i, and λ, which can be symbolically represented in the form

$$\Lambda v = 0, \tag{5}$$

where Λ is the coefficient matrix, which is a nonlinear function of λ, and v is the vector of unknowns A_i, B_i, C_i, and D_i. A nontrivial solution of the system of equations (5) exists only if the determinant of the coefficient matrix vanishes. This condition yields a characteristic equation for the (generally speaking, complex) eigenvalues λ, which must be solved. According to our current purposes, we are interested only in the values of λ which may lead to singularities of the stress field. This fact, together with the condition of continuity of the field of displacements at the common vertex of the regions, imply that we seek eigenvalues within the range $0 < \mathrm{Re}\lambda < 1$. Then, to find solutions of the eigenequations, we use a numerical method based on the inspection of the condition number of the coefficient matrix Λ (see [9] for more details).

If the problem is geometrically symmetric, then it is possible to split the elastic field into a symmetric part and a skew-symmetric part, namely, into a part due to the Mode-I deformation and a part due to the Mode-II deformation [10]. In the former case, the following symmetric conditions are applied at $\theta^* = 0, \pi$ instead of Eq. (4):

$$\tau_{r\theta}^i(r,\theta^*) = 0 \quad \text{and} \quad u_\theta^i(r,\theta^*) = 0. \tag{6}$$

In the latter case, the following skew-symmetric conditions are imposed:

$$\sigma_\theta^i(r,\theta^*) = 0 \quad \text{and} \quad u_r^i(r,\theta^*) = 0. \tag{7}$$

In this way, the problem is reduced to two factors: the former determines the eigenvalues corresponding to Mode-I deformation, whereas the latter determines the eigenvalues corresponding to Mode-II deformation.

Perfectly Bonded Trimaterial Junctions

The problem of a trimaterial junction schematically depicted in Fig. 1b is investigated. Two materials occupying two quarter planes with a specified ratio of their elastic moduli E_3/E_1 are joined by an intermediate material whose elastic modulus E_2 varies from E_1 to E_3. Different mechanical configurations characterized by the ratio E_3/E_1 are considered. The real and imaginary parts of the eigenvalues are numerically computed and shown in Fig. 2 as functions of the modular ratio E_2/E_1. The limit case given by $E_3/E_1 = 0$ corresponds to a bimaterial junction (Fig. 2a). The solutions for trimaterial junctions characterized by $E_3/E_1 = 100$ and $E_3/E_1 = 1000$ are also illustrated in Figs. 2b, c. For these problems, the limit cases given either by $E_2/E_1 = 1$ or by $E_2/E_1 = E_3/E_1$ correspond to bimaterial junctions whose eigenvalues are presented in [10, 11].

Due to the geometric symmetry of the problems, the roots of the eigenequations corresponding to Mode-I or Mode-II deformation can be computed separately. It is worth noting that, regardless of the value of the elastic modular ratio E_3/E_1, there are some mechanical configurations admitting only one root of the eigenequation (Figs. 2b, c). More specifically, if the elastic modulus of the intermediate material is close to the elastic modulus of the softer component, no stress singularities due to Mode-I deformation are found. On the other hand, as the Young's modulus of the intermediate material approaches the Young's modulus of the third material, the Mode-II stress singularities are avoided. These trends are completely general for this type of trimaterial junction and can be observed for any given value of the parameter E_3/E_1 different from one.

In the case of Mixed-Mode deformation, a criterion of minimum singularity would suggest that a more suitable material configuration is attained for the case corresponding to the maximum eigenvalues. In fact, for $\mathrm{Re}\,\lambda = 1$, the singularity disappears. As a consequence, for the analyzed trimaterial junction problems, the optimum configuration would correspond to the stiffness ratio such that $\mathrm{Re}\,\lambda^{\mathrm{I}} = \lambda^{\mathrm{II}}$ (Figs. 2b, c). By considering trimaterial junctions with different values of the ratio E_3/E_1, the value of E_2/E_1 is computed according to the criterion presented above and depicted in Fig. 2d vs. the ratio of the elastic moduli of materials 1 and 3. From these results, the following best-fit curve is obtained:

$$\frac{E_2}{E_1} = 0.933\left(\frac{E_3}{E_1}\right)^{0.467} \cong \sqrt{\frac{E_3}{E_1}}. \tag{8}$$

In practice, this condition rules out intermediate materials whose elastic moduli are close to the elastic modulus of the stiffer material. Furthermore, it is worth noting that all roots are necessary to describe the stress state in both materials, since it cannot be concluded that any root is somehow less dominant.

Cracked Trimaterial Junctions

Debonding along interfaces in trimaterial junctions was studied in [8]. However, little attention was paid to the problem of transgranular cracks. Hence, some of the previous results obtained for the order of the stress singularity are extended to the case of a crack placed along the symmetry line inside material 1 (or 3).

If we consider, as a representative case, the trimaterial junction characterized by the ratio $E_3/E_1 = 100$ (as in the uncracked problem analyzed in Fig. 2b), then the Young's modulus of the intermediate material is an independent variable of the problem and varies from E_1 to E_3. A crack is then considered either in the softer material (Fig. 3a) or in the stiffer material (Fig. 3b). In each studied case, the real and imaginary parts of Mode-I and Mode-II eigenvalues are computed.

In the limit case where the crack lies in the softer material and $E_2/E_1 = 1$, the problem reduces to a cracked bimaterial junction with $E_2/E_1 = 1/100$. Similarly, if the trimaterial junction is characterized by $E_2/E_1 = 100$, then a different bimaterial problem is addressed. The same reasoning holds when the crack lies in the stiffer material and the limit cases $E_2/E_1 = 1$ and $E_2/E_1 = 100$ are considered.

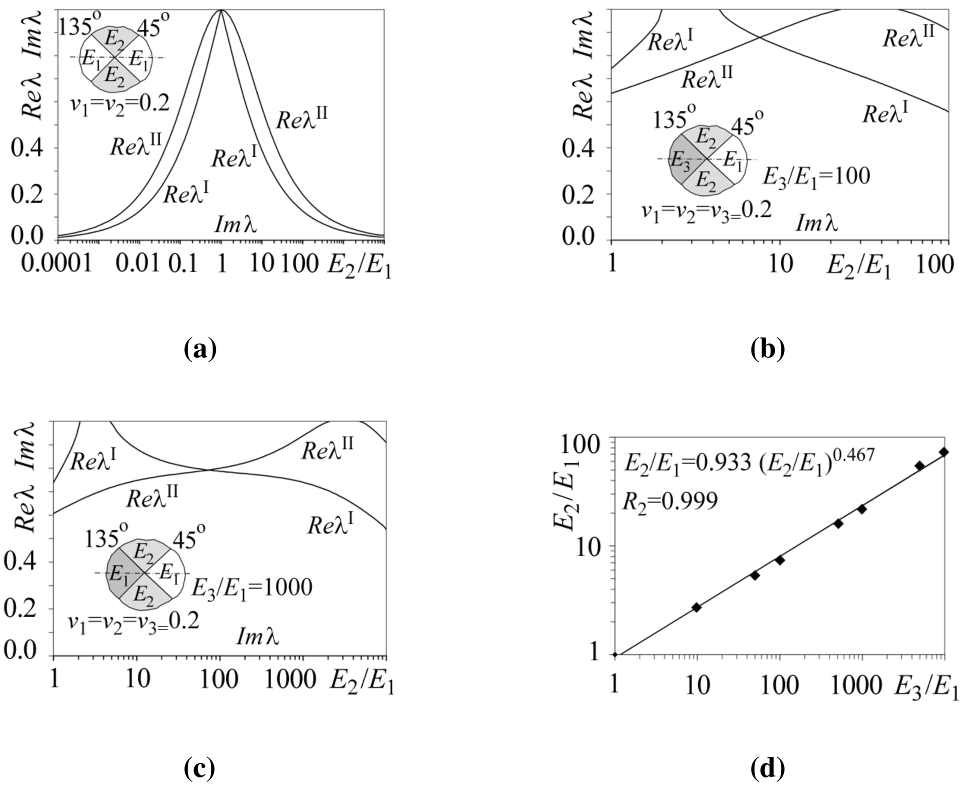

Fig. 2. Eigenvalues for perfectly bonded trimaterial junctions under the conditions of plane strain characterized by $E_3/E_1 = 1$ (a), $E_3/E_1 = 100$ (b), and $E_3/E_1 = 10{,}000$ (c); the relationship between E_2/E_1 and E_3/E_1 according to the minimum singularity criterion (d).

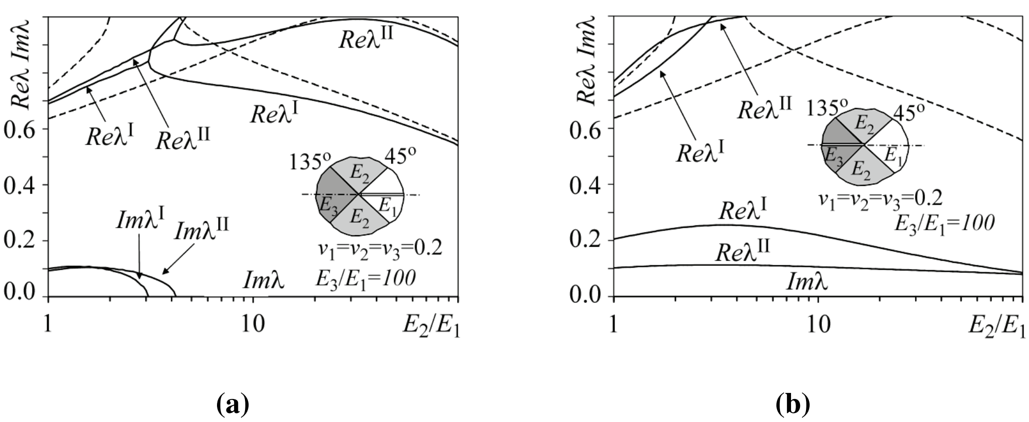

Fig. 3. Eigenvalues for cracked trimaterial junctions characterized by $E_3/E_1 = 100$. The crack lies either in the softer material (a) or in the stiffer material (b).

The asymptotic results concerning bimaterial problems of this sort are presented in [11]. For the direct comparison between cracked and uncracked solutions, the eigenvalues of the corresponding uncracked problems are shown as the dashed lines in Fig. 3.

If the crack lies in the softer material (Fig. 3a), then the Mode-I and Mode-II eigenvalues remain greater than 0.5, as in the uncracked problem. The main difference is represented by the fact that, in the range $3 <$

$E_2/E_1 < 4$, the Mode-II eigenvalues are complex and two real roots are found for Mode-I deformation. Furthermore, for E_2/E_1 approximately equal to 3, a transition from two real roots to a complex conjugate couple also occurs for Mode-I eigenvalues.

On the other hand, a completely different behavior is observed for a crack in the stiffer material (Fig. 3b). The eigenvalues computed for this problem are real and two additional roots are found within the range $1 < E_2/E_1 < 4$ as compared to the uncracked configurations. These roots are much lower than 0.5, i.e., the corresponding order of the stress singularity is more severe than in the case of a crack inside a homogeneous material. For $E_2/E_1 > 4$, the eigenvalues greater than 0.5 disappear and the stress state is characterized solely by the lower roots.

As also observed for cracked bimaterial junctions [11], these results imply that the asymptotic behavior of the singular stress field can be strongly influenced by cracks in the stiffer material. Both Mode-I and Mode-II stress singularities are equally important for the description of the stress state in the trimaterial regions.

Discussion and Conclusions

In the present work, the formulation of the general problem of multimaterial junctions is performed according to the mathematical formalism of the eigenfunction-expansion method. The numerical results are obtained for trimaterial junctions perfectly bonded along their interfaces. The materials are linearly elastic, homogeneous, and isotropic. Moreover, they are in the plane stress (or strain) state. Thanks to the geometric symmetry of the considered problems, the singularities due to the Mode-I and Mode-II types of deformation are separated. For some of the above-mentioned geometries, the influence of transgranular cracks on the asymptotic stress state is investigated.

From the engineering point of view, the obtained results contain useful information about the variation of the order of the stress singularity as the stiffness of one material varies relative to the stiffnesses of the other materials. Contrary to the results obtained by Pageau, et al. [9] suggesting that the intermediate material increases the order of the stress singularity in trimaterial junction geometries when one material occupies a half-plane region, we show that the third material can be favorable for the investigated trimaterial junctions. The criterion of minimum singularity suggests that the best choice is given by the intermediate material whose elastic modulus is closer to the elastic modulus of the softer component. The results concerning cracked geometries demonstrate that trimaterial junctions are significantly influenced by cracks in the stiffer material. The numerical results show that all eigenvalues must be taken into account for the accurate analysis of elastic stresses.

Finally, it should be emphasized that the presence of trimaterial junctions in the material microstructure can have an important implication to the size–scale effects on the strength of materials. In fact, for a linear elastic structure with a characteristic size b, the components of the stress field at the vertex of a trimaterial junction are

$$\sigma_{ij} = K^* e^{\mathrm{Re}\lambda - 1} f_{ij}(\theta), \tag{9}$$

where the eigenvalue $\mathrm{Re}\,\lambda$ is determined according to the asymptotic analysis. If the dimensional analysis is applied, then the generalized stress-intensity factor can be expressed as follows [12, 13]:

$$K^* = \sigma b^{1-\mathrm{Re}\lambda} g, \tag{10}$$

where σ is the nominal stress applied to the structure, b is a characteristic structural size, and g is a shape factor depending on the geometry of the structure and the topology of the junction.

The failure stress σ_f is reached when the generalized stress-intensity factor K^* becomes critical, i.e., for $K^* = K_c^*$:

$$\sigma_f = \frac{K_c^* b^{\mathrm{Re}\lambda - 1}}{g}. \tag{11}$$

In the logarithmic form, Eq. (11) turns into

$$\sigma_f = (\ln K_c^* - \ln g) - (1 - \mathrm{Re}\,\lambda)\ln b. \tag{12}$$

On the bilogarithmic plot, Eq. (12) implies that the strength is a linear function of the structural size with a slope of $\mathrm{Re}\,\lambda - 1$. If the singularity vanishes, i.e., $\mathrm{Re}\,\lambda = 1$, then the scale effect disappears. Therefore, the minimum singularity criterion can also be regarded as the minimum scale-effect criterion.

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